Uncertainty Traps*

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Abstract

We develop a theory of endogenous uncertainty and business cycles in which short-lived shocks can generate long-lasting recessions. In the model, higher uncertainty about fundamentals discourages investment. Since agents learn from the actions of others, information flows slowly in times of low activity and uncertainty remains high, further discouraging investment. The unique equilibrium of this economy displays uncertainty traps — self-reinforcing episodes of high uncertainty and low activity. While the economy recovers quickly after small shocks, large temporary shocks may have nearly permanent effects on the level of activity. The economy is subject to an information externality but uncertainty traps remain even in the efficient allocation. We extend our framework to include additional features of standard business cycles models and show that uncertainty traps can substantially worsen recessions and increase their duration, even under optimal policy interventions.

JEL Classification: E32, D83

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1 Introduction

One of the central features of macroeconomic activity is its high persistence. The NBER definition of business cycles implies that it takes for the U.S. economy close to 40 months on average to recover from a through until the next peak. Business cycles are also asymmetric: it takes about 17 months for the economy to move from peak to through, so that recoveries last on average more than twice as long as downfalls. These features have been visible during in the 2007-2009 crisis. The unemployment rate increased from 4.4% in May 2007 to 10% in October 2009 and has barely decreased to a level 7.9% in early 2013.

What explains these prolonged declines in economic activity? In this paper, we develop a quantitative theory of endogenous uncertainty and business cycles to explain these phenomena. The theory captures two key forces. First, higher uncertainty about fundamentals discourages investment. Second, economic agents can learn from the actions of others. The interaction between these two forces creates room for uncertainty traps — self reinforcing episodes of high uncertainty and low activity. In times of low activity information flows slowly and uncertainty stays high, further discouraging investment. This explains why low activity may persist under good fundamentals.

We first develop a baseline theory that includes only the essential features of the mechanism, and then we extend the model in various dimensions for a quantitative evaluation. In the model, firms choose to undertake an irreversible investment whose return depends on an imperfectly observed fundamental. Beliefs about that fundamental are common to all firms, but can be regularly updated using various signals. Formally, we define uncertainty as the variance of the prior about the fundamental. Information, in turn, diffuses through a simple social learning channel: the higher the number of firms that invests, the larger the number of signals received by firms and the stronger the reduction in their uncertainty.

This environment naturally produces an interaction between beliefs and economic activity. Firms are more likely to invest if they hold more optimistic or less uncertain beliefs about the fundamental. Therefore, low uncertainty is associated with a high investment rate. At the same time, the law of motion for beliefs depends on the investment rate through social learning. When few firms invest, uncertainty rises and the firms' optimism, captured by the mean of the beliefs distribution, is less likely to fluctuate.

Using this setup we demonstrate the existence of uncertainty traps. Formally, we define an uncertainty trap as the coexistence of multiple stationary points in the joint dynamics of uncertainty and economic activity for a given mean of the distribution of beliefs. We refer to these fixed points as regimes. Due to the complementarity between investment and information diffusion, in high-activity regimes there is low uncertainty and in low-activity regimes there is high uncertainty. Despite this multiplicity, the recursive equilibrium is uniquely pinned down by the stochastic evolution of the mean level of beliefs. But, because of it, the unique equilibrium is prone to nonlinear dynamics and asymmetries. For example, the long-run response to a temporary negative shock becomes considerably more protracted when its magnitude is above some threshold. The economy quickly recovers after a small temporary shock, but it may permanently shift into a low activity regime after a large shock of the same duration. In turn, a positive temporary shock of sufficient magnitude can put the economy back on track.

As in other theories of social learning, there are inefficiently low levels of investment because agents do not internalize the effect of their actions on common information. This inefficiency naturally creates room for welfare-enhancing policy interventions. To find these policies, we study the problem of a constrained planner that is subject to the same informational constraints as private agents. We find that the socially constrained-efficient allocation can be implemented with aggregate-beliefs dependent subsidies. For example, it could be desirable to subsidize investment in times of high uncertainty and low activity. However, under certain conditions, the optimal policy does not eliminate the uncertainty traps. Therefore, while policy interventions may be desirable, they do not necessarily eradicate the nonlinearities generated by the complementarity between uncertainty and economic activity.

After characterizing the model, we evaluate the quantitative importance of the uncertainty traps. For that, we extend the baseline model to bring it closer to general real business cycle models. Among other features, we generalize the capital accumulation process by adding an intensive margin. We also introduce a risk-averse representative household with endogenous labor supply. To estimate the importance of uncertainty traps, we compare the outcomes from our extended model with a restricted setup in which uncertainty is not allowed to adjust endogenously. In preliminary numerical exercises, we find that uncertainty traps make economic downturns more persistent and pronounced relative to a framework with fixed uncertainty.

The emphasis on the wait-and-see effect of uncertainty on investment is shared with a recent literature that studies how changes in the volatility of productivity shocks affects the economy, such as Arellano et al. (2012), Bachmann and Bayer (2009), Bloom (2009), Bloom et al. (2012), and Schaal (2012). Two features set us apart from that literature. First, these papers focus on uncertainty induced by time-varying volatility in productivity. In contrast, our learning approach enables us to dissociate subjective uncertainty from volatility in fundamentals. While in our setup volatility generates uncertainty, there can also be periods of high uncertainty with constant volatility. Second, in our analysis the movements in uncertainty are endogenous. That literature focuses, in contrast, on exogenous volatility shocks to productivity. These two distinguishing features create the additional propagation of shocks that we explore in the paper.

The notion of uncertainty in this paper seems justified in the face of systematic references by businessmen and commentators to high levels of uncertainty in the aftermath the 2007-2009 recession despite the decline in several measures of volatility.[1] Indeed, our theory allows for uncertainty to persist in a context with low volatility. The advantage of allowing for endogenous uncertainty movements (as opposed to exogenous volatility shocks) is that endogenous uncertainty is better able to deliver persistent macroeconomic series. Because high volatility events are short-lived, models that focus on that type of shock are hard to reconcile with the persistence of recessions. In contrast,

[1]Measures of aggregate and idiosyncratic volatility such as the VIX volatility index have substantially declined since 2009, as shown in Schaal (2012). Another interesting source of uncertainty suggested in the literature, from which we abstract in this paper, is policy uncertainty. See Baker et al. (2012); Fernández-Villaverde et al. (2011).

subjective uncertainty traps can deliver persistence in a low-volatility context.

Endogenous movements in uncertainty can be modeled in different ways. We make this notion operative using a simple concept of social learning. Intuitively, we envision firms holding bits and pieces of information about a shared fundamental; when a firm invests or hires, its actions reveal information about the state of the economy to other agents. Hence, our analysis relates to papers on fads and herding in the tradition of Banerjee (1992), Bikhchandani et al. (1992), and Chamley and Gale (1994). A number of studies, such as Cunningham (2004), Kaustia and Knüpfer (2009), Khang (2012), and Patnam (2011), empirically document the relevance of social learning in various contexts such as investment in the semiconductor industry, stock market entry decisions, housing purchases, and R&D expenditures. Social learning about technology has also been demonstrated to be important in other contexts such as economic development, as shown by Foster and Rosenzweig (1995) and Hausmann and Rodrik (2003).

Our analysis also relates to a theoretical macroeconomic literature that studies environments with learning from market outcomes such as Rob (1991), Caplin and Leahy (1993), Zeira (1994), Veldkamp (2005), Ordonez (2009), and Amador and Weill (2010), as well as to papers on endogenous volatility over the business cycle, such as Bachmann and Moscarini (2011) and DErasmo and Boedo (2011). Specially related is the analysis in Van Nieuwerburgh and Veldkamp (2006). In their model, agents hold beliefs about a fundamental and the signal-to-noise ratio varies procyclically; this delays recoveries because agents discount new information more heavily in recessions. However, in that paper, uncertainty about the fundamental provides a weak feedback and the economy quickly learns its way out of a recession. The key feature that distinguishes our analysis is the presence of irreversible investments. The option value created by irreversibilities offers a strong additional motive for agents to defer investment in uncertain times. The interaction between social learning and irreversible investment leads, in our setup, to nearly permanent effects of uncertainty on economic activity.

The paper is structured as follows. Section 2 presents the baseline model and the definition of the recursive equilibrium. Section 3 characterizes the partial-equilibrium investment decision of an individual firm, the uniqueness of the equilibrium, the existence of uncertainty traps, and the welfare implications. Section 4 features the preliminary quantitative analysis using an extended model. Section 5 concludes. Proofs are relegated to the appendix.

2 Suggestive Evidence

The central channel in the theory is the feedback between uncertainty and investment. We argue that the inactivity of firms during recessions slows down the diffusion of information, creating uncertainty and discouraging further investment. Therefore, the model predicts that recessions are times where both uncertainty and firm inactivity are high, and that these features may persist even if productivity has recovered. In this section we provide first-pass evidence consistent with these features of the model.

2.1 Uncertainty over the Business Cycle

The literature that studies the impact of uncertainty shocks establishes that the variance in idiosyncratic shocks to productivity increases during bad times. Bloom et al. (2012) demonstrate that the dispersion of plant- and industry- level shocks to productivity is counter-cyclical and peaks in recessions. Other commonly used measures of firm-level volatility, such as the VIX index of volatility in stock market returns or the dispersion in firm level sales, reproduce the same pattern.

Our theory is more specifically concerned with subjective uncertainty. Direct measures of subjective uncertainty are also available and exhibit similar counter-cyclical patterns. Different surveys ask respondents to assess the main reasons why they prefer to postpone economic decisions. According to the National Federation of Independent Business (2012), 40% of answers rank "economic uncertainty" as the most critical problem that they faced in 2012. A more systematic evidence comes from the Michigan Survey of Consumers, which shows a peak during recessions in the percent of consumers who state that "uncertain future" is the main reason to postpone purchases of durable goods in the United States (see figure 1). A similar pattern is observed in the UK, where firm's perceived uncertainty increases during recessions according to the CBI Industrial Trends Survey. Leduc and Liu (2012) argue that these subjective measures of uncertainty are countercyclical.

Other measures of subjective uncertainty with strong counter-cyclical patterns are the variance of ex-post forecast errors about economic conditions and the dispersion of beliefs featured in Bachmann et al. (2013). In bad times, agents hold more heterogeneous beliefs about future economic conditions. Our model allows for short-lived dispersion of beliefs among economic agents within each period, and predicts that the within-period variance of beliefs is larger when uncertainty is high and economic activity is low.

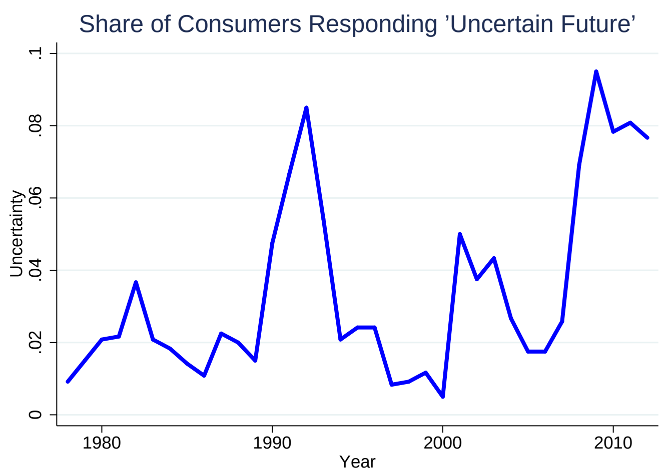

Figure 1: Subjective Uncertainty over Time (Source: Michigan Survey of Consumers)

2.2 Share of Zeros in Investment over the Business Cycle

A second piece of evidence consistent with the basic mechanism that we present concerns the incidence of firm inactivity during recessions. While aggregate investment naturally is counter-cyclical, we emphasize a microeconomic channel based on the inactivity of firms. Because firms face indivisible investment choices, their incentives to invest are low when uncertainty is high. In turn, we posit that lack of activity slows down diffusion of information. Therefore, we expect to see a higher fraction of firms that do not invest when the level of activity is low.

The literature on lumpy adjustment studies the distribution of investment rates over the business cycle, but it focuses more on investment spikes rather than zeros. In this literature, Cooper and Haltiwanger (2006) report that 8% of plant-year observations in the United States between 1972 and 1998 have investment rates below 1% in absolute value. For us, the important question is how this fraction varies over the business cycle. Gourio and Kashyap (2007) report the share of investment zeros for the US and Chile between 1975 and 2000, arguing that in both countries the share of exact or near-zeros is "strongly countercyclical". They report a correlation between aggregate investment and the share of investment rates close to zero of -0.94 in the US and -0.56 in Chile. Since aggregate investment is strongly procyclical, this implies that the share of zeros or near-zeros in firm-level investment is countercyclical.

We complement this evidence with data on the prevalence of exact and near-zeros in investment for a longer time series that includes the current recession. For that, we use quarterly data from Compustat between 1975 and 2012. We follow Eisfeldt and Rampini (2006) in using the variable Property, Plant and Equipment as proxy for physical capital at the firm level. Figure 2 shows the share of firms with zero or near-zero investment rates. To compute these figures, we first restrict the Compustat dataset to firms with non-missing investment rates at the quarterly level and to quarters with at least 500 such firms. Then, we calculate the share of zeros or near-zeros in each quarter. The figure shows the average shares across all quarters in each year, distinguishing between all firms and firms in manufacturing only.

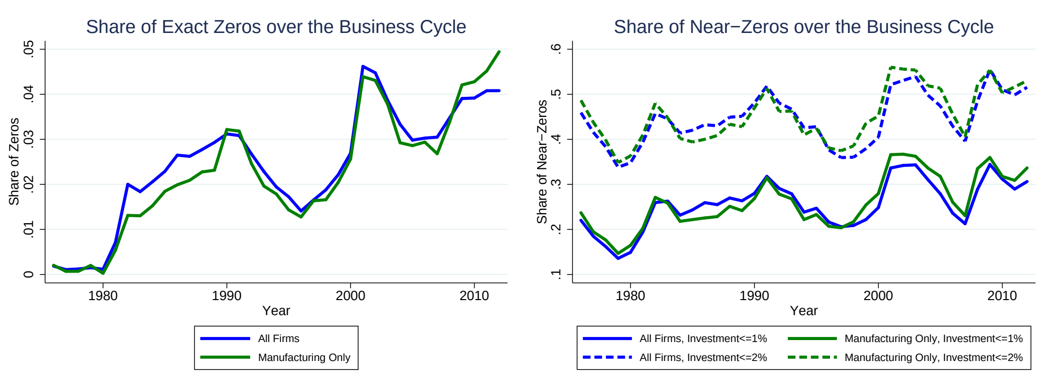

Figure 2: Share of firms with Zero or Near-Zero Investment (Source: Compustat)

On average, 2% of firms display zero investment rates at the quarterly level. In turn, the share of inactive firms is countercyclical. For the years in which the series overlaps with the Gourio and Kashyap (2007) data, both series display similar properties.[2] Interestingly, for the recent recession investment inactivity spikes and remains relatively high after economic activity has recovered. In 2012, 4.9% of firms display zero change in capital. Similar patterns are observed for the share of firms near zero investment in absolute value. For example, the average share of firms with investment rate below 1% is 33%, and with investment rate below 2% is 53%, and both measures peak in bad times. Figure 3 shows the positive correlation between uncertainty from the Michigan Consumer Survey and the share of firms with zero investment from Compustat. The years corresponding to the 2007-2009 crisis and its aftermath appear on the upper right area of the graph.

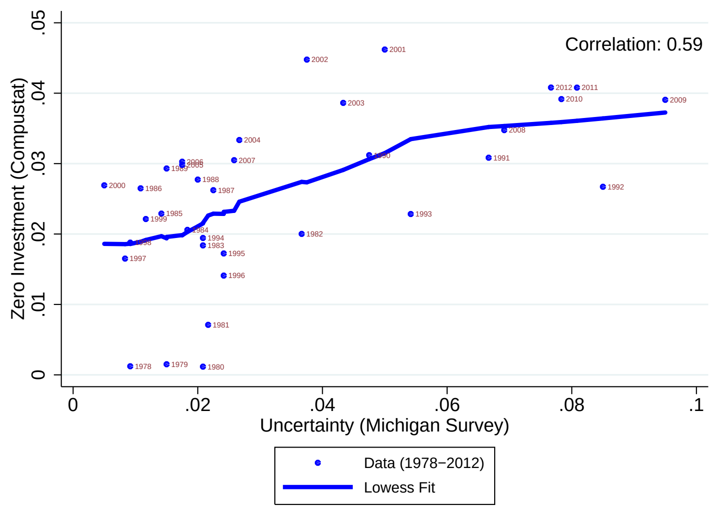

Figure 3: Uncertainty and Share of Firms with Zero Investment

3 Baseline Model

We present a stylized model that features only the necessary ingredients to generate uncertainty traps. The intuitions from this simple model as well as the laws of motion governing the dynamics of uncertainty carry through to the full model that we use for our numerical analysis.

3.1 Population and Technology

Time is discrete. There is a large, fixed number of firms $\bar{N}$ indexed by $n \in \{1, \ldots, \bar{N}\}$. Each firm has a unique investment opportunity that produces output x_n which is the sum of two components: a persistent common component θ which denotes the economy's fundamental as well as

[2] Gourio and Kashyap (2007) use establishment level data from the Census Bureau's Annual Survey of Manufacturers. The correlation between the share of firms with investment rate below 2% in their data and in Compustat is 0.65.

an idiosyncratic transitory component ε_n^x,

$$x_n = \theta + \varepsilon_n^x.$$

The common component follows a random walk, so that the next period's fundamental is

$$\theta' = \theta + \varepsilon^\theta. \tag{1}$$

The innovations $\left(\varepsilon^\theta, \varepsilon_n^x\right)$ are independent and normally distributed,

$$\varepsilon^\theta \sim \mathcal{N}\left(0, \gamma_\theta^{-1}\right) \text{ and } \varepsilon_n^x \sim \mathcal{N}\left(0, \gamma_x^{-1}\right).$$

To produce, a firm must pay a fixed cost f, drawn each period from the continuous cumulative distribution F with mean $\bar{f}$ and variance σ^f. Once production has taken place, the firm exits the economy and is immediately replaced by a new firm with an investment opportunity. This assumption ensures that the mass of firms in the economy remains constant.[3]

Upon exit, the firm receives the flow payoff $u(x_n)$. Firms have constant absolute risk-aversion[4],

$$u\left(x_n\right) = \frac{1}{a}\left(1 - e^{-a x_n}\right),$$

where a is the coefficient of absolute risk aversion.

3.2 Timing and Information

At the beginning of each period, firms decide whether to invest or not without knowing their productivity x_n. Their decision therefore depends on their beliefs about the unobserved fundamental θ. As time unfolds, they learn about θ in various ways. First, firms learn about the fundamental from a public signal Y observed at the end of each period,

$$Y = \theta + \varepsilon^y \tag{2}$$

where $\varepsilon^y \sim \mathcal{N}(0, \gamma_y^{-1})$. This signal captures the information released by statistical agencies or the media. Second, they learn by observing total production in the economy: when firm n invests, its output x_n is observed by all the other firms. Since θ cannot be distinguished from the idiosyncratic term ε_n^x, production x_n acts as a noisy signal of the fundamental. Because of the normality assumption, a sufficient statistic for the information provided by each firm's individual output is the public signal

$$X \equiv \frac{1}{N} \sum_{n \in I} x_n = \theta + \varepsilon_N^X, \tag{3}$$

[3]The assumption that firms exit when they invest is for tractability of the baseline model and it can be relaxed.

[4]The assumption of risk aversion is not necessary for the results. We include it for technical reasons in the general-equilibrium uniqueness proof. In the simulation of the numerical model, we show that the main mechanisms carry through with risk neutrality.

where $N \in \{1, \ldots, \bar{N}\}$ is the endogenous number of firms that invest, I is the set of such firms and where

$$\varepsilon_N^X \equiv \frac{1}{N} \sum_{n \in I} \varepsilon_n^x \sim \mathcal{N}\left(0, (N\gamma_x)^{-1}\right).$$

It is important to note that the precision of this signal, $N\gamma_x$, increases with the number of investing firms, N. The higher N is, the more precise is the information collected by each individual firm by observing production. Social learning takes place through this channel in the economy.

Under the assumption of a common initial prior, and because all signals become public at the end of a period, beliefs are common across firms. Our normality assumptions about the signals and the fundamental allow us to describe beliefs about θ by

$$\theta \mid \mathcal{I} \sim \mathcal{N}\left(\mu, \gamma^{-1}\right),$$

where $\mathcal{I}$ is the information set at the beginning of the period. The mean of the distribution μ captures the optimism of agents about the state of the economy, while γ represents the precision of their beliefs about the fundamental. Precision γ is inversely related to the amount of uncertainty in the economy: as γ increases, the variance of beliefs decreases and uncertainty declines. The aggregate state space of the economy therefore reduces to the common beliefs (μ, γ).

The timing of events is summarized in Figure 4.

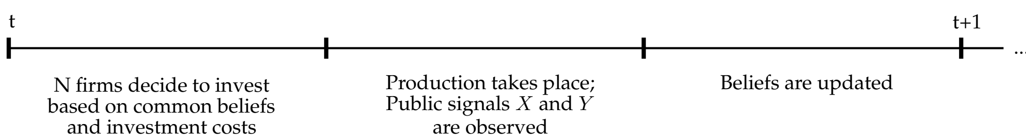

Figure 4: Timing of events

3.3 Firm Problem

In each period, given fixed cost f and beliefs about the fundamental (μ, γ), a firm can either wait or invest. Its value is

$$V(\mu, \gamma, f) = \max\left\{V^W(\mu, \gamma), V^I(\mu, \gamma) - f\right\}, \tag{4}$$

where $V^W(\mu, \gamma)$ is the value of the firm if it waits until the next period and $V^I(\mu, \gamma)$ is the value of the firm after incurring the investment cost f.

We assume that the number of firms $\bar{N}$ is large enough so that firms behave competitively. Specifically, they do not internalize the impact of their decisions on aggregate information. The firm's problem yields an optimal investment rule $\chi(\mu, \gamma, f) \in \{0, 1\}$, such that[5]

$$\chi(\mu, \gamma, f) = \begin{cases} 1 & \text{if invests } \Leftrightarrow V^I(\mu, \gamma) - f \geq V^W(\mu, \gamma) \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

When a firm waits, it starts the next period with new beliefs (μ', γ') about the fundamental and a new draw of the fixed cost f'. Therefore, the value of waiting is

$$V^W(\mu, \gamma) = \beta \mathbb{E}\left[V\left(\mu', \gamma', f'\right) | \mu, \gamma\right]. \tag{6}$$

In turn, when a firm invests it receives output x and exits. Therefore, the value of investing, net of the fixed cost, is

$$\begin{aligned} V^I(\mu, \gamma) &= \mathbb{E}\left[u(x) | \mu, \gamma\right] \\ &= \mathbb{E}\left[\frac{1}{a}\left(1 - e^{-a \cdot x}\right) | \mu, \gamma\right] = \frac{1}{a}\left(1 - e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)}\right). \end{aligned} \tag{7}$$

3.4 Law of Motion for Common Beliefs (μ, γ)

Firms start the period with beliefs (μ, γ). By period's end, they have observed the public signals X and Y defined in (3) and (2). Standard rules for Bayesian updating imply that the common posterior about θ is normally distributed with mean and precision of information equal to

$$\begin{aligned} \mu_{post} &= \frac{\gamma\mu + \gamma_y Y + N\gamma_x X}{\gamma + \gamma_y + N\gamma_x}, \\ \gamma_{post} &= \gamma + \gamma_y + N\gamma_x. \end{aligned}$$

These standard updating rules have a straightforward interpretation: the mean of the posterior belief is a precision-weighted average of the past belief μ and the new signals, Y and X, whereas its precision is the sum of the precision of the prior belief, γ, and the precision of the the new signals.

The pair $(\mu_{post}, \gamma_{post})$ describes the beliefs by period's end. Between the two periods, the fundamental θ evolves according to the random walk (1). Therefore, beliefs about next period's fundamental θ' are normally distributed with mean and precision equal to

$$\mu' = \mu_{post} = \frac{\gamma\mu + \gamma_y Y + N\gamma_x X}{\gamma + \gamma_y + N\gamma_x}, \tag{8}$$

$$\gamma' = \left(\frac{1}{\gamma_{post}} + \frac{1}{\gamma_\theta}\right)^{-1} = \left(\frac{1}{\gamma + \gamma_y + N\gamma_x} + \frac{1}{\gamma_\theta}\right)^{-1} \equiv \Gamma(N, \gamma). \tag{9}$$

Conditions (8) and (9) are the laws of motions for beliefs. The first moment, μ', depends on the aggregate public signals X and Y. The number of investing firms, N, determines the quality

[5] We assume that firms choose to invest in the case of indifference. This assumption is innocuous as these events happen with probability 0.

of the public signal X. In turn, the evolution of the precision of information γ' only depends on N and γ. The higher N is, the more precise the public signal X is and the higher the precision of the beliefs about θ'. We let $\Gamma(N,\gamma)$ in (9) be the law of motion of the precision of information.

3.5 Law of Motion for the Number of Investing Firms N

From (5), a firm invests if its cost f_n falls below the threshold $V^I(\mu,\gamma)-V^W(\mu,\gamma)$. Therefore, the process for the number of investing firms N satisfies

$$N\left(\mu,\gamma,\{f_n\}_{1\le n\le\bar N}\right)=\sum_{n=1}^{\bar N}\chi(\mu,\gamma,f_n)=\sum_{n=1}^{\bar N}\mathbb{I}\left(V^I(\mu,\gamma)-f_n\ge V^W(\mu,\gamma)\right). \tag{10}$$

Since the investment rule $\chi(\mu,\gamma,f_n)$ is a function of the random fixed cost, the number of investing firms is a random variable that depends on the realization of the shocks $\{f_n\}_{1\le n\le\bar N}$. As these costs are i.i.d., the ex-ante probability of investment for each firm is

$$p(\mu,\gamma)\equiv F\left(V^I(\mu,\gamma)-V^W(\mu,\gamma)\right).$$

and the ex-ante distribution of N is binomial,

$$N\sim \text{Bin}\left(\overline{N},p(\mu,\gamma)\right).$$

Note that because the shocks $\{f_n\}_{1\le n\le\overline{N}}$ are independent from the fundamental θ and investment decisions are made before the observation of $\{x_n\}_{1\le n\le\overline{N}}$, the law governing N is a simple function of beliefs (μ,γ). In particular, there is nothing to learn from the non-investment of firms, nor from the realization of N itself.

3.6 Recursive Equilibrium

We are ready to define a recursive equilibrium:

Definition 1. A recursive equilibrium consists of a policy function $\chi(\mu,\gamma,f)$, value functions $V(\mu,\gamma,f)$, $V^W(\mu,\gamma)$, $V^I(\mu,\gamma)$, laws of motions for aggregate beliefs $\{\mu',\gamma'\}$, and a number of investing firms $N\left(\mu,\gamma,\{f_n\}_{1\le n\le\bar N}\right)$, such that

1. The value function $V(\mu,\gamma,f)$ solves (4), with $V^W(\mu,\gamma)$ and $V^I(\mu,\gamma)$ defined according to (6) and (7), with the corresponding policy function $\chi(\mu,\gamma,f)$ in (5);
2. The aggregate beliefs (μ,γ) evolve according to (8) and (9); and
3. The number $N\left(\mu,\gamma,\{f_n\}_{1\le n\le\bar N}\right)$ of firms that invest is given by (10).

4 Equilibrium Characterization

We first characterize the optimal investment decision of a firm given the laws of motions for beliefs. Because of the irreversibility of their investment, we show that firms are less likely to invest when uncertainty is high through a standard option value effect. We then establish the existence and uniqueness of a recursive equilibrium and characterize its key properties. In particular, we examine the interaction between firms' behavior in the face of uncertainty and social learning. This interaction leads to episodes of self-sustaining uncertainty and low activity, which we call uncertainty traps. We provide sufficient conditions on the parameters that guarantee the existence of uncertainty traps. We then discuss the type of aggregate dynamics that they imply. The response of the economy to shocks is highly non-linear: large, short-lived shocks may plunge the economy into long-lasting recessions. Finally, we characterize the planning problem and show that uncertainty traps still exist in the efficient allocation. We also discuss policy implications.

4.1 Investment Rule Given the Evolution of Beliefs

The optimal investment rule $\chi(\mu, \gamma, f)$ crucially depends on how beliefs evolve in the economy. We therefore begin by establishing two simple lemmas about the dynamics of aggregate beliefs.

Evolution of the Mean of Beliefs Using (8), we can characterize the stochastic process for the mean of beliefs about the fundamental.

Lemma 1. *For a given number of investing firms N, the mean of beliefs μ follows a random walk with time-varying volatility s,*

$$\mu^{'}|\mu, \gamma = \mu + s(N, \gamma)\,\varepsilon,$$

where $s(N, \gamma) \equiv \left(\frac{1}{\gamma} - \frac{1}{\gamma+\gamma_y+N\gamma_x}\right)^{\frac{1}{2}}$ *and* $\varepsilon \sim \mathcal{N}(0, 1)$.

Proof. The full statements and proofs of propositions are relegated to the Appendix. □

The mean of beliefs captures the optimism of agents about the fundamental. It evolves stochastically due to the the arrival of new information. Its volatility $s(N, \gamma)$ is time-varying because the amount of information that firms collect over time is endogenous. In particular, it depends negatively on the current precision of beliefs γ. In times of low uncertainty, when γ is high, agents trust their current information and put less weight on new signals, making the mean of beliefs more stable. The volatility of the mean of beliefs also depends positively on the number of active firms, N. When N is large, more information flows to agents, making beliefs more likely to jump.

Evolution of Uncertainty The precision of beliefs captures the uncertainty of agents about the fundamental and its dynamics play a key role for the existence of uncertainty traps. It is random as a result of the finiteness of the number of firms. Conditioning on the realization of N, the dynamics of γ is deterministic and allows for a simple analytical characterization.

Lemma 2. *The precision of next-period beliefs γ' increases with N and γ. For a given number of investing firms N, the law of motion for the precision of beliefs $\gamma' = \Gamma(N, \gamma)$ admits a unique stable fixed point in γ.*

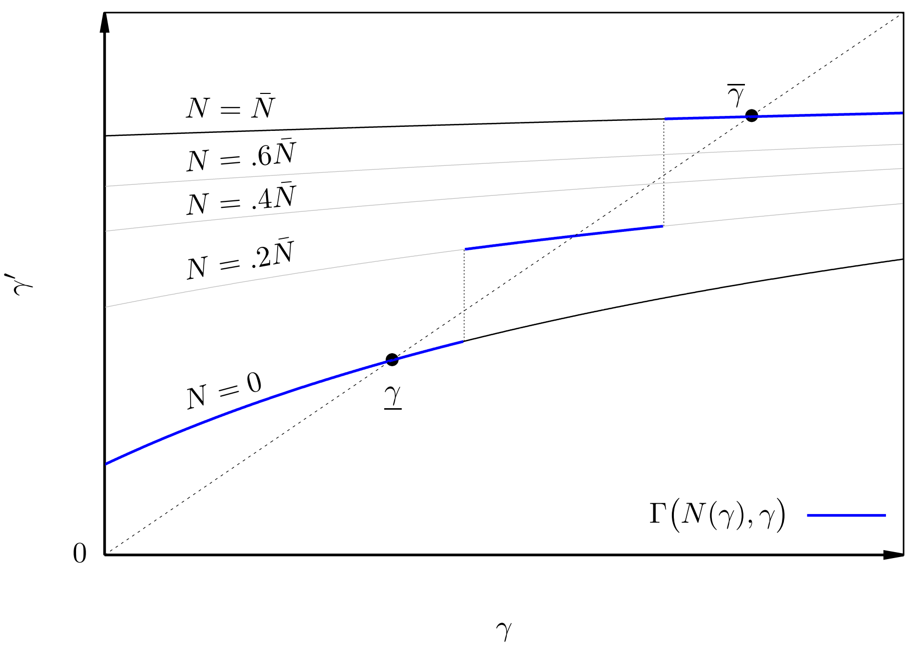

Figure 5: Example of dynamics for belief precision γ

The thin solid curves on Figure 5 depict $\Gamma(N, \gamma)$ for different values of N. As stated in Lemma 2, an increase in the level of activity raises the next period precision of information γ' for any level of γ in the current period. Since N is between 0 and $\overline{N}$, the support of the ergodic distribution of γ must be between the two bounds $\underline{\gamma}$ and $\bar{\gamma}$ defined as the fixed points $\underline{\gamma} \equiv \Gamma(0, \underline{\gamma})$ and $\bar{\gamma} \equiv \Gamma(\bar{N}, \bar{\gamma})$. Intuitively, $\underline{\gamma}$ is the precision level reached if no firm invests for a long period of time, while $\bar{\gamma}$ is the level reached if all firms invest during a long period of time.

In equilibrium, N is endogenous and varies with γ. Assuming for the moment that $N(\gamma)$ is an increasing step function, the figure illustrates how the feedback from uncertainty to investment opens up the possibility of multiple stationary points in the dynamics of the precision, and therefore uncertainty. For the chosen path of $N(\gamma)$, the function $\gamma' = \Gamma(N(\gamma), \gamma)$ depicted by the solid curve in Figure 5 has three stable fixed points. Below, we formally establish that this type of multiplicity is a generic feature of the recursive equilibrium.

Optimal Timing of Investment How does the individual investment decision depend on beliefs? Intuitively, a more optimistic firm is more likely to invest since the expected return on investment is higher. In turn, uncertainty may reduce investment for two reasons. First, risk averse firms dislike uncertain payoffs. Second, since investment is costly and irreversible, there is an option value of waiting: in the face of uncertainty, firms prefer to delay investment to gather additional information and avoid downside risks.

The following proposition formally establishes the validity of this intuition and provides a characterization of the optimal investment behavior.

Proposition 1. *Under Assumption 1 and for some $p(\mu, \gamma) \in \mathcal{P}$ defined in the appendix, for a perceived information flow characterized by $N \sim Bin\left(\overline{N}, p\left(\mu, \gamma\right)\right)$ and γ_x sufficiently small, there exists a unique cutoff for the fixed costs $f_c\left(\mu, \gamma\right) \in \mathbb{R} \cup \{-\infty, \infty\}$ such that firms invest if and only if $f \leq f_c\left(\mu, \gamma\right)$. The cutoff $f_c\left(\mu, \gamma\right)$ is strictly increasing in μ and γ.*

This partial equilibrium result characterizes the investment rule given the random number of investing firms $N\left(\mu, \gamma\right)$ and the laws of motion for μ and γ. Firms invest if and only if the idiosyncratic fixed cost falls below the threshold $f_c\left(\mu, \gamma\right)$. An important result for what follows is that the probability of investment decreases with uncertainty.

4.2 Existence and Uniqueness

We have described in Lemmas 1 and 2 how beliefs depend on the number of investing firms, and, in Proposition 1, how firms' investment decisions are affected by beliefs. In the latter, firms make their decisions with the perception that N follows a binomial distribution $Bin\left(\overline{N}, p\left(\mu, \gamma\right)\right)$ for some function $p\left(\mu, \gamma\right) \in \mathcal{P}$. We now close the equilibrium by requiring that the perceived behavior of firms is consistent with their actual investment decisions: $p\left(\mu, \gamma\right) = F\left(f_c\left(\mu, \gamma\right)\right)$. The next proposition shows that such a general equilibrium exists and is unique.

Proposition 2. *Under the regularity conditions in Assumptions 1 and 2 stated in the appendix, and for γ_x small enough, a recursive equilibrium exists and is unique. The expected number of investing firms is increasing in the mean of beliefs μ and increasing in precision γ.*

Figure 6 depicts the expected number of investing firms as a function of beliefs (μ, γ). The partial equilibrium results from Proposition 1 carry through to the general equilibrium: the number of investing firms increases as they become more optimistic about the fundamental (μ high) or less uncertain (γ high). In turn, as we illustrated in the example of figure 5, the last result opens the possibility of multiple fixed points in the dynamics of uncertainty.

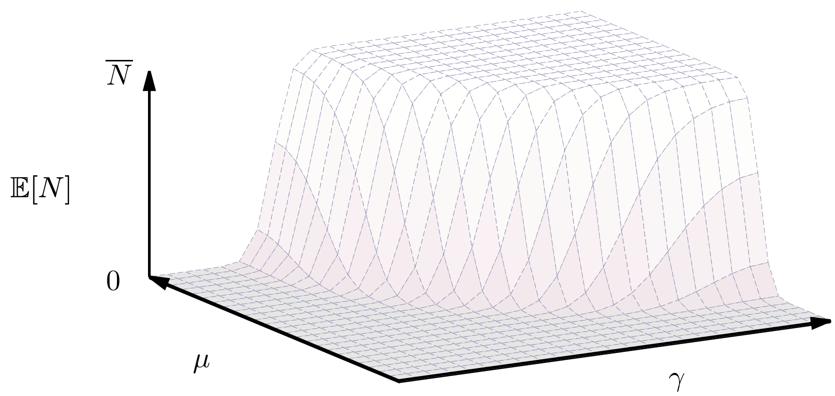

Figure 6: Example of aggregate investment pattern

5 Uncertainty Traps

We now describe the key mechanism at work behind uncertainty traps.

5.1 Definition and Existence

We assume at this point that the total number of firms $\bar{N}$ is large enough, so that

$$n(\mu,\gamma) \equiv \frac{N(\mu,\gamma)}{\bar{N}} \simeq F\left(f_c(\mu,\gamma)\right).$$

With this assumption, we can treat n as a deterministic function of beliefs, ignoring fluctuations due to the finiteness in the number of firms.[6] We are now ready to define an uncertainty trap:

Definition 2. Given some mean beliefs μ_0, there is an *uncertainty trap* if there are at least two locally stable fixed points in the dynamics of beliefs precision $\gamma' = \Gamma\left(N\left(\mu_0,\gamma\right),\gamma\right).$

We refer to these fixed points in the dynamics of γ as *regime*s to emphasize that they are not distinct equilibria. Indeed, the presence of uncertainty traps does not imply that there are multiple equilibria in the model. In fact, Proposition 2 establishes that there is a unique rational expectation equilibrium. While multiple values of γ may satisfy the equation $\gamma = \Gamma\left(N\left(\mu,\gamma\right),\gamma\right)$, the regime that prevails at any given time is uniquely determined by the history of past aggregate shocks.

[6]We must be careful when taking the limit as $\overline{N} \to \infty$. If we kept γ_x constant, a law of large number would apply and θ would be fully revealed through the social learning channel. We instead assume that the precision of each individual signal goes to 0 as we add firms: $\gamma_x\left(\overline{N}\right) = \gamma_x/\overline{N}$. This ensures that all our previous results as well as the laws of motion for information are maintained, substituting N with $n = N/\overline{N}$. In particular, the precision of X remains the same: $N\gamma_x\left(\bar{N}\right) = n\gamma_x$.

However, because of shocks to μ, there is always a strictly positive probability that the economy switches endogenously between regimes.

Do uncertainty traps always arise? The following proposition establishes formally that uncertainty traps exist for a range of mean of beliefs μ. An important condition for this result is that the dispersion in the distribution of fixed costs, σ^f, is not too large. This ensures a strong enough feedback from information to investment.[7]

Proposition 3. *Under the conditions of Proposition 2 and for σ^f small enough, there exists a non-empty interval $[\mu_l, \mu_h]$ such that, for all $\mu_0 \in (\mu_l, \mu_h)$, the economy features an uncertainty trap with at least two regimes $\gamma_l(\mu_0) < \gamma_h(\mu_0)$. Regime γ_l is characterized by high uncertainty and low investment while regime γ_h is characterized by low uncertainty and high investment.*

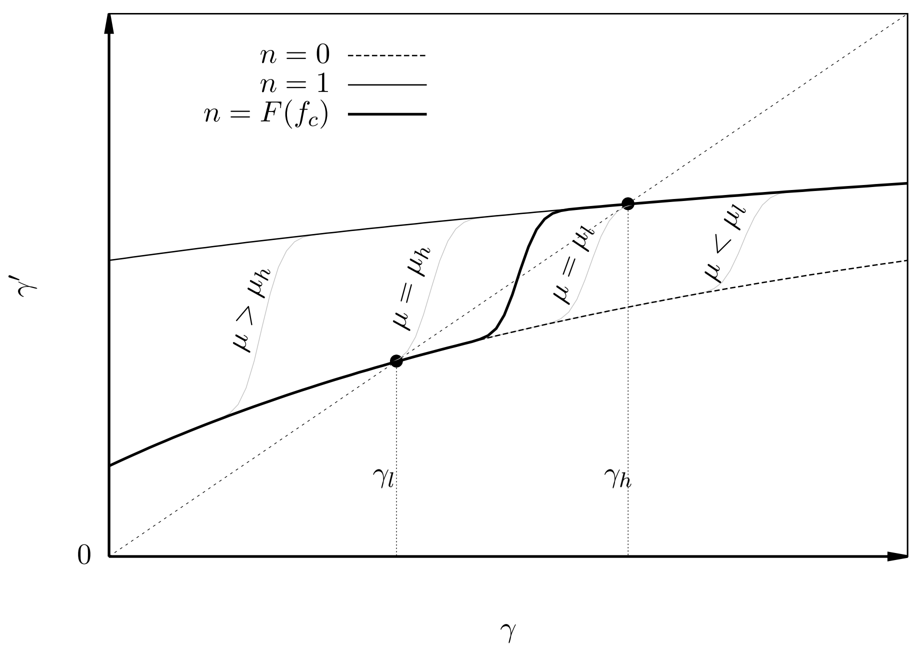

Figure 7: Dynamics for precision γ for different values of μ

Figure 7 offers an example of the law of motion of γ for different values of μ. To understand the mechanism at work, let us first focus on the thick solid curve, which shows the function $\gamma' = \Gamma(N(\mu,\gamma),\gamma)$ for a fixed intermediate value of μ. For small values of γ, uncertainty is high and firms do not invest. As a result, they do not learn anything by observing aggregate activity and the precision of beliefs γ' remains low. As uncertainty decreases, firms are sufficiently certain about the fundamental to start investing. As N increases, they learn more and more from each others' production and γ' starts to increase as well.

Notice that the thick curve crosses the 45° lines in three locus points. The middle intersection is unstable but the other two are locally stable regimes, γ_l and γ_h, so that the economy might settle

[7] Intuitively, as the distribution of fixed costs becomes less dispersed, the fraction of investing firms $n(\mu,\gamma)$ reacts quickly to changes in beliefs. See the appendix for details.

for some time in their neighborhood. At γ_l, uncertainty is high and investment is low while the opposite is true when the economy is at γ_h.

5.2 Dynamics: Non-linearity and Persistence

We have only considered so far the evolution of γ for a fixed μ, but as random shocks hit the economy, and firms receive random signals, the mean of beliefs μ will start to change. In this context, how stable are the regimes associated to γ_l and γ_h? Figure 7 provides the answer. Notice that as long as μ stays between the two depicted values μ_l and μ_h, which correspond to the range $[\mu_l, \mu_h]$ from Proposition 3, the two regimes γ_l and γ_h remain stable. Therefore, as long as the shocks to μ are not too large, the economy remains at the regime it is currently in.

For values of μ above μ_h, the dynamics of beliefs only admits the high-activity regime, while for values below μ_l, it only admits the low-activity regime. Shocks to μ can therefore make one regime unstable and force the economy to move to the other regime, if they are large enough or last for a sufficiently long time. Proposition 3 establishes that the situation depicted in Figure 7 is a generic feature of the equilibrium.

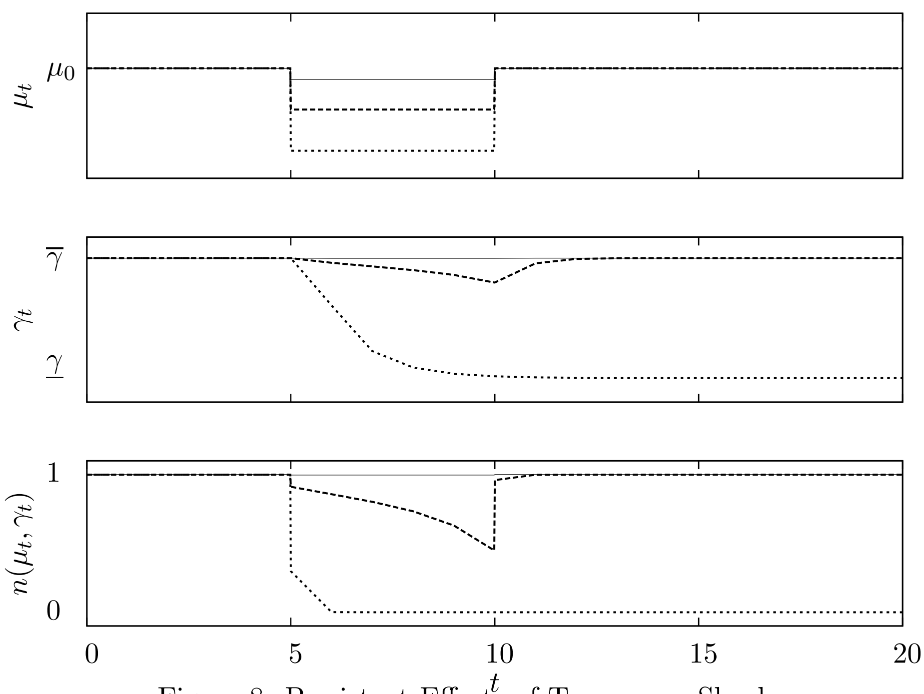

Figure 8: Persistent Effects of Temporary Shocks

As we have seen, uncertainty traps give rise to non-linear aggregate dynamics: the economy reacts very differently to large shocks in comparison to small ones. Figure 8 shows various simulations that illustrate this feature using the example from Figure 7. The top panel presents three different series of shocks to the mean of beliefs μ. The three series start from the high activity/low uncertainty regime. The economy is then hit at $t = 5$ by a negative shock to μ, due to a particularly bad realization of either the public signals or the fundamental. The economy returns to normal at

$t = 10$. Across the three series, the magnitude of the initial shock is different.

The middle and bottom panels show the response of belief precision γ and the fraction of investing firms n. The fraction of firms that invest prior to the shock is 1. The solid gray line represents a small temporary shock. After the shock hits, firms still find it profitable to invest, the fraction of investing firms remains equal to 1, and the precision of beliefs is unaffected. When the economy is hit by a temporary shock of slightly larger magnitude, some firms stop investing, leading to a gradual increase in uncertainty. As uncertainty rises, investment goes down even further and the economy starts to drift towards the low regime. However, when mean beliefs recover, the precision of information and the number of active firms quickly return to the high-activity regime. In contrast, when the economy is hit by an even larger temporary shock, such as described by the dotted line, the number of firms delaying investment is large enough to produce a self-sustaining increase in uncertainty. The economy quickly shifts to the low-activity regime and remains there even after mean beliefs recover to the initial position. The economy has fallen into an uncertainty trap.

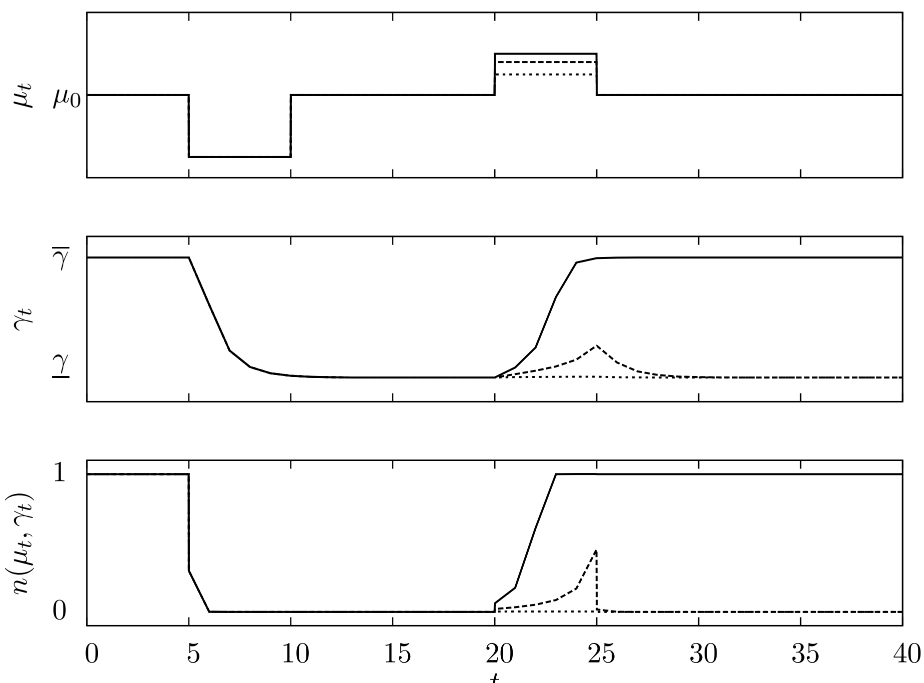

Figure 9: Escaping an Uncertainty Trap

How does the economy escape from an uncertainty trap? Figure 9 depicts the evolution of the economy after it is hit by the large shock from Figure 8. As before, the large negative shock hitting the economy from periods 5 to 10 pushes the economy into the low activity-high uncertainty regime. Eventually, the economy receives positive signals that lead to a temporary increase in mean beliefs between periods 20 and 25, possibly because of a recovery in the fundamental. When the temporary increase in average beliefs is not sufficiently strong, the recovery is short-lived. However, when the temporary increase is sufficiently large, the economy reverts back to the high-activity regime. Once again, temporary shocks of sufficient magnitude to the fundamental may lead to nearly permanent effects on the economy.

A number of additional lessons can be drawn from these simulations:

1. In this framework uncertainty is a by-product of recessions. This result echoes some of the empirical findings by Bachmann et al. (2013) who show that uncertainty is caused by recessions and thereby conclude that it is of secondary importance for the business cycle. We show however that uncertainty may still have a large impact on the economy, even if it is not what triggers recessions.

2. As in models with learning in the spirit of Van Nieuwerburgh and Veldkamp (2006), this theory provides an explanation for asymmetries in business cycles In good times, since agents receive a large flow of information, they react faster to shocks than in bad times.

3. The simulations also highlight that agents can be uncertain about the fundamental without necessarily being uncertain about endogenous variables. For example, when the economy is trapped in the low activity regime, firms are *certain that they are uncertain*: they can accurately predict the low level of investment and output. This highlights a potential difficulty with identifying uncertainty in the data. As implied by the model, the accuracy of forecasts about variables like output may possibly be a bad proxy for uncertainty about fundamentals.

5.3 Policy Implications

The process through which information is released in this economy raises the question of efficiency. In the decentralized equilibrium firms invest less often than they should from a social welfare perspective because they do not internalize that their investments release information to the rest of the economy. Proposition 4 shows that the decentralized economy is constrained inefficient. But a simple policy instrument such as an investment subsidy that only depends on current beliefs (μ, γ) is sufficient to make firms internalize their impact on the rest of the economy and implements the efficient allocation.

Proposition 4. *The decentralized competitive equilibrium is constrained inefficient and the efficient utilitarian allocation can be implemented with positive investment subsidies $\tau(\mu, \gamma)$ and a uniform tax. In turn, when γ_x and σ^f are small, the efficient allocation is still subject to uncertainty traps.*

Proposition 4 implies that firms are more likely to invest in the efficient allocation, but uncertainty traps can still arise. If the planner does not have any additional information than agents in the economy, it is still optimal to wait when uncertainty is too high. Hence, there still exists a strong complementarity between information and the level of activity in the constrained efficient-allocation, although uncertainty traps are less likely to arise in that case.

6 Numerical Example

To illustrate the impact of uncertainty traps on business cycles fluctuations, we enrich the model in several dimensions. We include a Cobb-Douglas production function that combines labor and capital as inputs, and a representative household that supplies labor and owns the firms. We also generalize the stochastic process followed by the fundamental. To see if the high persistence predicted by the baseline model survives to optimal policy interventions, we focus on the problem of the social planner. Since the planner internalizes the information externality, firms invest more in the optimal allocation than in the competitive equilibrium. We can therefore be confident that if uncertainty traps arise in the optimal allocation they will also arise in the competitive equilibrium.

6.1 Extended model

There are $\bar{N}$ firms that operate a Cobb-Douglas technology to produce the unique consumption good. A firm n employing l units of labor and k units of capital produces output

$$q_n = (A + Y)\, k_n^{\alpha} l_n^{1-\alpha},$$

where

$$\begin{aligned} Y &= \theta + \varepsilon^y \\ \theta' &= \rho_\theta \theta + \varepsilon^\theta \end{aligned}$$

with distributions $\varepsilon^\theta \sim \mathcal{N}\left(0, \left(1 - \rho_\theta^2\right) \gamma_\theta^{-1}\right)$ and $\varepsilon^y \sim \mathcal{N}\left(0, \gamma_y^{-1}\right)$ and where $A > 0$ is the unconditional mean of total factor productivity. As in the baseline model, the stochastic process θ is the fundamental of the economy but it now follows an AR(1) process instead of a random walk. Notice that this assumptions should weaken the uncertainty traps since agents now know that, after a bad shock, the fundamental will reverse towards its mean. We use $(A + Y)$ as total factor productivity instead of the perhaps more common exponential form to make sure that a change in the perceived variance of Y does not influence the expectation of productivity.[8]

Firms can only invest if they have an investment opportunity. At the end of the period, a firm without an opportunity will gain one with probability q. A firm cannot have more than one opportunity at any time. A firm with an opportunity can invest an amount $i > 0$, per unit of capital that it owns, at a cost $f > 0$, also per unit of capital. Therefore, the capital of a firm n evolves according to

$$k_n' = (1 - \delta + i)\, k_n,$$

where δ denotes the depreciation rate. By making the investment costs proportional to the current capital stock, we can aggregate the economy easily. Allowing firms to invest only if they have an

[8] While productivity could theoretically be negative, the variance of the distributions considered below are such that it never happens in any of the simulations.

investment opportunity generates the irreversibility of investment required to generate the option value of waiting.

There is a risk-neutral representative consumer who supplies one unit of labor inelastically and who discounts future utility at rate β. Risk neutrality implies that uncertainty only matters through the irreversibility of investment.

The information structure and the timing of events are as follow:

1. All firms share the same prior distribution over the fundamental $\theta \mid \mathcal{I} \sim \mathcal{N}\left(\mu, \gamma^{-1}\right)$ where μ and γ denote the mean and the precision of beliefs.

2. Firms that hold an investment opportunity decide whether to invest or not.

3. Firms that invest receive a private signal $x_n = \theta + \epsilon_n^x$, where $\epsilon_n^x \sim \mathcal{N}\left(0, \tilde{\gamma}_x^{-1}\right)$, and then choose labor l_n.

4. The common shock y is revealed and observed by everyone. All firms produce and all actions are observed.

5. Firms that do not hold an investment opportunity receive one with probability q.

6. Agents update their beliefs for next period.

As before, firms decide whether or not to invest based solely on the common information (μ, γ). Once investment has taken place, each firm observes a private signal and, based on that signal, hires workers. Since the number of workers hired is an increasing function of the signal received, and because labor is observable, all firms can figure out all the private signals received by others. Because of the normality of the random variables, this is equivalent to observing the public signal

$$X = \frac{1}{\tilde{N}} \sum_{n \in I} x_n = \theta + \varepsilon_N^X$$

where I is the set of firms investing, $\tilde{N}$ is the number of firms that invest and $\varepsilon_N^X \sim \mathcal{N}(0, (\tilde{N}\tilde{\gamma}_x)^{-1})$. Importantly, the precision of this aggregate public signal is increasing in $\tilde{N}$. Firms also learn about the fundamental by observing the aggregate productivity y. Here, it is the observation of the firms' labor decision, instead output as in the baseline model, that reveals their private information. This assumption allows us to keep the analysis tractable.

6.2 Social Planner

Because of the constant returns to scale in production and the overall linear structure of the model we can aggregate the economy analytically. To do so we take the limit as the total number of firms $\bar{N}$ goes to infinity as the precision of each individual signal goes to zero. We do so in a way that keeps the precision of public information unchanged. More precisely, we assume that the precision of each private signal is $\tilde{\gamma}_x = \gamma_x / \bar{N}$, and then take the limit as $\bar{N} \to \infty$. In the limit, the mass of each firm also goes to zero so that we have a mass 1 of firms.

Under these assumptions, the laws of motion for the beliefs of agents replicate those in the baseline model. The only difference is that, now, the next-period prior is adjusted to take into account the fact that the fundamental θ follows an AR(1) process instead of a random walk,

$$\mu' = \rho \frac{\gamma\mu + \gamma_y Y + N\gamma_x X}{\gamma + \gamma_y + N\gamma_x} \tag{11}$$

$$\gamma' = \left(\frac{\rho^2}{\gamma + \gamma_y + N\gamma_x} + (1-\rho^2)\sigma_\theta^2 \right)^{-1} \tag{12}$$

where N is now the fraction of firms investing; the limit of $\tilde{N}/\bar{N}$ as we move to the continuum.

We can now write the social planner's problem. Denoting aggregate capital by K, the fraction of firms with an investment opportunity by Q and the number of firms investing by N, the social planner solves

$$V(\mu, \gamma, K, Q) = \max_{0<N<Q} (A + \mu) K^\alpha - KNf + \mathbb{E}\left[\beta V\left(\mu', \gamma', K', Q'\right) | \mu, \gamma\right]$$

subject to

$$\begin{aligned} K' &= K(1 - \delta + Ni) \\ Q' &= Nq + (Q - N) + (1 - Q)q \end{aligned}$$

and to the laws of motion from equations 11 and 12.

It is clear from the planner's problem that without the fluctuating investment opportunities, which corresponds to having q and Q both equal to 1, the model would be a standard business cycles model without any irreversibility of investment. The dynamics of Q, and the impact that N has on it, is therefore the element that makes the option value of waiting a function of uncertainty.

6.3 Simulations

To illustrate that uncertainty traps can create additional persistence in the macroeconomic aggregates, we proceed to a series of simulations. We first parametrize the model with the values shown in Table 1. The time period is one month. The values for the discount rate β, the depreciation rate δ and the share of capital in production α are standard.

Let us first look at some properties of the policy function. In Figure 10 we can see the investment decision of firms as a function of the mean of beliefs μ for three levels of uncertainty. As in the benchmark model, firms are more likely to invest when μ is high and uncertainty is low. Notice also that when μ is at its steady state firms keep investing unless uncertainty is high, in which case they are sufficiently worried about the state of the fundamental that they prefer to wait. This behavior opens the possibility for uncertainty traps to exists. The dynamics of the precision of beliefs γ, shown in Figure 11, suggests that we have indeed three stable points similar to those that exist in the baseline model. These points are however only stable as long as μ, K and Q remains at their

Parameter	Value
Total factor productivity	$A = 1$
Discount factor	$\beta = (0.95)^{1/12}$
Depreciation rate	$\delta = 1 - (0.9)^{1/12}$
Share of capital in production	$\alpha = 0.4$
Probability of receiving an investment opportunity	$q = 0.25$
Cost of investment	$f = 0.15$
Size of investment	$i = 0.1$
Persistence of fundamental	$\rho_\theta = 0.99$
Precision of ergodic distribution of fundamental	$\gamma_\theta = 400$
Precision of public signal	$\gamma_y = 100$
Precision of aggregated private signals when $N = 1$	$\gamma_x = 15000$

Table 1: Parameters values for the numerical simulations

steady-state value. As K declines, which would be the case in a recession, the return on capital increases and firms invest more. As a result, more information is released and the low stable point disappears. This is one important difference from the benchmark model. Here the uncertainty trap is not globally stable if capital depreciates. We can however expect the mechanism to increase the duration of recessions. To see how strong this effect is we now proceed to a series of simulations.

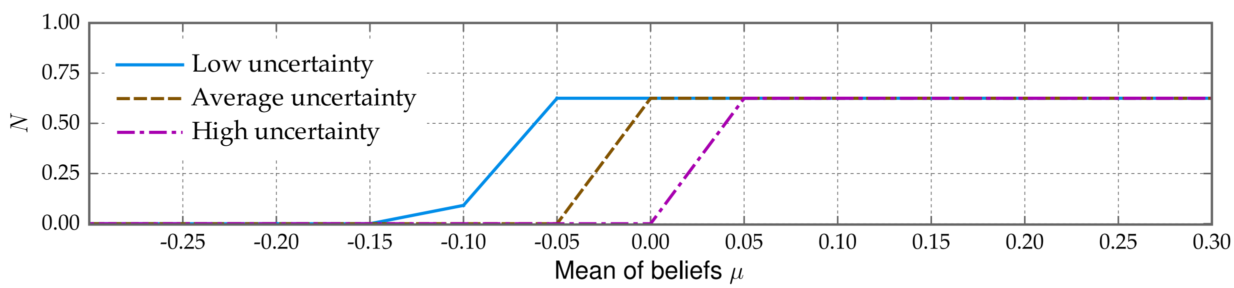

Figure 10: Investment decision $N(\mu, \gamma, K, Q)$ for K and Q constant at their steady-state level.

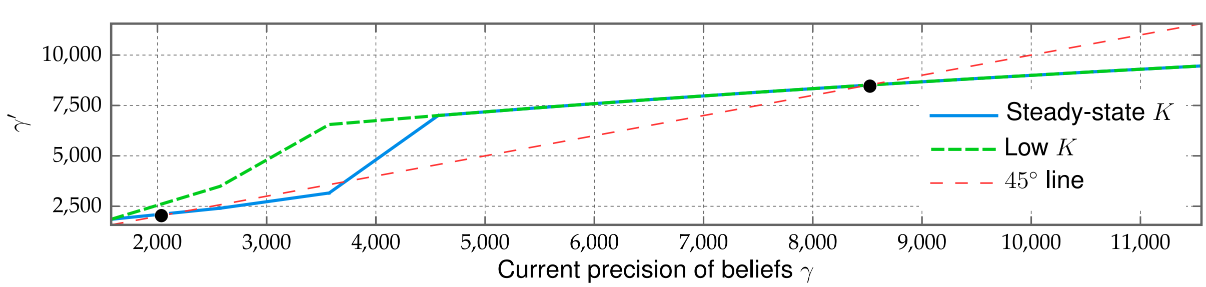

Figure 11: Dynamics of precision γ for μ and Q constant at their steady-state value.

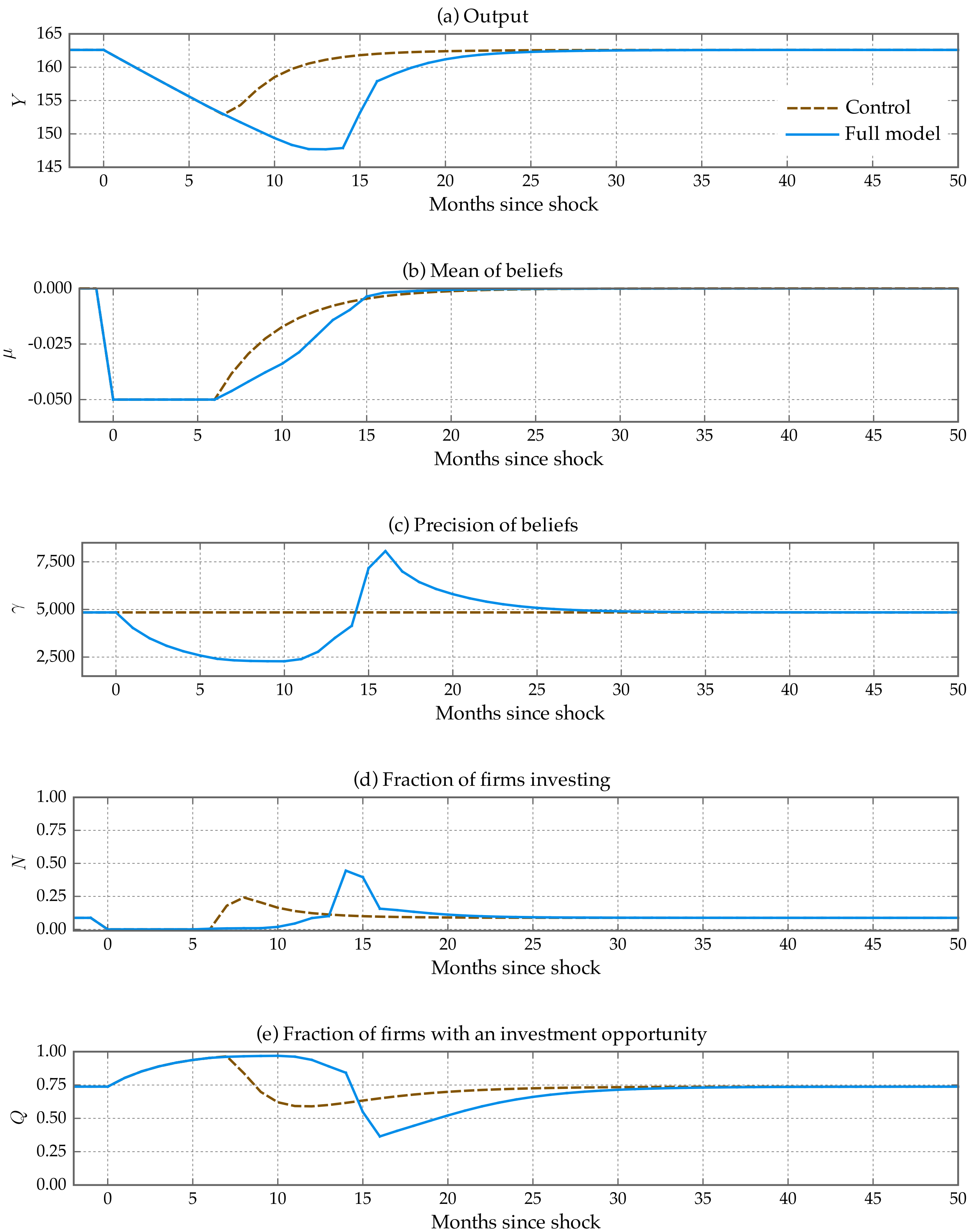

Figure 12: Evolution of the economy after a 5% negative shock to μ. The solid curve shows the evolution of the economy according to the full model while the dashed curve shows the evolution of a control economy in which the flow of public information is fixed at its steady-state level.

We consider the evolution of an economy that suffers from a -5% shock to the mean of beliefs μ that lasts 6 months. This shock could originate from a bad signal X or a bad productivity draw Y. The impulse response functions are represented by the solid curves in Figure 12. The dashed curves represent the response of a control economy in which we prevent the investment decision of firms from affecting the information flow. Explicitly, we keep the N that enters in equations 11 and 12 constant at its steady-state level. Therefore, the difference between the two curves corresponds to the the additional impact of the endogenous uncertainty.

Let us consider the full model first. On impact, firms become worried that their current and future productivity will be low. The expected return adding capital becomes lower than its cost and firms cut back on investment. As a result, less private signals are generated in the economy and the precision of beliefs starts falling as can be seen in Panel (c). Once the shock is over, agents start receiving signals suggesting that the fundamental is actually better than expectations. Firms update their beliefs accordingly and, as shown in Panel (b), they become more optimistic. The recovery is however impeded by the high degree of uncertainty which leads firms to delay investment. Since investment remains low, firms receive little information about the fundamental which maintains the high degree of uncertainty and delays the recovery.

We can evaluate the impact of the endogenous uncertainty by comparing the impulse response functions of the full model to those of the control economy. There, since the flow of information coming from the public signal X remains constant, firms quickly learn that the fundamental is better than expected and the economy recovers quickly. We see a drop in output of about 6% in the control economy while production shrinks by 9% in the full model. The trough of the recession also happens 7 months later and the economy takes substantially more time to recover to its steady state.

To understand the nonlinear nature of the model, it is useful to compare this 5% shock to μ with a shock half as big. The impulse response functions of output to both shocks are shown in Figure 13. We see that output barely reacts to the small shock. This shock is not sufficiently strong to affect the investment decision in a significant way. As a result, the amount of information received by firms does not change much and the economy recovers quickly.

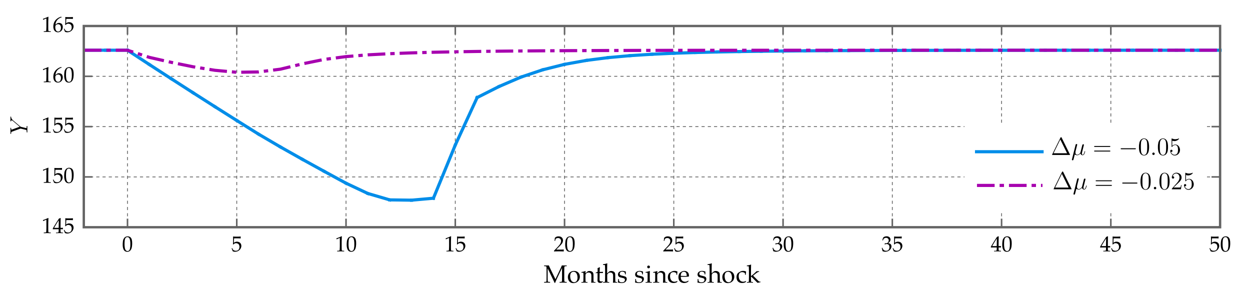

Figure 13: Evolution of output after a 5% and 2.5% negative shock to μ.

In the simulations of Figure 12, the recovery is faster in the control economy because the flow of information is kept constant at its steady-state level. In the full model, this flow depends endogenously on the optimal investment policy $N(\mu, \gamma, K, Q)$ and therefore on the full state space. Since this object obviously depends on μ in an important way, one could imagine that the recovery is slow in the full model only because μ remains low and that the uncertainty channel emphasized in this paper is only a side-show. This is not so. To show this, we proceed to a new simulation in which we keep γ constant at its steady-state level while allowing the flow of information in the control group to vary endogenously with the other variables of the state space. Explicitly, we use $N(\mu, \gamma^{ss}, K, Q)$, where γ^{ss} is the steady-state level of γ, in equations 11 and 12. The impulse response functions for output are shown in Figure 14. The solid curve shows output in the full model once again. The brown dashed curve shows the same economy with constant flow of information as in Figure 12 while the purple curve shows the economy where information evolves with $N(\mu, \gamma^{ss}, K, Q)$. As we can see, the uncertainty channel is the main contributes significantly to the depth and persistence of the recession.

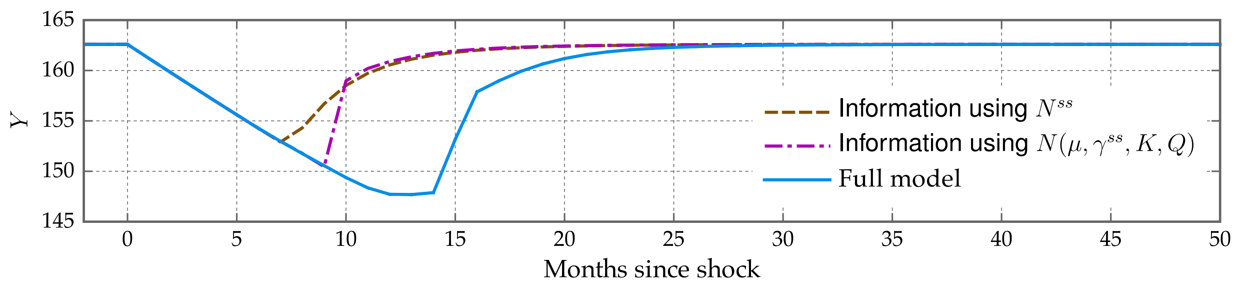

Figure 14: Evolution of output after a 5% negative shock to μ in the full model, the economy with constant flow of information and a control economy in which the flow of information evolves with $N(\mu, \gamma^{ss}, K, Q)$.

Finally, let us consider shocks to the precision of beliefs. As uncertainty only fluctuates endogenously in the model, shocking γ directly violates our assumption of rational expectations. Keeping this in mind, it is still an interesting exercise to understand the policy functions. Figure 15 shows the impulse response functions of output, the mean of beliefs and the precision of beliefs when the standard deviation of the prior distribution increases by 50%. We once again keep the flow of information constant at N^{ss} in the control economy. As expected, output drops in both cases. In the both economies, γ recovers according to equation 12, but since N is relatively high in the control economy, the economy is faster. We can see that, in the full model, short shocks to uncertainty persists for a relatively long time since they also slow the learning process.

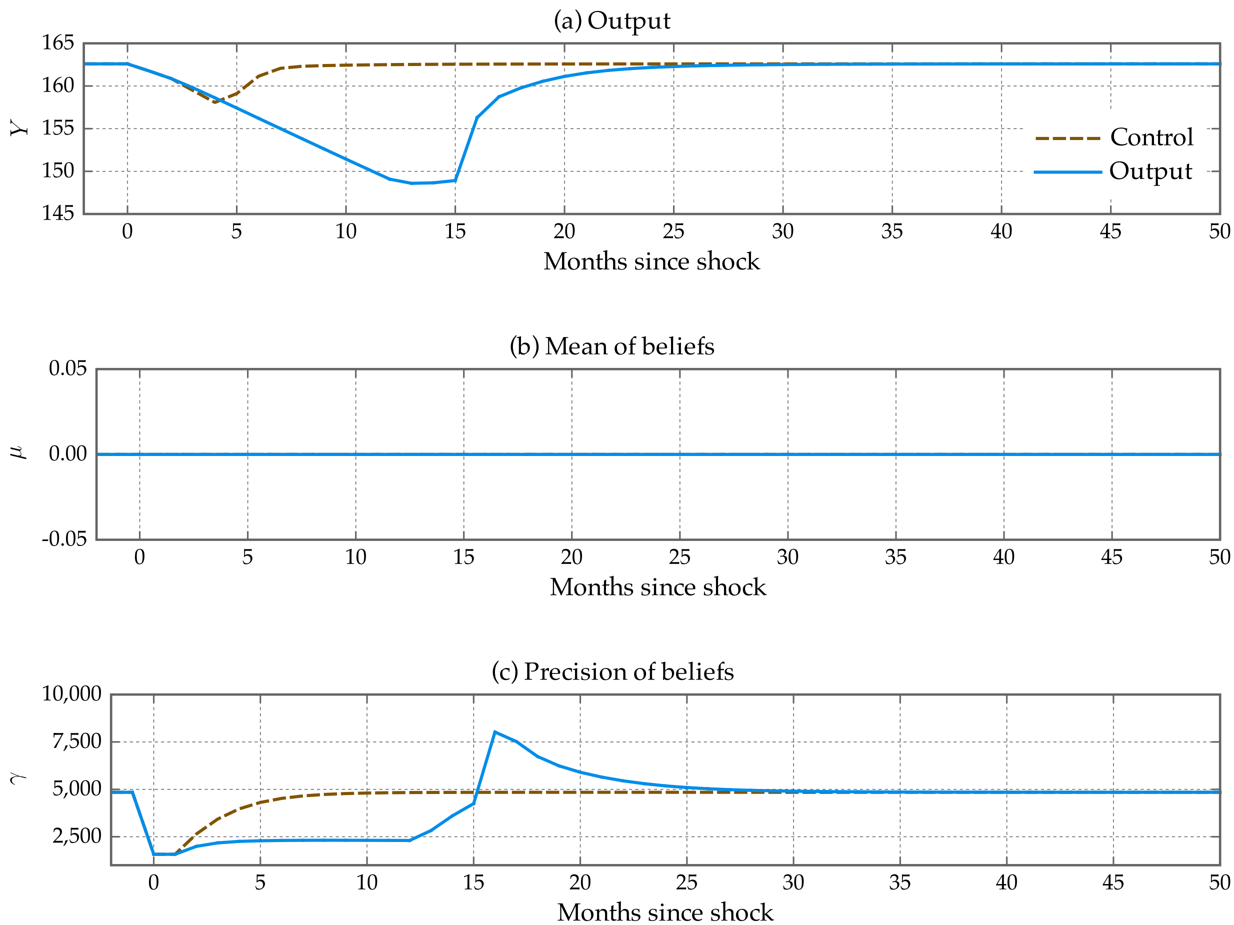

Figure 15: Endogenous vs exogenous uncertainty. Evolution of the economy after a 50% negative shock to the standard deviation of the prior. The solid curve shows the evolution of the full model while the dashed curve shows the evolution of a control economy in which the flow of public information is fixed at its steady-state level.

7 Conclusion

We developed a quantitative theory of endogenous uncertainty that embeds social learning in a standard model of irreversible investment. In the model, agents receive private information about some fundamental and learn from the actions of others. During periods of high economic activity, a large amount of information is revealed. In periods of low economic activity, information diffuses slowly and uncertainty rises. High uncertainty encourages firms to defer investment in the future, slowing the diffusion of information further.

We show that this interaction between social learning and the option value of investment creates a powerful complementarity between information and economic activity. We derive conditions under which this complementarity is strong enough to sustain two distinct locally stable steady states in the dynamics of activity and uncertainty for a given level of optimism: a high activity/low uncertainty regime and a low activity/high uncertainty one. We demonstrate that the equilibrium of the model is still unique, but that this multiplicity creates nonlinear dynamics in which recessions can have a near permanent impact on the economy.

We explore the robustness of this mechanism in a quantitative version of the model that nests the key components of our setup in a more traditional model of business cycles. Despite being at a very preliminary stage, our simulations show some encouraging evidence that the effects of uncertainty are robust and able to provide a sizable propagation mechanism with substantial persistence of recessions. We are currently working on a full-fledged calibration and an empirical study to complement these results.

References

Amador, M. and P.-O. Weill (2010): "Learning from Prices: Public Communication and Welfare," *Journal of Political Economy*, 118.

Arellano, C., Y. Bai, and P. Kehoe (2012): "Financial Markets and Fluctuations in Uncertainty," Working paper, Federal Reserve Bank of Minneapolis and NBER; Arizona State University; University of Minnesota and Federal Reserve Bank of Minneapolis.

Bachmann, R. and C. Bayer (2009): "Firm-Specific Productivity Risk over the Business Cycle: Facts and Aggregate Implications," Working paper, University of Michigan; IGIER, Universitá Bocconi.

Bachmann, R., S. Elstner, and E. R. Sims (2013): "Uncertainty and Economic Activity: Evidence from Business Survey Data," *American Economic Journal: Macroeconomics*, 5, 217–49.

Bachmann, R. and G. Moscarini (2011): "Business cycles and endogenous uncertainty," *manuscript, Yale University, July.*

Baker, S. R., N. Bloom, and S. J. Davis (2012): "Measuring economic policy uncertainty," *manuscript, Stanford University.*

Banerjee, A. V. (1992): "A simple model of herd behavior," *The Quarterly Journal of Economics*, 107, 797–817.

Bikhchandani, S., D. Hirshleifer, and I. Welch (1992): "A theory of fads, fashion, custom, and cultural change as informational cascades," *Journal of political Economy*, 992–1026.

Bloom, N. (2009): "The Impact of Uncertainty Shocks," *Econometrica*, 77, 623–685.

Bloom, N., M. Floetotto, N. Jaimovich, I. Saporta-Eksten, and S. Terry (2012): "Really Uncertain Business Cycles," Nber working paper no.18245, NBER.

Caplin, A. and J. Leahy (1993): "Sectoral shocks, learning, and aggregate fluctuations," *The Review of Economic Studies*, 60, 777–794.

Chamley, C. and D. Gale (1994): “Information revelation and strategic delay in a model of investment,” *Econometrica: Journal of the Econometric Society*, 1065–1085.

Cooper, R. W. and J. C. Haltiwanger (2006): “On the nature of capital adjustment costs,” *The Review of Economic Studies*, 73, 611–633.

Cunningham, R. (2004): “Investment, Private Information and Social Learning: A Case Study of the Semiconductor Industry,” .

DErasmo, P. N. and H. J. M. Boedo (2011): “Intangibles and endogenous firm volatility over the business cycle,” *manuscript, University of Virginia*.

Eisfeldt, A. L. and A. A. Rampini (2006): “Capital reallocation and liquidity,” *Journal of Monetary Economics*, 53, 369–399.

Fernández-Villaverde, J., P. A. Guerrón-Quintana, K. Kuester, and J. Rubio-Ramírez (2011): “Fiscal volatility shocks and economic activity,” Tech. rep., National Bureau of Economic Research.

Foster, A. D. and M. R. Rosenzweig (1995): “Learning by Doing and Learning from Others: Human Capital and Technical Change in Agriculture,” *Journal of Political Economy*, 103, pp. 1176–1209.

Gourio, F. and A. K. Kashyap (2007): “Investment Spikes: New Facts and a General Equilibrium Exploration,” Working Paper 13157, National Bureau of Economic Research.

Hausmann, R. and D. Rodrik (2003): “Economic development as self-discovery,” *Journal of development Economics*, 72, 603–633.

Kaustia, M. and S. Knüpfer (2009): “Learning from the outcomes of others: Stock market experiences of local peers and new investors market entry,” *Helsinki School of Economics Working Paper*.

Khang, I. G. (2012): “Word of mouth and investments: Evidence from new neighbor assignments,” *Available at SSRN 2172553*.

Leduc, S. and Z. Liu (2012): “Uncertainty shocks are aggregate demand shocks,” *Federal Reserve Bank of San Francisco Working Paper*, 10.

Ordonez, G. L. (2009): “Larger crises, slower recoveries: the asymmetric effects of financial frictions,” Tech. rep., Federal Reserve Bank of Minneapolis.

Patnam, M. (2011): “Corporate networks and peer effects in firm policies: Evidence from India,” *Department of Economics, University of Cambridge (mimeo& graph)*.

Rob, R. (1991): “Learning and capacity expansion under demand uncertainty,” *The Review of Economic Studies*, 58, 655–675.

Schaal, E. (2012): "Uncertainty, Productivity and Unemployment in the Great Recession," working paper, New York University.

Van Nieuwerburgh, S. and L. Veldkamp (2006): "Learning asymmetries in real business cycles," *Journal of monetary Economics*, 53, 753–772.

Veldkamp, L. L. (2005): "Slow boom, sudden crash," *Journal of Economic Theory*, 124, 230–257.

Zeira, J. (1994): "Informational cycles," *The Review of Economic Studies*, 61, 31–44.

A Appendix

Proofs of the lemmas and propositions from the theoretical part are provided here. We start with a number of assumptions and definitions.

Assumption 1. *Parameters are such that* $\beta e^{\frac{a^2}{2\gamma_\theta}} < 1$.

This assumption guarantees that a number of effects highlighted in the baseline model are unambiguous and in particular that the option value of waiting is strong enough to dominate other forces. It states in particular the following: risk aversion (a) and aggregate risk ($\frac{1}{\gamma_\theta}$) cannot be too large, otherwise the incentives for waiting are greatly reduced (the condition states that the risk induced by waiting never outweighs the option value of waiting).

Assumption 2. *F is continuous, twice differentiable with bounded first and second derivatives.*

We impose a number of regularity conditions on the cumulative distribution of investment costs to guarantee that the equilibrium number of investing firms $N(\mu, \gamma) \sim Bin\left(\bar{N}, F\left(f_c\left(\mu, \gamma\right)\right)\right)$ is well-behaved.

Definition 3. Define the following bounds and set:

1. Let $\bar{\gamma}$ be the unique strictly positive solution of

$$\bar{\gamma} = \left(\frac{1}{\bar{\gamma} + \gamma_y + \bar{N}\gamma_x} + \frac{1}{\gamma_\theta}\right)^{-1} = \Gamma\left(\bar{N}, \bar{\gamma}\right),$$

   and $\underline{\gamma}$ the unique strictly positive solution of

$$\underline{\gamma} = \left(\frac{1}{\underline{\gamma} + \gamma_y} + \frac{1}{\gamma_\theta}\right)^{-1} = \Gamma\left(0, \underline{\gamma}\right),$$

2. Let $\mathcal{S} = [\underline{\mu}, \bar{\mu}] \times [\underline{\gamma}, \bar{\gamma}]$, where $\underline{\mu}$ and $\bar{\mu}$ are some arbitrary but large bounds on μ.

We restrict (μ, γ) to be in the domain $\mathcal{S}$. Unfortunately, since shocks are normally distributed, imposing bounds on μ restricts the equilibrium and makes the Bayesian learning formulas not completely consistent with rationality. We ignore that matter for practical purposes as these bounds are necessary to the fixed point theorems. With bounds large enough, this should be a good enough approximation.

We define the set $\mathcal{P}$ in which the probability $p(\mu, \gamma) = F\left(f_c\left(\mu, \gamma\right)\right)$ that a firm invests will lie:

Definition 4. Let $\mathcal{P}$ be the set of twice-continuously differentiable functions $p : (\mu, \gamma) \in \mathcal{S} \longrightarrow \mathbb{R}$ such that p has bounded first and second derivatives: $\forall (\mu, \gamma) \in \mathcal{S}, |p_\mu(\mu, \gamma)| \leq \bar{p}_\mu$, $|p_\gamma(\mu, \gamma)| \leq \bar{p}_\gamma$, and $|p_{xy}(\mu, \gamma)| \leq \bar{p}_{xy}$ for $(x, y) \in \{\mu, \gamma\}^2$.

We also define the set $\mathcal{G}$ in which the firm's surplus of waiting compared to investing will lie:

Definition 5. Let $\mathcal{G}$ be the set of twice-continuously differentiable functions G of $(\mu, \gamma, f) \in \mathcal{S} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

1. G is weakly decreasing and convex in μ,
2. G is weakly decreasing in γ,
3. G is Lipschitz continuous of constant 1 in f,
4. G has bounded first and second derivatives: $\forall (\mu, \gamma, f), |G_x(\mu, \gamma)| \leq \bar{G}_x$ and $|G_{xy}(\mu, \gamma)| \leq \bar{G}_{xy}$ for $(x, y) \in \{\mu, \gamma, f\}^2$.

The bounds on the derivatives of N depend on those of G and vice-versa in equilibrium. In the final version, we will need clean expressions (TO DO). In fact, differentiability may not obtain, so we will have to rewrite the proofs with Lipschitz-continuous functions or perhaps semi-differentiability.

Let us now turn to the proofs.

Lemma 1. *Mean beliefs μ follow a random walk with time-varying volatility,*

$$\mu^{'} = \mu + s(N, \gamma)\, \varepsilon^{\mu},$$

where $s(N, \gamma) = \left(\frac{1}{\gamma} - \frac{1}{\gamma + \gamma_y + N\gamma_x}\right)^{\frac{1}{2}}$ *and* $\varepsilon \sim \mathcal{N}(0, 1)$.

Proof. Conditioning on the realization of N and on current information, expression (8) tells us that μ' is normally distributed (it is the sum of a constant, μ, and two normally distributed signals, Y and X. Let us characterize its mean and variance:

$$\mathbb{E}[\mu'|\mu, \gamma, N] = \mathbb{E}\left[\frac{\gamma \cdot \mu + \gamma_y \cdot Y + N\gamma_x \cdot X}{\gamma + \gamma_y + N\gamma_x}|\mu, \gamma, N\right] = \mu,$$

$$\begin{aligned} Var(\mu'|\mu, \gamma, N) &= Var\left(\frac{(\gamma_y + N\gamma_x)\theta + \gamma_y \varepsilon^y + N\gamma_x \varepsilon_N^X}{\gamma + \gamma_y + N\gamma_x}|\mu, \gamma, N\right) \\ &= \left(\frac{\gamma_y + N\gamma_x}{\gamma + \gamma_y + N\gamma_x}\right)^2 \frac{1}{\gamma} + \left(\frac{\gamma_y}{\gamma + \gamma_y + N\gamma_x}\right)^2 \frac{1}{\gamma_y} + \left(\frac{N\gamma_x}{\gamma + \gamma_y + N\gamma_x}\right)^2 \frac{1}{N\gamma_x} \\ &= \frac{1}{\gamma} - \frac{1}{\gamma + \gamma_y + N\gamma_x}. \end{aligned}$$

We can therefore write μ' as the sum of μ plus some normal innovation ε^{μ} with a variance that depends on N and γ, N being random, distributed according to a binomial law. □

Proposition 1 (full). *For $p(\mu, \gamma) \in \mathcal{P}$ such that $N(\mu, \gamma) \sim Bin\left(\overline{N}, p(\mu, \gamma)\right)$, under assumption 1 and for γ_x sufficiently low, for all $(\mu, \gamma) \in \mathcal{S}$, there exists a unique cutoff value $f_c(\mu, \gamma) \in \mathbb{R} \cup \{-\infty, \infty\}$, such that firms invest if and only if $f \leq f_c(\mu, \gamma)$. In addition, $f_c(\mu, \gamma)$ is a strictly increasing function of μ and γ.*

Proof. Using the definition 4, write the mapping $\mathcal{T}^p : \mathcal{G} \longrightarrow \mathcal{G}$ for $G(\mu,\gamma,f) = V(\mu,\gamma,f) - \left(V^I(\mu,\gamma) - f\right)$ as follows:

$$\mathcal{T}^p G(\mu,\gamma) = \max\left\{0, \beta E\left[G(\mu',\gamma',f') + V^I(\mu',\gamma') - f\right] - V^I(\mu',\gamma') + f\right\}.$$

Substitute with the stopping value $V^I(\mu,\gamma) = \frac{1}{a}\left(1 - e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\right)$:

$$\begin{aligned}
\mathcal{T}^p G(\mu,\gamma,f) &= \max\left\{0, \beta E\left[G(\mu',\gamma',f') + \frac{1}{a}\left(1 - e^{-a\mu' + \frac{a^2}{2}\left(\frac{1}{\gamma'}+\frac{1}{\gamma_x}\right)}\right) - f\right] - V^I(\mu,\gamma) + f\right\} \\
&= \max\left\{0, \beta E\left[G(\mu',\gamma',f') + \frac{1}{a}\left(1 - e^{-a(\mu+s\cdot\varepsilon) + \frac{a^2}{2}\left(\frac{1}{\gamma+\gamma_y+N\gamma_x}+\frac{1}{\gamma_\theta}+\frac{1}{\gamma_x}\right)}\right) - f\right]\right. \\
&\qquad\left. -\frac{1}{a}\left(1 - e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\right) + f\right\} \\
&= \max\left\{0, \beta\left[E\left(G(\mu',\gamma',f')\right) + \frac{1}{a}\left(1 - e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma+\gamma_y+N\gamma_x}+\frac{1}{\gamma_\theta}+\frac{1}{\gamma_x}+s^2\right)}\right) - \bar{f}\right]\right. \\
&\qquad\left. -\frac{1}{a}\left(1 - e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\right) + f\right\} \\
&= \max\left\{0, \underbrace{\frac{1}{a}e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + f - (1-\beta)\frac{1}{a} - \beta\bar{f} + \beta E\left[G(\mu',\gamma',f')\right]}_{\equiv C^p(G)\ \text{(continuation value)}}\right\}
\end{aligned}$$

where the notation $s(N(\mu,\gamma),\gamma) = \left(\frac{1}{\gamma} - \frac{1}{\gamma+\gamma_y+N(\mu,\gamma)\gamma_x}\right)^{\frac{1}{2}}$ denotes the standard deviation of μ' given (μ,γ) and N. Under assumption 1, the first term in the braces is positive: it is a strictly decreasing, convex function of μ, and strictly decreasing function of γ.

We are going to show that $\mathcal{T}^p$ is a well defined mapping from $\mathcal{G}$ to $\mathcal{G}$. We show in addition that it satisfies the Blackwell conditions, so that it is a contraction with a unique fixed point. Indeed, $\mathcal{T}^p$ satisfies the following properties:

1. Monotonicity: if $G_1(\mu,\gamma,f) \leq G_2(\mu,\gamma,f)$ for all $(\mu,\gamma) \in \mathcal{S}$, $f \in \mathbb{R}$ then

$$\begin{aligned}
&\frac{1}{a}e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + f - (1-\beta)\frac{1}{a} - \beta\bar{f} + \beta E\left[G_1(\mu',\gamma',f')\right] \leq \\
&\frac{1}{a}e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + f - (1-\beta)\frac{1}{a} - \beta\bar{f} + \beta E\left[G_2(\mu',\gamma',f')\right]
\end{aligned}$$

so that $(\mathcal{T}^p G_1)(\mu,\gamma,f) \leq (\mathcal{T}^p G_2)(\mu,\gamma,f)$ for all $(\mu,\gamma) \in \mathcal{S}$, $f \in \mathbb{R}$;

2. Discounting: for $K \geq 0$,

$$\begin{aligned}
&\tfrac{1}{a}e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + f - (1-\beta)\tfrac{1}{a} - \beta\bar{f} + \beta E\left[G(\mu',\gamma',f') + K\right] \\
=\ &\tfrac{1}{a}e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + f - (1-\beta)\tfrac{1}{a} - \beta\bar{f} + \beta E\left[G(\mu',\gamma',f')\right] + \beta K \\
\leq\ &(\mathcal{T}^p G)(\mu,\gamma,f) + \beta K
\end{aligned}$$

so that $\left[\mathcal{T}^p(G+K)\right](\mu,\gamma,f) \leq \left(\mathcal{T}^p G\right)(\mu,\gamma,f) + \beta K$ for all $(\mu,\gamma) \in \mathcal{S},\ f \in \mathbb{R}$.

Therefore, $\mathcal{T}^p$ is a contraction mapping that admits a unique fixed point. We only need to check that it is well defined, i.e. that it remains in set $\mathcal{G}$.

- **Decreasing in** μ

We want to show that $\mathcal{T}^p G$ decreases with μ. The term $\frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)$ decreases with μ, so we only need to check that $\beta E\left[G\left(\mu',\gamma',f'\right)\right]$ does as well. Denote $\pi_N^{\bar{N}}(p) = \binom{\bar{N}}{N}p^N(1-p)^{\bar{N}-N}$. Pick $\mu_1 < \mu_2$. We use the following notation:

$$E_i\left[G\left(\mu_j + s\varepsilon, \Gamma, f'\right)\right] = \sum_{N=1}^{\bar{N}} \pi_N^{\bar{N}}\left(p\left(\mu_i,\gamma\right)\right) \int G\left(\mu_j + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right) d\Phi(\varepsilon)\, dF\left(f'\right)$$

with $s(N,\gamma) = \left(\frac{1}{\gamma} - \frac{1}{\gamma+\gamma_y+N\gamma_x}\right)^{\frac{1}{2}}$ and $\Gamma(N,\gamma) = \left(\frac{1}{\gamma+\gamma_y+N\gamma_x} + \frac{1}{\gamma_\theta}\right)^{-1}$ and $\varepsilon \sim \mathcal{N}(0,1)$ for $i,j = 1,2$. A change in μ has a direct effect on G as a function of μ and another effect through the expectation $p(\mu,\gamma)$. We are interested in the sign of

$$\begin{aligned} & E_2\left[G\left(\mu_2 + s\varepsilon, \Gamma, f'\right)\right] - E_1\left[G\left(\mu_1 + s\varepsilon, \Gamma, f'\right)\right] \\ = \; & \underbrace{E_2\left[G\left(\mu_2 + s\varepsilon, \Gamma, f'\right)\right] - E_2\left[G\left(\mu_1 + s\varepsilon, \Gamma, f'\right)\right]}_{\leq 0 \text{ because } G \text{decreasing}} + E_2\left[G\left(\mu_1 + s\varepsilon, \Gamma, f'\right)\right] - E_1\left[G\left(\mu_1 + s\varepsilon, \Gamma, f'\right)\right]. \end{aligned}$$

The first term is negative, which is good for us. The only nuisance term is the second one coming from the change in beliefs. We are going to show that it is a $O(\gamma_x)$, i.e. negligible when γ_x is small. Using the notation $g_N(\mu,\gamma) \equiv \int G\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right) d\Phi(\varepsilon)\, dF\left(f'\right)$ and $\Pi_N(p) = \sum_{n=1}^{N} \pi_n^{\bar{N}}(p)$, rewrite the following:

$$E_p\left[G\left(\mu + s\varepsilon, \Gamma, f'\right)\right] = \sum_{N=1}^{\bar{N}} \pi_N^{\bar{N}}(p) \underbrace{\int G\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right) d\Phi(\varepsilon)\, dF\left(f'\right)}_{\equiv g_N(\mu,\gamma)}$$

Summing by parts:

$$E_p\left[G\left(\mu + s\varepsilon, \Gamma, f'\right)\right] = g_{\bar{N}}(\mu,\gamma) - \sum_{N=1}^{\bar{N}-1} \Pi_N(p) \cdot \left(g_{N+1} - g_N\right)(\mu,\gamma).$$

Taking the derivative with respect to p, we obtain

$$\frac{\partial}{\partial p} E_p\left[G\left(\mu + s\varepsilon, \Gamma, f'\right)\right] = -\sum_{N=1}^{\bar{N}-1} \Pi_N'(p) \cdot \left(g_{N+1} - g_N\right)(\mu,\gamma)$$

Let us now control the terms $(g_{N+1} - g_N)(\mu, \gamma)$. To that end, let us look at the following term:

$$\begin{aligned} & \left|G\left(\mu + s(N+1,\gamma)\varepsilon, \Gamma(N+1,\gamma), f'\right) - G\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right)\right| \\ \leq & \left|G\left(\mu + s(N+1,\gamma)\varepsilon, \Gamma(N+1,\gamma), f'\right) - G\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N+1,\gamma), f'\right)\right| \\ & + \left|G\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N+1,\gamma), f'\right) - G\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right)\right| \\ \leq & \bar{G}_\mu \left|\varepsilon\left(s(N+1,\gamma) - s(N,\gamma)\right)\right| + \bar{G}_\gamma \left|\Gamma(N+1,\gamma) - \Gamma(N,\gamma)\right| \end{aligned}$$

where we have used the fact that G is bi-Lipschitz continuous. Therefore,

$$\begin{aligned} \left|(g_{N+1} - g_N)(\mu,\gamma)\right| & \leq \bar{G}_\mu \left|\left(s(N+1,\gamma) - s(N,\gamma)\right)\right| \int |\varepsilon|\, d\Phi(\varepsilon) + \bar{G}_\gamma \left|\Gamma(N+1,\gamma) - \Gamma(N,\gamma)\right| \\ & \leq \bar{G}_\mu \left|\left(s(N+1,\gamma) - s(N,\gamma)\right)\right| + \bar{G}_\gamma \left|\Gamma(N+1,\gamma) - \Gamma(N,\gamma)\right|. \end{aligned}$$

We just need to check that these terms are small:

$$|s(N+1,\gamma) - s(N,\gamma)| \leq \frac{\gamma_x}{(\underline{\gamma} + \gamma_y)^2} = O(\gamma_x)$$

$$\begin{aligned} |\Gamma(N+1,\gamma) - \Gamma(N,\gamma)| & = \frac{\gamma_\theta^2 \gamma_x}{(\gamma_\theta + \gamma + \gamma_y + N\gamma_x)(\gamma_\theta + \gamma + \gamma_y + (N+1)\gamma_x)} \\ & \leq \frac{\gamma_\theta^2 \gamma_x}{(\gamma_\theta + \underline{\gamma} + \gamma_y)^2} = O(\gamma_x). \end{aligned}$$

We can now go back to the nuisance term

$$\left|\frac{\partial}{\partial p} E_p\left[G\left(\mu + s\varepsilon, \Gamma, f'\right)\right]\right| = O(\gamma_x),$$

where we have used the fact that $\Pi'_N(p)$ is a polynomial of degree $\bar{N} - 1$ and is thus bounded on the interval $[0,1]$. We may now conclude. For γ_x small enough, the continuation value $C^p(G)$ will be strictly decreasing. Its derivative,

$$\frac{\partial}{\partial \mu} C^p(G) = \underbrace{-e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right)}_{< -e^{-a\bar{\mu} + \frac{a^2}{2}\frac{1}{\gamma}}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) < 0} + \beta \underbrace{E_{p(\mu,\gamma)}\left[\frac{\partial G}{\partial \mu}\right]}_{\leq 0} + \beta \underbrace{\frac{\partial p}{\partial \mu}}_{\leq \bar{p}_\mu} \cdot \underbrace{\frac{\partial}{\partial p} E_p\left[G\left(\mu + s\varepsilon, \Gamma, f'\right)\right]}_{O(\gamma_x)}$$

is strictly negative for γ_x small enough. Therefore, for a small enough γ_x, $\mathcal{T}^p G$ is weakly decreasing in μ.

- **Decreasing in γ**

The proof is based on the same argument as the one we just developed. The derivative of the continuation value with respect to γ is:

$$\frac{\partial}{\partial\gamma}C^{p}\left(G\right)=\underbrace{-\frac{a}{2\gamma^{2}}e^{-a\mu+\frac{a^{2}}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_{x}}\right)}\left(1-\beta e^{\frac{a^{2}}{2\gamma_{\theta}}}\right)}_{<-\frac{a}{2\bar{\gamma}}e^{-a\bar{\mu}+\frac{a^{2}}{2}\frac{1}{\bar{\gamma}}}\left(1-\beta e^{\frac{a^{2}}{2\gamma_{\theta}}}\right)<0}+\beta\underbrace{E_{p(\mu,\gamma)}\left[\frac{\partial G}{\partial\gamma}\right]}_{\leq0}+\beta\underbrace{\frac{\partial p}{\partial\gamma}}_{\leq\bar{p}_{\gamma}}\cdot\underbrace{\frac{\partial}{\partial p}E_{p}\left[G\left(\mu+s\varepsilon,\Gamma,f'\right)\right]}_{O(\gamma_{x})}.$$

Therefore, for γ_x small enough, the derivative is strictly negative, bounded away from 0. The whole mapping $\mathcal{T}^pG$ is thus weakly decreasing in γ.

- **Lipschitz in f of constant 1**

Pick $f_1 < f_2$. This is trivially true:

$$\left|\mathcal{T}^{p}G\left(\mu,\gamma,f_{2}\right)-\mathcal{T}^{p}G\left(\mu,\gamma,f_{1}\right)\right|\leq\left|\left[C^{p}\left(G\right)\right]\left(\mu,\gamma,f_{2}\right)-\left[C^{p}\left(G\right)\right]\left(\mu,\gamma,f_{1}\right)\right|=\left|f_{2}-f_{1}\right|.$$

- **Convex in μ**

Take the second derivative of the continuation value:

$$\begin{aligned}\frac{\partial^{2}}{\partial\mu^{2}}C^{p}\left(G\right) &= \underbrace{ae^{-a\mu+\frac{a^{2}}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_{x}}\right)}\left(1-\beta e^{\frac{a^{2}}{2\gamma_{\theta}}}\right)}_{\underbrace{\geq ae^{-a\bar{\mu}+\frac{a^{2}}{2}\frac{1}{\bar{\gamma}}}\left(1-\beta e^{\frac{a^{2}}{2\gamma_{\theta}}}\right)>0}_{\text{independent of}_{\gamma_{x}}}}+\beta\underbrace{E_{p(\mu,\gamma)}\left[\frac{\partial^{2}G}{\partial\mu^{2}}\right]}_{\geq0}\\ &\quad+\beta\underbrace{\frac{\partial p}{\partial\mu}}_{\leq\bar{p}_{\mu}}\cdot\frac{\partial}{\partial p}E_{p}\left[\frac{\partial}{\partial\mu}G\left(\mu+s\varepsilon,\Gamma,f'\right)\right]+\beta\underbrace{\frac{\partial^{2}p}{\partial\mu^{2}}}_{\leq\bar{p}_{\mu\mu}}\cdot\frac{\partial^{2}}{\partial p^{2}}E\left[G\left(\mu+s\varepsilon,\Gamma,f'\right)\right].\end{aligned}$$

The claim is proved by showing that the terms $\frac{\partial}{\partial p}E_p\left[\frac{\partial}{\partial\mu}G\left(\mu+s\varepsilon,\Gamma,f'\right)\right]$ and $\frac{\partial^2}{\partial p^2}E\left[G\left(\mu+s\varepsilon,\Gamma,f'\right)\right]$ are $O\left(\gamma_x\right)$. The argument extends the one developed in the proof for monotonicity.

$$\frac{\partial^{2}}{\partial p^{2}}E_{p}\left[G\left(\mu+s\varepsilon,\Gamma,f'\right)\right]=-\sum_{N=1}^{\bar{N}-1}\Pi_{N}''\left(p\right)\cdot\underbrace{\left(g_{N+1}-g_{N}\right)\left(\mu,\gamma\right)}_{=O(\gamma_{x})}=O\left(\gamma_{x}\right)$$

since $\Pi_N''(p)$ is a polynomial of degree $\bar{N}-2$ in p, therefore bounded on $[0,1]$. Defining $g_N^{(\mu)}(\mu,\gamma)\equiv\int G_\mu\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f'\right)d\Phi\left(\varepsilon\right)dF\left(f'\right)$, we have

$$E_{p}\left[\frac{\partial}{\partial\mu}G\left(\mu+s\varepsilon,\Gamma,f'\right)\right]=\sum_{N=1}^{\bar{N}}\pi_{N}^{\bar{N}}\left(p\right)g_{N}^{(\mu)}\left(\mu,\gamma\right)=g_{\bar{N}}^{(\mu)}\left(\mu,\gamma\right)-\sum_{N=1}^{\bar{N}-1}\Pi_{N}\left(p\right)\cdot\left(g_{N+1}^{(\mu)}-g_{N}^{(\mu)}\right)\left(\mu,\gamma\right).$$

Taking the derivative:

$$\frac{\partial}{\partial p}E_{p}\left[\frac{\partial}{\partial\mu}G\left(\mu+s\varepsilon,\Gamma,f'\right)\right]=-\sum_{N=1}^{\bar{N}-1}\Pi_{N}'\left(p\right)\cdot\left(g_{N+1}^{(\mu)}-g_{N}^{(\mu)}\right)\left(\mu,\gamma\right).$$

By a similar argument as before

$$\begin{aligned} & \left|G_\mu\left(\mu+s\left(N+1,\gamma\right)\varepsilon,\Gamma\left(N+1,\gamma\right),f'\right)-G_\mu\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f'\right)\right| \\ \leq\ & \bar{G}_{\mu\mu}\left|\varepsilon\left(s\left(N+1,\gamma\right)-s\left(N,\gamma\right)\right)\right|+\bar{G}_{\mu\gamma}\left|\Gamma\left(N+1,\gamma\right)-\Gamma\left(N,\gamma\right)\right|, \end{aligned}$$

so that

$$\left|\left(g_{N+1}^{(\mu)}-g_N^{(\mu)}\right)(\mu,\gamma)\right| \leq \bar{G}_{\mu\mu}\left|\left(s\left(N+1,\gamma\right)-s\left(N,\gamma\right)\right)\right|+\bar{G}_{\mu\gamma}\left|\Gamma\left(N+1,\gamma\right)-\Gamma\left(N,\gamma\right)\right|=O\left(\gamma_x\right).$$

The whole term $E_p\left[\frac{\partial}{\partial\mu}G\left(\mu+s\varepsilon,\Gamma,f'\right)\right]$ is thus a $O\left(\gamma_x\right)$, so for γ_x small enough, the second derivative of the continuation value $C^p\left(G\right)$ is strictly positive, uniformly bounded away from 0. Convexity (weak) is thus preserved by the mapping $\mathcal{T}^pG$.

- **Regularity conditions: continuity, differentiability and boundedness**

Skip for now.

- **Existence and monotonicity of $f_c\left(\mu,\gamma\right)$**

The existence of the cutoff is trivially guaranteed by the fact that the continuation utility is linear in f. Firms invest iff

$$\begin{aligned} C^p\left(G\right)=\ & \frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+f-\left(1-\beta\right)\frac{1}{a}-\beta\bar{f}+\beta E\left[G\left(\mu',\gamma',f'\right)\right]\leq 0 \\ \Leftrightarrow f\leq\ & -\frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+\left(1-\beta\right)\frac{1}{a}+\beta\bar{f}-\beta E\left[G\left(\mu',\gamma',f'\right)\right]\equiv f_c\left(\mu,\gamma\right). \end{aligned}$$

Notice that $f_c\left(\mu,\gamma\right)=f-\left[C^p\left(G\right)\right]\left(\mu,\gamma,f\right)$, so that the threshold inherits a number of properties from the continuation value. In particular, $f_c\left(\mu,\gamma\right)$ is strictly increasing in μ and γ, strictly concave in μ for γ_x small enough. □

We now turn to the general equilibrium results. Define the mapping:

Definition 6. Let $\mathcal{M}$ be the mapping from $p:\mathcal{P}\longrightarrow\mathcal{P}$ such that

$$\forall\left(\mu,\gamma\right)\in\mathcal{S},\quad\left(\mathcal{M}p\right)\left(\mu,\gamma\right)=F\left(f_c^p\left(\mu,\gamma\right)\right)$$

where f_c is defined as

$$f_c^p\left(\mu,\gamma\right)=-\frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+\left(1-\beta\right)\frac{1}{a}+\beta\bar{f}-\beta E\left[G^p\left(\mu',\gamma',f'\right)\right]$$

and G^p is the unique fixed point of the mapping

$$G^p=\mathcal{T}^pG^p.$$

Proposition 2 (full). *Under assumption 1, 2 and for γ_x small enough, an equilibrium exists and is unique, i.e. there exists a unique fixed point $p^* \in \mathcal{P}$ such that $\mathcal{M}p^* = p^*$. In addition, the investment decision of firms is characterized by the threshold $f_c^*(\mu,\gamma)$, strictly increasing in μ and γ.*

Proof. Unfortunately for our purpose, the mapping $\mathcal{M}$ does not satisfy the Blackwell conditions. In particular, monotonicity does not obtain under the current assumptions. If $p_1(\mu,\gamma) \leq p_2(\mu,\gamma)$ for all $(\mu,\gamma) \in \mathcal{S}$, even though the effect on G^{p_i} is small, it is in general ambiguous and I cannot conclude anything with regard to f_c^p. We are still going to show that $\mathcal{M}$ defines a contraction from $\mathcal{P}$ to $\mathcal{P}$.

First, let us check that it is a well-defined mapping. Recall the definition of f_c^p:

$$f_c^p(\mu,\gamma) = -\frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+(1-\beta)\frac{1}{a}+\beta\bar{f}-\beta E\left[G^p\left(\mu',\gamma',f'\right)\right]$$

Given that $G^p \in \mathcal{G}$, f_c^p inherits a number of nice properties: it is twice-continuously differentiable, with bounded first and second derivatives. Under assumption 2,$\mathcal{M}p = F\left(f_c^p\right)$ preserves these properties. So, $\mathcal{M}$ is a well-defined mapping from $\mathcal{P}$ to $\mathcal{P}$. TO DO: check that the actual bounds $\bar{p}_\mu$, $\bar{p}_\gamma$, etc., are consistent with our assumptions on $\mathcal{G}$.

We will now show that $\mathcal{M}$ defines a contraction. Take $p_1, p_2 \in \mathcal{P}$. We must first control the term$\| G^{p_1} - G^{p_2} \|$. Start with some $G \in \mathcal{G}$, we have:

$$\begin{aligned}\mathcal{T}^{p_i}G(\mu,\gamma,f) &= \max\Big\{0, \frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+f-(1-\beta)\frac{1}{a}-\beta\bar{f} \\ &\quad +\beta E_{p^i}\left[G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right]\Big\}\end{aligned}$$

How does $\mathcal{T}^{p_2}G$ compare to $\mathcal{T}^{p_1}G$? First, remember that using the notation $\Pi_N(p) = \sum_{n=1}^{N}\pi_n^{\bar{N}}(p)$ we can write the following:

$$\begin{aligned}E_p\left[G\left(\mu+s\varepsilon,\Gamma,f'\right)\right] &= \sum_{N=1}^{\bar{N}}\pi_N^{\bar{N}}(p)\underbrace{\int G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)d\Phi(\varepsilon)\,dF\left(f'\right)}_{\equiv g_N(\mu,\gamma)} \\ &= g_{\bar{N}}(\mu,\gamma) - \sum_{N=1}^{\bar{N}-1}\Pi_N(p)\cdot\left(g_{N+1}-g_N\right)(\mu,\gamma)\end{aligned}$$

so that we can control the term:

$$
\begin{aligned}
& \left|E_{p_2}\left[G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_1}\left[G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right| \\
= & \left|\sum_{N=1}^{\bar{N}-1}\left[\Pi_N(p_2)-\Pi_N(p_1)\right]\cdot(g_{N+1}-g_N)(\mu,\gamma)\right| \\
\leq & \gamma_x\left(\frac{\bar{G}_\mu}{(\underline{\gamma}+\gamma_y)^2}+\frac{\bar{G}_\gamma\gamma_\theta^2}{(\gamma_\theta+\underline{\gamma}+\gamma_y)^2}\right)\sum_{N=1}^{\bar{N}-1}\left|\Pi_N(p_2)-\Pi_N(p_1)\right|
\end{aligned}
$$

where I have used some results established in proposition 1. The probability $\Pi_N(p)$ is a polynomial in p of degree $\bar{N}$, it is continuous on the compact $[0,1]$ and therefore uniformly continuous: $\| \Pi_N(p_2)-\Pi_N(p_1) \|\leq B_N \| p_2-p_1 \|$. Therefore, there exists a constant $B\geq 0$ such that

$$
\left|E_{p_2}\left[G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_1}\left[G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right|\leq B\gamma_x \| p_2-p_1 \| .
$$

What does it imply for $(\mathcal{T}^{p_2}G-\mathcal{T}^{p_1}G)(\mu,\gamma,f)$? Assume WLOG that

$$
E_{p_2}\left[G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\geq E_{p_1}\left[G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right].
$$

Then three cases arise:

1. Either the max is reached at 0 for both and $(\mathcal{T}^{p_2}G-\mathcal{T}^{p_1}G)(\mu,\gamma,f)=0$;

2. Or:

$$
\begin{aligned}
\frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+f-(1-\beta)\frac{1}{a}-\beta\bar{f}+\beta E_{p_2}\left[G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right] &> 0 \\
\frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+f-(1-\beta)\frac{1}{a}-\beta\bar{f}+\beta E_{p_1}\left[G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right] &\leq 0
\end{aligned}
$$

   so that $(\mathcal{T}^{p_1}G)(\mu,\gamma,f)=0$. In that case,

$$
\begin{aligned}
\left|(\mathcal{T}^{p_2}G-\mathcal{T}^{p_1}G)(\mu,\gamma,f)\right| &\leq \beta\left\{E_{p_2}\left[G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_1}\left[G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right\} \\
&\leq \beta B\gamma_x \| p_2-p_1 \|
\end{aligned}
$$

3. Or, last case, the max is reached on the right-hand side and the inequality directly applies

$$
\left|(\mathcal{T}^{p_2}G-\mathcal{T}^{p_1}G)(\mu,\gamma,f)\right|\leq\beta B\gamma_x \| N_2-N_1 \| .
$$

Now, we must find out what happens after a number of iterations. To lighten notation, let us denote $G_n^i\equiv(\mathcal{T}^{p_i})^n G$.

$$
\begin{aligned}
& \left|E_{p_2}\left[G_1^2\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_1}\left[G_1^1\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right| \\
= & \left|E_{p_2}\left[G_1^2\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_2}\left[G_1^1\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right. \\
& \left.+E_{p_2}\left[G_1^1\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_1}\left[G_1^1\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right| \\
\leq & \beta B\gamma_x \parallel p_2-p_1 \parallel + \left|E_{p_2}\left[G_1^1\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_1}\left[G_1^1\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right| \\
& \text{(apply same argument as above)} \\
\leq & (1+\beta)B\gamma_x \parallel p_2-p_1 \parallel
\end{aligned}
$$

Recursively, one can show that for $n \geq 1$:

$$
\parallel (\mathcal{T}^{p_2})^n G - (\mathcal{T}^{p_1})^n G \parallel \leq \beta \frac{1-\beta^n}{1-\beta} B\gamma_x \parallel p_2-p_1 \parallel
$$

Since operators $\mathcal{T}^{p_i}$ are contractions, we can take the limit:

$$
\parallel G^{p_2} - G^{p_1} \parallel \leq \frac{\beta}{1-\beta} B\gamma_x \parallel p_2-p_1 \parallel . \tag{13}
$$

Now, what does this imply for $\parallel f_c^{p_2} - f_c^{p_1} \parallel$? Recall the definition of the threshold:

$$
f_c^{p_i}(\mu,\gamma) = -\frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+(1-\beta)\frac{1}{a}+\beta\bar{f}-\beta E_{p_i}\left[G^{p_i}\left(\mu',\gamma',f'\right)\right]
$$

Therefore, we have:

$$
\begin{aligned}
|f_c^{p_2}(\mu,\gamma)-f_c^{p_1}(\mu,\gamma)| &= \beta\left|E_{p_2}\left[G^{p_2}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_1}\left[G^{p_1}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right| \\
&= \beta\left|E_{p_2}\left[G^{p_2}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_2}\left[G^{p_1}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right. \\
&\quad \left.+E_{p_2}\left[G^{p_1}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_1}\left[G^{p_1}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right| \\
&\leq \beta\left(\frac{\beta}{1-\beta}B\gamma_x \parallel p_2-p_1 \parallel + B\gamma_x \parallel p_2-p_1 \parallel\right) \\
&\leq \frac{\beta}{1-\beta}B\gamma_x \parallel p_2-p_1 \parallel
\end{aligned}
$$

The mapping is thus also continuous in f_c, since we have:

$$
\parallel f_c^{p_2} - f_c^{p_1} \parallel \leq \frac{\beta}{1-\beta} B\gamma_x \parallel p_2-p_1 \parallel .
$$

We just have to conclude now with the mapping $\mathcal{M}$:

$$\begin{aligned}
\left|(\mathcal{M}p_2 - \mathcal{M}p_1)(\mu,\gamma)\right| &= \left|F\left(f_c^{p_2}(\mu,\gamma)\right) - F\left(f_c^{p_1}(\mu,\gamma)\right)\right| \\
&= \left|F'\left(\tilde{f}\right)\left(f_c^{p_2}(\mu,\gamma) - f_c^{p_1}(\mu,\gamma)\right)\right| \\
&\quad \left(\text{for some } \tilde{f} \in \left[f_c^{p_1}(\mu,\gamma), f_c^{p_2}(\mu,\gamma)\right]\right) \\
&\leq \frac{\beta}{1-\beta} B\gamma_x \parallel F' \parallel \cdot \parallel p_2 - p_1 \parallel .
\end{aligned}$$

This tells us that the mapping $\mathcal{M}$ is continuous as long as F' is bounded, which is guaranteed by assumption 2. But the best is yet to come: we can choose γ_x such that

$$\frac{\beta}{1-\beta} B\gamma_x \parallel F' \parallel < 1.$$

In that case, the mapping $\mathcal{M}$ is actually a contraction! By the contraction mapping theorem, this guarantees the existence and uniqueness of the equilibrium $\mathcal{N}^*$ for small values of γ_x. □

Proposition 3 (full). *Under assumption 1, 2 and for γ_x small enough, there exists a non-empty interval $[\mu_l, \mu_h]$, such that for all $\mu \in (\mu_l, \mu_h)$ and for σ^f low enough, there exists at least two locally stable stationary points $\gamma_l(\mu) < \gamma_h(\mu)$ in the dynamics of γ. Equilibrium γ_l (resp. γ_h) is characterized by high uncertainty (resp. low) and low investment (resp. high).*

Proof. Define the function

$$\varphi_\mu^N(\gamma) = \left(\frac{1}{\gamma + \gamma_y + N(\mu,\gamma)\gamma_x} + \frac{1}{\gamma_\theta}\right)^{-1} - \gamma.$$

By continuity of $N = \bar{N} \cdot p(\mu,\gamma)$, $\varphi_\mu^N(\gamma)$ is a continuous function. Notice first that $\varphi_\mu^N(\underline{\gamma}) \geq 0$ and $\varphi_\mu^N(\bar{\gamma}) \leq 0$:

$$\begin{aligned}
\varphi_\mu^N(\underline{\gamma}) &= \left(\frac{1}{\underline{\gamma} + \gamma_y + N(\mu,\underline{\gamma})\gamma_x} + \frac{1}{\gamma_\theta}\right)^{-1} - \underline{\gamma} \\
&\geq \left(\frac{1}{\underline{\gamma} + \gamma_y} + \frac{1}{\gamma_\theta}\right)^{-1} - \underline{\gamma} = 0
\end{aligned}$$

$$\begin{aligned}
\varphi_\mu^N(\bar{\gamma}) &= \left(\frac{1}{\bar{\gamma} + \gamma_y + N(\mu,\bar{\gamma})\gamma_x} + \frac{1}{\gamma_\theta}\right)^{-1} - \bar{\gamma} \\
&\leq \left(\frac{1}{\bar{\gamma} + \gamma_y + \gamma_x} + \frac{1}{\gamma_\theta}\right)^{-1} - \bar{\gamma} = 0
\end{aligned}$$

We are going to show that when σ_f is low, there exists a range $[\mu_l, \mu_h]$ such that for any $\mu^* \in (\mu_l, \mu_h)$, we can always find two points $\gamma_1 < \gamma_2$ with $\gamma_1, \gamma_2 \in (\underline{\gamma}, \bar{\gamma})$ such that $\varphi_{\mu^*}^N(\gamma_1) < 0$ and $\varphi_{\mu^*}^N(\gamma_2) > 0$. This will imply, by the Intermediate Value Theorem, that there exists two values $\gamma_l^* < \gamma_h^*$ with

$\underline{\gamma} \leq \gamma_l^* < \gamma_1$ and $\gamma_2 < \gamma_h^* \leq \bar{\gamma}$ such that $\varphi_\mu^N(\gamma_l^*) = \varphi_\mu^N(\gamma_h^*) = 0$, i.e. two distinct stationary points in the dynamics of precision γ.

An important step in this proof is established in lemma 3 below, where we prove that as σ^f goes to 0 the cutoff $f_c^{\sigma^f}$ converges uniformly towards some limit f_c^0 and that the number of investing firms converges pointwise to the limit $N^0(\mu, \gamma) = \bar{N}\mathbb{I}\left(\bar{f} \leq f_c^0(\mu, \gamma)\right)$.

We must first find a range of values for μ in which we are guaranteed to have multiple stationary points for γ. We are going to use the fact that $f_c^{\sigma^f}$ is strictly increasing in μ and γ at a bounded rate. Recall the definition:

$$f_c^{\sigma^f}(\mu, \gamma) = -\frac{1}{a} e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)} \left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + (1-\beta)\frac{1}{a} + \beta\bar{f} - \beta E_{\sigma^f}\left[G^{\sigma^f}(\mu', \gamma', f')\right].$$

Since G^{σ^f} has bounded derivatives, we can find upper and lower bounds for the derivatives of $f_c^{\sigma^f}$ in μ and γ that are strictly positive, as long as γ_x is low enough. Denote these bounds $\overline{f}_\mu$,$\underline{f}_\mu$ and $\overline{f}_\gamma$,$\underline{f}_\gamma$. The derivatives are:

$$\begin{aligned} 0 < \underline{f}_\mu \leq \frac{\partial}{\partial \mu} f_c^{\sigma^f}(\mu, \gamma) &= e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)} \left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + \beta E\left[\frac{\partial}{\partial \mu} G^{\sigma^f}\right] + O(\gamma_x) \leq \overline{f}_\mu \\ 0 < \underline{f}_\gamma \leq \frac{\partial}{\partial \gamma} f_c^{\sigma^f}(\mu, \gamma) &= \frac{a}{2\gamma^2} e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)} \left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + \beta E\left[\frac{\partial}{\partial \gamma} G^{\sigma^f}\right] + O(\gamma_x) \leq \overline{f}_\gamma \end{aligned}$$

Since f_c^0 is the uniform limit of continuous functions, it is continuous. The limit f_c^0 may not be differentiable, but it is bi-Lipschitz continuous with Lipschitz constants $\left(\underline{f}_\mu, \overline{f}_\mu\right)$ and $\left(\underline{f}_\gamma, \overline{f}_\gamma\right)$. We know therefore that for μ low, $f_c^0(\mu, \bar{\gamma}) < \bar{f}$ (remember that $\bar{f}$ is the mean of the fixed cost distribution), and that for μ high, $f_c^0(\mu, \bar{\gamma}) > \bar{f}$. By the Intermediate Value theorem, we know that there exists a point μ_l at which $f_c^0(\mu_l, \bar{\gamma}) = \bar{f}$. Since f_c^0 is strictly increasing in γ, we have that $f_c^0(\mu_l, \underline{\gamma}) < \bar{f}$. Using the fact that f_c^0 is bi-Lipschitz continuous, we have the following inequality:

$$f_c^0(\mu, \underline{\gamma}) \quad \leq \quad f_c^0(\mu_l, \underline{\gamma}) + \bar{f}_\mu \cdot (\mu - \mu_l).$$

Define $\mu_h = \mu_l + \frac{\bar{f} - f_c(\mu_l, \underline{\gamma})}{\bar{f}_\mu} > \mu_l$. Then, for any $\mu \in (\mu_l, \mu_h)$:

$$f_c^0(\mu, \underline{\gamma}) \leq f_c^0(\mu_l, \underline{\gamma}) + \bar{f}_\mu \cdot (\mu - \mu_l) < \bar{f} < f_c^0(\mu, \bar{\gamma}).$$

We will now show that the interval (μ_l, μ_h) is a range of values for μ in which we are guaranteed to have two steady-states. Pick any $\mu^* \in (\mu_l, \mu_h)$. Then $f_c^0(\mu^*, \underline{\gamma}) < \bar{f}$ (meaning that $N^0(\mu^*, \underline{\gamma}) = 0$) and $f_c^0(\mu^*, \bar{\gamma}) > \bar{f}$ (meaning $N^0(\mu^*, \bar{\gamma}) = \bar{N}$). By continuity of f_c^0, we can pick (γ_1, γ_2) with $\underline{\gamma} < \gamma_1 < \gamma_2 < \bar{\gamma}$, such that $f_c^0(\mu^*, \gamma_1) < \bar{f}$ and $f_c^0(\mu^*, \gamma_2) > \bar{f}$. Therefore, $N^0(\mu^*, \gamma_1) = 0$ and

$N^0(\mu^*, \gamma_2) = \bar{N}$. We have:

$$\begin{aligned}\varphi_{\mu^*}^{N^0}(\gamma_1) &= \left(\frac{1}{\gamma_1 + \gamma_y + N^0(\mu^*, \gamma_1)\gamma_x} + \sigma_\theta^2\right)^{-1} - \gamma_1 \\ &= \left(\frac{1}{\gamma_1 + \gamma_y} + \sigma_\theta^2\right)^{-1} - \gamma_1 < \left(\frac{1}{\underline{\gamma} + \gamma_y} + \sigma_\theta^2\right)^{-1} - \underline{\gamma} = 0\end{aligned}$$

$$\begin{aligned}\varphi_{\mu^*}^{N^0}(\gamma_2) &= \left(\frac{1}{\gamma_2 + \gamma_y + N^0(\mu^*, \gamma_2)\gamma_x} + \sigma_\theta^2\right)^{-1} - \gamma_2 \\ &= \left(\frac{1}{\gamma_2 + \gamma_y + \gamma_x} + \sigma_\theta^2\right)^{-1} - \gamma_2 > \left(\frac{1}{\bar{\gamma} + \gamma_y + \gamma_x} + \sigma_\theta^2\right)^{-1} - \bar{\gamma} = 0.\end{aligned}$$

Since $N^{\sigma^f}(\mu, \gamma) \underset{\sigma^f \to 0}{\longrightarrow} N^0(\mu, \gamma)$, for σ^f small enough, we will have: $\varphi_{\mu^*}^{N^{\sigma^f}}(\gamma_1) < 0$ and $\varphi_{\mu^*}^{N^{\sigma^f}}(\gamma_2) > 0$, which implies that there exists at least two locally stable steady-states γ_l^* and γ_h^* $\left(\varphi_{\mu^*}^{N^{\sigma^f}}(\gamma_l^*) = \varphi_{\mu^*}^{N^{\sigma^f}}(\gamma_h^*) = 0\right)$ with $\underline{\gamma} \leq \gamma_l^* < \gamma_1$ and $\gamma_2 < \gamma_h^* \leq \bar{\gamma}$ (one can pick at least 2 locally stable steady-states because $\varphi_{\mu^*}^{N^{\sigma^f}}$ must cross the x-axis from above at least twice). □

In what follows, we prove the technical lemma that establishes the continuity of the cutoff $f_c^{\sigma^f}$ in σ^f.

Lemma 3. *As $\sigma^f \to 0$, the equilibrium cutoff value $f_c^{\sigma^f}$ converges uniformly towards some limit f_c^0:*

$$\sup_{(\mu,\gamma)\in\mathcal{S}} \left| f_c^{\sigma^f}(\mu, \gamma) - f_c^0(\mu, \gamma) \right| \underset{\sigma^f \to 0}{\longrightarrow} 0$$

and the number of investing firms converges pointwise to the following limit:

$$\forall (\mu, \gamma), \quad N^{\sigma^f}(\mu, \gamma) = \bar{N} \cdot F^{\sigma^f}\left(f_c^{\sigma^f}(\mu, \gamma)\right) \underset{\sigma^f \to 0}{\longrightarrow} N^0(\mu, \gamma) \equiv \bar{N} \cdot \mathbb{1}\left(\bar{f} \leq f_c^0(\mu, \gamma)\right).$$

Proof. This proof is similar to the proof of continuity of the mapping $\mathcal{T}^p$ in p. Since $N = \bar{N}p$, we use N and p interchangeably from now on and abuse notation in saying that $\mathcal{M}$ is a mapping for $N : \mathcal{N} \longrightarrow \mathcal{N}$. Pick two different variances for the fixed cost σ_1^f and σ_2^f. The notation $\mathcal{T}^{N,\sigma_i^f}$ denotes the mapping $\mathcal{T}$ for the value function G when N is the aggregate number of investing firms perceived by agents and the fixed costs are distributed according to $F^{\sigma_i^f}$.

Outline of the proof: Starting with the sane initial aggregate law N, we compare the objects f_c^{N,σ_1^f} and f_c^{N,σ_2^f} after the first iteration of the mappings $\mathcal{M}^{\sigma_1^f}$ and $\mathcal{M}^{\sigma_2^f}$. In a second step, we establish a recursive relationship to compare the same objects after an arbitrary number of iterations. We then conclude that the limits of both contractions $N^{\sigma_i^f} = \lim_{n\to\infty}\left(\mathcal{M}^{\sigma_i^f}\right)^n N$ produce equilibrium cutoffs that are close in the following sense:

$$\parallel f_c^{N^{\sigma_2^f},\sigma_2^f} - f_c^{N^{\sigma_1^f},\sigma_1^f} \parallel \leq \bar{A}\left|\sigma_2^f - \sigma_1^f\right|$$

for some strictly positive constant $\bar{A}$, which suffices to establish the result.

• Start with some functions G and N, identical for both mappings. Denote $G_n^{N,\sigma_i^f} \equiv \left(\mathcal{T}^{N,\sigma_i^f}\right)^n G$. Let me prove by recursion that:

$$\left|\left(G_n^{N,\sigma_2^f} - G_n^{N,\sigma_1^f}\right)(\mu,\gamma,f)\right| \leq \beta\frac{1-\beta^n}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right|.$$

This is trivially true for $n = 0$. Assume it is true for until $n \geq 0$, then:

$$\begin{aligned}
&\left|\left(G_{n+1}^{N,\sigma_2^f} - G_{n+1}^{N,\sigma_1^f}\right)(\mu,\gamma,f)\right| \\
\leq\ & \beta\left|E_{\sigma_2^f}\left[G_n^{N,\sigma_2^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right)\right] - E_{\sigma_1^f}\left[G_n^{N,\sigma_1^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right)\right]\right| \\
\leq\ & \beta\left|E_{\sigma_2^f}\left[G_n^{N,\sigma_2^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right)\right] - E_{\sigma_1^f}\left[G_n^{N,\sigma_2^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right)\right]\right| \\
& +\beta\left|E_{\sigma_1^f}\left[G_n^{N,\sigma_2^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right)\right] - E_{\sigma_1^f}\left[G_n^{N,\sigma_1^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right)\right]\right| \\
\leq\ & \beta\left|\int\left(G_n^{N,\sigma_2^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), \bar{f} + \sigma_2^f\upsilon\right) - G_n^{N,\sigma_2^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), \bar{f} + \sigma_1^f\upsilon\right)\right)d\Phi(\varepsilon)\,d\Phi(\upsilon)\right| \\
& +\beta\times\beta\frac{1-\beta^n}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right| \\
\leq\ & \beta\int\left|\sigma_2^f - \sigma_1^f\right||\upsilon|\,d\Phi(\upsilon) + \beta^2\frac{1-\beta^n}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right| \quad \text{(Lipschitz of constant 1 in } f\text{)} \\
\leq\ & \beta\left|\sigma_2^f - \sigma_1^f\right| + \beta^2\frac{1-\beta^n}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right| = \beta\frac{1-\beta^{n+1}}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right|
\end{aligned}$$

which proves the recursion. Taking the limit $G^{N,\sigma_i^f} = \lim_{n\to\infty}\left(\mathcal{T}^{N,\sigma_i^f}\right)^n G$:

$$\left|G^{N,\sigma_2^f} - G^{N,\sigma_1^f}\right| \leq \frac{\beta}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right|. \tag{14}$$

Let us turn to the equilibrium cutoff rule:

$$\begin{aligned}
&\left|f_c^{N,\sigma_2^f}(\mu,\gamma) - f_c^{N,\sigma_1^f}(\mu,\gamma)\right| \\
=\ & \beta\left|E_{\sigma_2^f}\left[G^{N,\sigma_2^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right)\right] - E_{\sigma_1^f}\left[G^{N,\sigma_1^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right)\right]\right| \\
=\ & \beta\left|E_{\sigma_2^f}\left[G^{N,\sigma_2^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right)\right] - E_{\sigma_2^f}\left[G^{N,\sigma_1^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right)\right]\right. \\
& \left.+E_{\sigma_2^f}\left[G^{N,\sigma_1^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right)\right] - E_{\sigma_1^f}\left[G^{N,\sigma_1^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right)\right]\right| \\
\leq\ & \beta\left(\frac{\beta}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right|\right. \\
& \left.+\left|\int\left(G^{N,\sigma_1^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), \bar{f} + \sigma_2^f\upsilon\right) - G^{N,\sigma_1^f}\left(\mu + s(N,\gamma)\varepsilon, \Gamma(N,\gamma), \bar{f} + \sigma_1^f\upsilon\right)\right)d\Phi(\varepsilon)\,d\Phi(\upsilon)\right|\right) \\
\leq\ & \beta\left(\frac{\beta}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right| + \left|\sigma_2^f - \sigma_1^f\right|\right) \\
\leq\ & \frac{\beta}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right|
\end{aligned} \tag{15}$$

Let us now turn to the number of investing firms N. Denote $N_n^{\sigma_i^f} \equiv \left(\mathcal{M}^{\sigma_i^f}\right)^n N$.

$$\left|\left(N_1^{\sigma_2^f} - N_1^{\sigma_1^f}\right)(\mu,\gamma)\right| \leq \bar{N}\left|F^{\sigma_2^f}\left(f_c^{N,\sigma_2^f}(\mu,\gamma)\right) - F^{\sigma_1^f}\left(f_c^{N,\sigma_1^f}(\mu,\gamma)\right)\right|$$
$$\leq \bar{N}\left|F^{\sigma_2^f}\left(f_c^{N,\sigma_2^f}(\mu,\gamma)\right) - F^{\sigma_2^f}\left(f_c^{N,\sigma_1^f}(\mu,\gamma)\right) + F^{\sigma_2^f}\left(f_c^{N,\sigma_1^f}(\mu,\gamma)\right) - F^{\sigma_1^f}\left(f_c^{N,\sigma_1^f}(\mu,\gamma)\right)\right|$$

where we see that $N^{\sigma_2^f}$ may not always be close to $N^{\sigma_1^f}$ under the sup norm. The problem is that the above expression could be close to 1 for a few of points if σ_i^f is low and $f_c^{N,\sigma_2^f} \neq f_c^{N,\sigma_1^f}$. However, we now show that this is not a problem as they will be close on *average*. The only thing we need for the final result is pointwise convergence for N.

- We will now establish a recursive relationship to compare the two objects $f_c^{N_n^{\sigma_1^f},\sigma_1^f}$ and $f_c^{N_n^{\sigma_2^f},\sigma_2^f}$.

Assume that after n iterations of the mapping $\mathcal{M}$, we have two different functions $N_n^{\sigma_2^f}$ and $N_n^{\sigma_1^f}$ and that

$$\forall(\mu,\gamma), \quad \left|f_c^{N_n^{\sigma_2^f},\sigma_2^f}(\mu,\gamma) - f_c^{N_n^{\sigma_1^f},\sigma_1^f}(\mu,\gamma)\right| \leq A_n\left|\sigma_2^f - \sigma_1^f\right|.$$

Let us study the following term:

$$\begin{aligned}\left|\left(G^{N_{n+1}^{\sigma_2^f},\sigma_2^f} - G^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| &\leq \left|\left(G^{N_{n+1}^{\sigma_2^f},\sigma_2^f} - G^{N_{n+1}^{\sigma_2^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| + \left|\left(G^{N_{n+1}^{\sigma_2^f},\sigma_1^f} - G^{N_1^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| \\ &\leq \frac{\beta}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right| + \left|\left(G^{N_{n+1}^{\sigma_2^f},\sigma_1^f} - G^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| \qquad (16)\end{aligned}$$

where we have controlled the first term by the same argument as in (14). We need to study the second term:

$$\begin{aligned}\left|\left(G^{N_{n+1}^{\sigma_2^f},\sigma_1^f} - G^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| &= \left|\left(\lim_{m\to\infty}\left(\mathcal{T}^{N_{n+1}^{\sigma_2^f},\sigma_1^f}\right)^m G - \lim_{m\to\infty}\left(\mathcal{T}^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)^m G\right)(\mu,\gamma,f)\right| \\ &= \left|\left(\lim_{m\to\infty} G_m^{N_{n+1}^{\sigma_2^f},\sigma_1^f} - \lim_{m\to\infty} G_m^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right|.\end{aligned}$$

Starting with the first iteration:

$$
\begin{aligned}
& \left|\left(G_1^{N_{n+1}^{\sigma_2^f},\sigma_1^f} - G_1^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| \\
\leq\ & \beta\left|\int\left[G\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)-G\left(\mu+s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_1^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right| \\
\leq\ & \beta\left[\int\left[G\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)-G\left(\mu+s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right. \\
& \left.+\int\left[G\left(\mu+s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)-G\left(\mu+s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_1^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right| \\
\leq\ & \beta\left[\int\bar{G}_\mu|\varepsilon|\left|s\left(N_{n+1}^{\sigma_2^f},\gamma\right)-s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\right|d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')+\int\bar{G}_\gamma\left|\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right)-\Gamma\left(N_{n+1}^{\sigma_1^f},\gamma\right)\right|d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right] \\
\leq\ & \beta\left[\bar{G}_\mu\left|s\left(N_{n+1}^{\sigma_2^f},\gamma\right)-s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\right|+\bar{G}_\gamma\left|\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right)-\Gamma\left(N_{n+1}^{\sigma_1^f},\gamma\right)\right|\right] \\
\leq\ & \beta\left(\bar{G}_\mu\overline{\frac{\partial s}{\partial N}}+\bar{G}_\gamma\overline{\frac{\partial\Gamma}{\partial N}}\right)\left|\left(N_{n+1}^{\sigma_2^f}-N_{n+1}^{\sigma_1^f}\right)(\mu,\gamma)\right| \\
\leq\ & \beta B\gamma_x\left|\left(N_{n+1}^{\sigma_2^f}-N_{n+1}^{\sigma_1^f}\right)(\mu,\gamma)\right|
\end{aligned}
$$

where $B=\frac{1}{\underline{\gamma}+\gamma_y}\left(\left|\bar{G}_\mu\right|+\left|\bar{G}_\gamma\right|\gamma_\theta\right)$ is a constant similar to the one we used in proposition 2. We now establish recursively that for $m\geq 2$:

$$
\left|\left(G_m^{N_{n+1}^{\sigma_2^f},\sigma_1^f}-G_m^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right|\leq\beta B\gamma_x\left|\left(N_{n+1}^{\sigma_2^f}-N_{n+1}^{\sigma_1^f}\right)(\mu,\gamma)\right|+\beta^2\frac{1-\beta^{m-1}}{1-\beta}B\gamma_x\left(A_nC+D\sigma_1^f\sigma_2^f\right)\left|\sigma_2^f-\sigma_1^f\right|
$$

where constants C and D are those coming from lemma 4 below. Assuming the relationship is true until $m\geq 2$, we have:

$$
\begin{aligned}
& \left|\left(G_{m+1}^{N_{n+1}^{\sigma_2^f},\sigma_1^f}-G_{m+1}^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right|\leq\beta\left|E\left[G_m^{N_{n+1}^{\sigma_2^f},\sigma_1^f}-G_m^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right](\mu,\gamma,f)\right| \\
& \leq\beta\left|\int\left[G_m^{N_{n+1}^{\sigma_2^f},\sigma_1^f}\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)-G_m^{N_1^{\sigma_1^f},\sigma_1^f}\left(\mu+s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_1^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right| \\
& \leq\beta\left|\int\left[G_m^{N_{n+1}^{\sigma_2^f},\sigma_1^f}\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)-G_m^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right| \\
& +\beta\left|\int\left[G_m^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)-G_m^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\left(\mu+s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_1^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right| \\
& \leq\beta\int\left|\left(G_m^{N_{n+1}^{\sigma_2^f},\sigma_1^f}-G_m^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right| \\
& +\beta\int\left|\frac{\partial s}{\partial N}\varepsilon\frac{\partial G_1^{N_1^{\sigma_1^f},\sigma_1^f}}{\partial\mu}+\frac{\partial\Gamma}{\partial N}\frac{\partial G_1^{N_1^{\sigma_1^f},\sigma_1^f}}{\partial\gamma}\right|\left|\left(N_1^{\sigma_2^f}-N_1^{\sigma_1^f}\right)(\mu,\gamma)\right|d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f') \\
& \leq\beta\left(\beta B\gamma_x\int\left|\left(N_{n+1}^{\sigma_2^f}-N_{n+1}^{\sigma_1^f}\right)\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right)\right)\right|d\Phi(\varepsilon)+\beta^2\frac{1-\beta^{m-1}}{1-\beta}B\gamma_x\left(A_nC+D\sigma_1^f\sigma_2^f\right)\left|\sigma_2^f-\sigma_1^f\right|\right) \\
& +\beta B\gamma_x\left|\left(N_{n+1}^{\sigma_2^f}-N_{n+1}^{\sigma_1^f}\right)(\mu,\gamma)\right|
\end{aligned}
$$

Using lemma 4, we can control the term:

$$\int \left| \left(N_{n+1}^{\sigma_2^f} - N_{n+1}^{\sigma_1^f} \right) \left(\mu + s\left(N_{n+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \gamma^{'} \left(N_{n+1}^{\sigma_2^f}, \gamma \right) \right) \right| d\Phi\left(\varepsilon\right) \leq \left(A_n C + D\sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|.$$

Therefore:

$$\left| \left(G_{m+1}^{N_{n+1}^{\sigma_2^f}, \sigma_1^f} - G_{m+1}^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \right) (\mu, \gamma, f) \right| \leq \beta B \left| \left(N_{m+1}^{\sigma_2^f} - N_{m+1}^{\sigma_1^f} \right) (\mu, \gamma) \right| + \beta^2 \frac{1-\beta^m}{1-\beta} B \left(A_n C + D\sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|$$

which establishes the recursion. Taking the limit as $m \to \infty$:

$$\begin{aligned} & \left| \left(G^{N_{n+1}^{\sigma_2^f}, \sigma_1^f} - G^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \right) (\mu, \gamma, f) \right| \\ \leq \; & \beta B \left| \left(N_{n+1}^{\sigma_2^f} - N_{n+1}^{\sigma_1^f} \right) (\mu, \gamma) \right| + \frac{\beta^2}{1-\beta} B \left(A_n C + D\sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|. \end{aligned} \tag{17}$$

We see that G may not converge pointwise. However, the expectation of G will, which is what we need for our final result. Going back to equation (16):

$$\begin{aligned} & \left| \left(G^{N_{n+1}^{\sigma_2^f}, \sigma_2^f} - G^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \right) (\mu, \gamma, f) \right| \\ \leq \; & \left| \left(G^{N_{n+1}^{\sigma_2^f}, \sigma_2^f} - G^{N_{n+1}^{\sigma_2^f}, \sigma_1^f} \right) (\mu, \gamma, f) \right| + \left| \left(G^{N_{n+1}^{\sigma_2^f}, \sigma_1^f} - G^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \right) (\mu, \gamma, f) \right| \\ \leq \; & \frac{\beta}{1-\beta} \left| \sigma_2^f - \sigma_1^f \right| + \beta B \gamma_x \left| \left(N_1^{\sigma_2^f} - N_1^{\sigma_1^f} \right) (\mu, \gamma) \right| + \frac{\beta^2}{1-\beta} B \gamma_x \left(A_n C + D\sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|. \end{aligned}$$

where I have used equations (14) and (17). Let us turn to the cutoff value:

$$\begin{aligned} & \left| f_c^{N_{n+1}^{\sigma_2^f}, \sigma_2^f} (\mu, \gamma) - f_c^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} (\mu, \gamma) \right| \\ = \; & \beta \left| E_{\sigma_2^f} \left[G^{N_{n+1}^{\sigma_2^f}, \sigma_2^f} \left(\mu + s\left(N_{n+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \Gamma\left(N_{n+1}^{\sigma_2^f}, \gamma \right), f' \right) \right] - E_{\sigma_1^f} \left[G^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \left(\mu + s\left(N_{n+1}^{\sigma_1^f}, \gamma \right) \varepsilon, \Gamma\left(N_{n+1}^{\sigma_1^f}, \gamma \right), f' \right) \right] \right| \\ \leq \; & \beta \left| E_{\sigma_2^f} \left[G^{N_{n+1}^{\sigma_2^f}, \sigma_2^f} \left(\mu + s\left(N_{n+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \Gamma\left(N_{n+1}^{\sigma_2^f}, \gamma \right), f' \right) \right] - E_{\sigma_2^f} \left[G^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \left(\mu + s\left(N_{n+1}^{\sigma_1^f}, \gamma \right) \varepsilon, \Gamma\left(N_{n+1}^{\sigma_1^f}, \gamma \right), f' \right) \right] \right| \\ & + \beta \left| E_{\sigma_2^f} \left[G^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \left(\mu + s\left(N_{n+1}^{\sigma_1^f}, \gamma \right) \varepsilon, \Gamma\left(N_{n+1}^{\sigma_1^f}, \gamma \right), f' \right) \right] - E_{\sigma_1^f} \left[G^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \left(\mu + s\left(N_{n+1}^{\sigma_1^f}, \gamma \right) \varepsilon, \Gamma\left(N_{n+1}^{\sigma_1^f}, \gamma \right), f' \right) \right] \right| \\ \leq \; & \beta \left(\frac{\beta}{1-\beta} + \beta B \gamma_x \left(A_n C + D\sigma_1^f \sigma_2^f \right) + \frac{\beta^2}{1-\beta} B \gamma_x \left(A_n C + D\sigma_1^f \sigma_2^f \right) \right) \left| \sigma_2^f - \sigma_1^f \right| + \beta \left| \sigma_2^f - \sigma_1^f \right| \\ \leq \; & \beta \left(\frac{1}{1-\beta} + \frac{\beta}{1-\beta} B \gamma_x \left(A_n C + D\sigma_1^f \sigma_2^f \right) \right) \left| \sigma_2^f - \sigma_1^f \right| \\ \leq \; & \underbrace{\left[\frac{\beta}{1-\beta} \left(1 + \beta \gamma_x B D \sigma_1^f \sigma_2^f \right) + \frac{\beta^2}{1-\beta} \gamma_x B C A_n \right]}_{\equiv A_{n+1}} \left| \sigma_2^f - \sigma_1^f \right| \end{aligned}$$

This expression defines a recursive relationship:

$$A_{n+1} = \frac{\beta}{1-\beta} \left(1 + \beta \gamma_x B D \sigma_1^f \sigma_2^f \right) + \frac{\beta^2}{1-\beta} \gamma_x B C A_n$$

which converges to a unique limit $\bar{A}$ as long as $\frac{\beta^2}{1-\beta}\gamma_x BC < 1$ which is true if γ_x is chosen sufficiently small. Taking the limit as $n \to \infty$, we have:

$$\left| f_c^{N^{\sigma_2^f},\sigma_2^f}(\mu,\gamma) - f_c^{N^{\sigma_1^f},\sigma_1^f}(\mu,\gamma) \right| = \left| f_c^{\lim_{n\to\infty} N_n^{\sigma_2^f},\sigma_2^f}(\mu,\gamma) - f_c^{\lim_{n\to\infty} N_n^{\sigma_1^f},\sigma_1^f}(\mu,\gamma) \right| \leq \bar{A}\left|\sigma_2^f - \sigma_1^f\right|.$$

This tells us that as $\sigma^f \to 0$, the equilibrium cutoff converges uniformly to some limit:

$$\forall(\mu,\gamma), \quad f_c^{N^{\sigma^f},\sigma^f}(\mu,\gamma) \to f_c^0(\mu,\gamma).$$

Turning to the equilibrium entry schedule, N converges pointwise towards the limit:

$$\forall(\mu,\gamma), \quad N^{\sigma_n^f}(\mu,\gamma) = \bar{N}\cdot F^{\sigma_n^f}\left(f_c^{N^{\sigma_n^f},\sigma_n^f}(\mu,\gamma)\right) \to N^0(\mu,\gamma) = \bar{N}\cdot \mathbb{1}\left(\bar{f} \leq f_c^0(\mu,\gamma)\right).$$

□

Lemma 4. *Suppose two functions f_1 and f_2 are such that sup* $|f_2(\mu,\gamma) - f_1(\mu,\gamma)| \leq A\left|\sigma_2^f - \sigma_1^f\right|$ *for some strictly positive constant A. Assume also that both f_i's are continuous, differentiable and that $\frac{\partial f_i}{\partial \mu} > \underline{f}_\mu$. Then, for $N^i = \bar{N}F^{\sigma_i^f}(f_i)$, there exists two strictly positive constants C and D such that for $i = 1,2$:*

$$\int \left|\left(N^2 - N^1\right)\left(\mu + s\left(N^i,\gamma\right)\varepsilon, \Gamma\left(N^i,\gamma\right)\right)\right| d\Phi(\varepsilon) \leq \left(AC + D\sigma_1^f\sigma_2^f\right)\left|\sigma_2^f - \sigma_1^f\right|.$$

Proof. Abusing notation slightly with the convention $s \equiv s\left(N_1^{\sigma_i^f},\gamma\right)$ and $\gamma' \equiv \Gamma\left(N_1^{\sigma_i^f},\gamma\right)$:

$$\begin{aligned}
& \int \left|\left(N^2 - N^1\right)\left(\mu + s\left(N^i,\gamma\right)\varepsilon, \Gamma\left(N^i,\gamma\right)\right)\right| d\Phi(\varepsilon) \\
= \; & \bar{N}\int \left|F^{\sigma_2^f}\left(f_2\left(\mu+s\varepsilon,\gamma'\right)\right) - F^{\sigma_1^f}\left(f_1\left(\mu+s\varepsilon,\gamma'\right)\right)\right| d\Phi(\varepsilon) \\
\leq \; & \underbrace{\bar{N}\int \left|F^{\sigma_2^f}\left(f_2\left(\mu+s\varepsilon,\gamma'\right)\right) - F^{\sigma_2^f}\left(f_1\left(\mu+s\varepsilon,\gamma'\right)\right)\right| d\Phi(\varepsilon)}_{\equiv A_1} \\
& + \underbrace{\bar{N}\int \left|F^{\sigma_2^f}\left(f_1\left(\mu+s\varepsilon,\gamma'\right)\right) - F^{\sigma_1^f}\left(f_1\left(\mu+s\varepsilon,\gamma'\right)\right)\right| d\Phi(\varepsilon)}_{\equiv A_2}
\end{aligned}$$

Let us take care of the first term:

$$\begin{aligned}
A_1 &= \bar{N}\int \left|F^{\sigma_2^f}\left(f_2\left(\mu+s\varepsilon,\gamma'\right)\right) - F^{\sigma_2^f}\left(f_1\left(\mu+s\varepsilon,\gamma'\right)\right)\right| d\Phi(\varepsilon) \\
&\leq \bar{N}\int \left[F^{\sigma_2^f}\left(f_2\left(\mu+s\varepsilon,\gamma'\right) + A\left|\sigma_2^f - \sigma_1^f\right|\right) - F^{\sigma_2^f}\left(f_2\left(\mu+s\varepsilon,\gamma'\right) - A\left|\sigma_2^f - \sigma_1^f\right|\right)\right] d\Phi(\varepsilon)
\end{aligned}$$

using equation (15). f_2 is a nicely continuous, differentiable, strictly increasing function of μ, so we can proceed to the change of variable $x = f_2\left(\mu + s\varepsilon, \gamma^{'}\right)$:

$$\begin{aligned} A_1 &\leq \bar{N} \int \left[F^{\sigma_2^f}\left(x + A\left|\sigma_2^f - \sigma_1^f\right|\right) - F^{\sigma_2^f}\left(x - A\left|\sigma_2^f - \sigma_1^f\right|\right)\right] \underbrace{\frac{\Phi^{'}\left(\left(f_2\right)^{-1}(x)\right)dx}{s\cdot\left(f_2\right)^{'}\left(\left(f_2\right)^{-1}(x)\right)}}_{\equiv d\varphi(x)} \\ &\leq \bar{N}\int_{x=-\infty}^{\infty}\int_{f=x-A\left|\sigma_2^f-\sigma_1^f\right|}^{x+A\left|\sigma_2^f-\sigma_1^f\right|} dF^{\sigma_2^f} d\varphi(x) \leq \int_{f=-\infty}^{\infty}\int_{x=f-A\left|\sigma_2^f-\sigma_1^f\right|}^{f+A\left|\sigma_2^f-\sigma_1^f\right|} dF^{\sigma_2^f} d\varphi(x) \\ &\leq \bar{N}\int_{f=-\infty}^{\infty}\left[\varphi\left(f + A\left|\sigma_2^f - \sigma_1^f\right|\right) - \varphi\left(f - A\left|\sigma_2^f - \sigma_1^f\right|\right)\right] dF^{\sigma_2^f} \\ &\leq \bar{N}\int_{f=-\infty}^{\infty}\left[\varphi'\left(\tilde{f}\right)2A\left|\sigma_2^f - \sigma_1^f\right|\right] dF^{\sigma_2^f} \quad \text{(mean value theorem)} \\ &\leq 2A\bar{N}\left|\sigma_2^f - \sigma_1^f\right|\int_{f=-\infty}^{\infty}\left|\varphi'\left(\tilde{f}\right)\right| dF^{\sigma_2^f} \\ &\leq A\cdot 2\bar{N}\frac{\sup\left|\Phi'\right|}{\inf\left|s\cdot\left(f_2\right)'\right|}\left|\sigma_2^f - \sigma_1^f\right| \equiv AC\left|\sigma_2^f - \sigma_1^f\right| \end{aligned}$$

where I have used the fact the cdf of a unit normal Φ is bounded, $s \equiv s\left(N_1^{\sigma_i^f}, \gamma\right)$ is uniformly bounded from below and away from 0, and the derivative of f_2 is strictly positive, uniformly bounded away from 0 for γ_x small enough. Notice that the upper bound we derived is uniform: it does not depend on μ, γ, γ_x, etc. Let us control the second term A_2:

$$\begin{aligned} A_2 &= \bar{N}\int\left|F^{\sigma_2^f}\left(f_1\left(\mu + s\varepsilon, \gamma^{'}\right)\right) - F^{\sigma_1^f}\left(f_1\left(\mu + s\varepsilon, \gamma^{'}\right)\right)\right| d\Phi(\varepsilon) \\ &\leq \bar{N}\int\left|F^{\sigma_2^f}(x) - F^{\sigma_1^f}(x)\right| d\varphi(x) \quad \text{(change of variable } x = f_1\left(\mu + s\varepsilon, \gamma^{'}\right)\text{)} \\ &\leq \bar{N}\int\left|\Phi\left(\frac{x-\bar{f}}{\sigma_2^f}\right) - \Phi\left(\frac{x-\bar{f}}{\sigma_1^f}\right)\right| d\varphi(x) \quad \text{(change of variable } x = \sigma_1^f\sigma_2^f\tilde{x} + \bar{f}\text{)} \\ &\leq \bar{N}\int\left|\Phi\left(\sigma_1^f\tilde{x}\right) - \Phi\left(\sigma_2^f\tilde{x}\right)\right|\sigma_1^f\sigma_2^f d\varphi\left(\sigma_1^f\sigma_2^f\tilde{x} + \bar{f}\right) \\ &\leq \left[\bar{N}\int\left|\Phi^{'}(\hat{x})\tilde{x}\right| d\varphi\left(\sigma_1^f\sigma_2^f\tilde{x} + \bar{f}\right)\right]\sigma_1^f\sigma_2^f\left|\sigma_2^f - \sigma_1^f\right| \equiv D\sigma_1^f\sigma_2^f\left|\sigma_2^f - \sigma_1^f\right|. \end{aligned}$$

□

Proposition 4. *The following results hold:*

1. *The decentralized competitive equilibrium is inefficient. The symmetric, socially efficient allocation can be implemented with positive investment subsidies $\tau(\mu, \gamma)$;*

2. *When γ_x and σ^f are small, the efficient allocation is still subject to uncertainty traps.*

Proof. 1. In the limit case where the number of firms is large enough that the approximation $N/\bar{N} = F(f_c)$ is valid, we can write the planner's decision as a choice over the optimal cutoff f_c^{eff}

under which firms should invest:

$$
\begin{aligned}
W(\mu,\gamma) \quad &= \quad \max_{f_c^{eff}} \bar{N}\int_{-\infty}^{f_c^{eff}} \left(E\left[u(x)\right]-\tilde{f}\right) dF\left(\tilde{f}\right) + \beta E\left[W\left(\mu',\gamma'\right)\right] \\
\text{s.t} \quad \mu' &= \frac{\gamma\mu+\gamma_y Y + N\gamma_x X}{\gamma+\gamma_y+N\gamma_x} \\
\gamma' &= \left(\frac{1}{\gamma+\gamma_y+N\gamma_x}+\frac{1}{\gamma_\theta}\right)^{-1} \\
N &= \bar{N}F\left(f_c^{eff}\right)
\end{aligned}
$$

The first order condition with respect to the cutoff is

$$
\bar{N}F'\left(f_c^{eff}\right)\left(E\left[u(x)\right]-f_c^{eff}+\beta\frac{d}{dN}E\left[W\left(\mu+s(N,\gamma),\Gamma(N,\gamma)\right)\right]\right)=0,
$$

so that we can derive an expression for the efficient cutoff:

$$
f_c^{eff}(\mu,\gamma)=E\left[u(x)\right]+\beta\frac{d}{dN}E\left[W\left(\mu+s(N,\gamma),\Gamma(N,\gamma)\right)\right].
$$

We show that this optimal cutoff is implementable using investment subsidies $\tau(\mu,\gamma)$. Let us write the problem of firms facing these subsidies:

$$
V^\tau(\mu,\gamma,f)=\max\left\{E\left[u(x)\right]-f+\tau(\mu,\gamma),\beta E\left[V^\tau\left(\mu',\gamma',f'\right)\right]\right\}
$$

which yields the individual cutoff rule f_c:

$$
f_c^\tau(\mu,\gamma)=E\left[u(x)\right]+\tau(\mu,\gamma)-\beta E\left[V^\tau\left(\mu',\gamma',f'\right)\right].
$$

To implement the efficient allocation, we must identify the two cutoffs

$$
\begin{aligned}
& f_c^\tau(\mu,\gamma)=f_c^{eff}(\mu,\gamma) \\
\Leftrightarrow \quad & \tau(\mu,\gamma)=\underbrace{\beta\frac{d}{dN}E\left[W\left(\mu+s(N,\gamma),\Gamma(N,\gamma)\right)\right]}_{\text{information externality}}+\underbrace{\beta E\left[V^\tau\left(\mu',\gamma',f'\right)\right]}_{\text{option value of waiting}}. \qquad (18)
\end{aligned}
$$

Expression (18) is a functional equation in τ because of the dependence of V^τ in τ. We now prove that this functional equation defines a contraction and therefore that it has a solution, which is unique. Indeed, this mapping satisfies the Blackwell conditions:

1. Monotonicity: Pick $\tau_1 \leq \tau_2$, then it is easy to show that the contraction mapping that defines

V^{τ_i} preserves the ordering, $V^{\tau_1} \leq V^{\tau_2}$. Thus,

$$\begin{aligned} & \beta \frac{d}{dN} E\left[W\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right)\right] + \beta E V^{\tau_1}\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right) \\ \leq \; & \beta \frac{d}{dN} E\left[W\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right)\right] + \beta E V^{\tau_2}\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right). \end{aligned}$$

2. Discounting: it is easy to show that $V^{\tau+a} \leq V^{\tau} + a$, then

$$\begin{aligned} & \beta \frac{d}{dN} E\left[W\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right)\right] + \beta E V^{\tau+a}\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right) \\ \leq \; & \beta \frac{d}{dN} E\left[W\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right)\right] + \beta E V^{\tau}\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right) + \beta a. \end{aligned}$$

Therefore, we can conclude that equation (18) defines a unique set of transfers. By construction, these transfers implement the efficient allocation. To complete the proof, we are now going to show that these transfers are positive and non-zero in non-trivial cases. To be more precise, rewrite the mapping satisfied by these transfers:

$$\tau\left(\mu,\gamma\right) = \underbrace{\beta \frac{d}{dN} E\left[W\left(\mu',\gamma'\right)\right]}_{\equiv A(\mu,\gamma)} + \beta \underbrace{E\left[V^{\tau}\left(\mu',\gamma',f'\right)\right]}_{\equiv B(\mu,\gamma)}.$$

Term B is non-negative. In fact, as long as the efficient allocation is not trivial, i.e. that there exists some (μ,γ,f) at which firms invest (which is guaranteed since f has an unbounded support), term $B\left(\mu,\gamma\right)$ is strictly positive for some (μ,γ).

We now prove that A is non-negative. We want to understand the purely informational impact on welfare of an exogenous arrival of information. It is useful for our purpose to rewrite the planner's problem in a sequential way. A strategy for the planner is a collection of cutoff functions $\{f_0, f_1, \ldots, f_t, \ldots\}$ such that for each date t, f_t maps the set of all past histories of signals up to time t, $\{Y_s, X_s\}_{s=0}^{t}$, to the real line. Pick some date t_0. We are going to show that the exogenous arrival of a signal S of precision γ_S at date t_0 allows the planner to do at least as well as without it, because of the simple reason that it can simply ignore it. Denote $\mathcal{F}_t$ the information set $\{Y_s, X_s\}_{s=0}^{t}$ of the planner at each date without the exogenous signal, and $\mathcal{F}_t^S$ the information set $\left\{Y_s, X_s^S\right\}_{s=0}^{t}$ of the planner when the arrival of the exogenous signal is known and anticipated. Let $\{f_{c,t}\}$ any strategy considered by the planner without the exogenous signal. Construct the following strategy for the case with exogenous arrival of information:

$$\begin{aligned} \forall t < t_0, \qquad & f_{c,t}^S\left(\{Y_s, X_s\}_{s=0}^{t}\right) = f_{c,t}\left(\{Y_s, X_s\}_{s=0}^{t}\right) \\ \forall t \geq t_0, \qquad & f_{c,t}^S\left(\{Y_s, X_s\}_{s=0}^{t}, S\right) = f_{c,t}\left(\{Y_s, X_s\}_{s=0}^{t}\right) \end{aligned}$$

so that the two strategies and the information sets $\mathcal{F}_t$ and $\mathcal{F}_t^S$ coincide up to time $t_0 - 1$. After date t_0, strategy f_c^S deliberately ignores the new information. Therefore, by the law of iterated expectations, the two strategies have the same ex-ante payoffs. Welfare can only be increased with the arrival of new information, hence term $A(\mu, \gamma)$ is non-negative.

We can now safely conclude that the symmetric, efficient allocation can be implemented with positive transfers. In non-trivial cases, these transfers are strictly positive, which implies that the decentralized economy without transfers is in general inefficient.

2. The proof that the efficient allocation is subject to uncertainty traps follows closely that of the decentralized case. Thus, we only state the major steps of the proof:

- The optimal cutoff for the planner is defined by:

$$f_c^{eff}(\mu, \gamma) = E[u(x)] + \beta \frac{d}{dN} E[W(\mu + s(N, \gamma), \Gamma(N, \gamma))].$$

  The first step of the proof is to show that $\frac{d}{dN} E[W(\mu + s(N, \gamma), \Gamma(N, \gamma))]$ is a $O(\gamma_x)$, so that for γ_x low enough f_c^{eff} is strictly increasing in μ and γ with derivatives that can be bounded away from 0;

- In a second step, show that when $\sigma^f \to 0$, then f_c^{eff,σ^f} converges uniformly to some limit $f_c^{eff,0}$ that is bi-Lipschitz continuous, strictly increasing in μ and γ with derivatives bounded away from 0. Thus, we have the pointwise limit:

$$\forall (\mu, \gamma), \qquad N^{eff,\sigma^f}(\mu, \gamma) \to N^{eff,0}(\mu, \gamma) = \bar{N} \cdot \mathbb{1}\left(\bar{f} \le f_c^{eff,0}(\mu, \gamma)\right);$$

- Conclude identically to proposition 3 that for σ^f sufficiently small there are at least two locally stable steady-states in the dynamics of γ.

$\square$