Trading and Information Diffusion in Over-the-Counter Markets*

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Abstract

We model trading and information diffusion in OTC markets, when dealers with private information can engage in many bilateral transactions at the same time, they trade strategically, and dealers' strategies are represented as quantity-price schedules. We show that information diffusion is effective, but not informationally efficient. While each bilateral price partially aggregates the private information of all the dealers in one round of trading, prices can be more informative even within the constraints imposed by our environment. This is not a result of dealers' market power, but arises from the interaction between decentralization and differences in dealers' valuation of the asset. Furthermore, dealers with more trading partners are ex post better informed, tend to trade and intermediate more, earn more profit per transaction, set smaller effective spreads, and trade at less dispersed prices. We also revisit alternative explanations behind the disruption of OTC markets in the recent financial crisis.

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1 Introduction

A vast proportion of assets is traded in over-the-counter (OTC) markets. The disruption of several of these markets during the financial crisis of 2008, has highlighted the crucial role that OTC markets play in the financial system. The defining characteristic of OTC markets is that trade is decentralized. Dealers trade bilaterally with a subset of other dealers, resulting in different prices for each transaction.

In this paper we explore a novel approach to model OTC markets. In our model dealers that have private information can engage in many bilateral transactions at the same time, trade strategically, and their strategies are represented as quantity-price schedules. Our paper has a dual focus.

On the theoretical side, our main focus is to study how much information is revealed through trading in OTC markets. We show that information diffuses effectively through the network of trades. In one round of trading, each bilateral price partially aggregates the private information of all the dealers in the market, even when they are not a counterparty in the respective transaction. Yet, typically, information diffusion is not informationally efficient. If each dealer puts less weight on her private information, prices can be more informative. We also show that this systematic distortion of the price informativeness is not an outcome of dealers' market power. Instead, it arises from the interaction between trade decentralization and differences in dealers' valuation of the asset.

On the applied side, we emphasize that our model generates a joint distribution of prices and quantities for every bilateral transaction. This implies an unusually rich set of predictions at the transaction level. Our main observation is that dealers with more trading partners are ex post better informed, tend to trade and intermediate more, earn more profit per transaction, set smaller effective spreads, and trade at less dispersed, but more volatile prices. This also implies that larger transactions are associated with smaller spreads, less price dispersion and higher profits. We also revisit alternative explanations behind the disruption of OTC markets in the recent financial crisis. According to our model, narratives emphasizing increased counterparty risk are more consistent to the observed stylized facts than those emphasizing increased informational frictions .

In our main specification, there are n risk-neutral dealers organized in a dealer network.

Intuitively, a link between i and j represents that they are potential counterparties in a trade. There is a single risky asset in zero net supply. The final value of the asset is uncertain and interdependent across dealers with an arbitrary correlation coefficient controlling the relative importance of the common and private components. Each dealer observes a private signal about her value, and all dealers have the same quality of information. Since values are interdependent, inferring each others' signals is valuable. Values and signals are drawn from a known multivariate normal distribution. Each dealer simultaneously chooses her trading strategy, understanding her price effect given other dealers' strategies. For any private signal, each dealer's trading strategy is a generalized demand function which specifies the quantity of the asset she is willing to trade with each of her counterparties depending on the vector of prices in the transactions she engages in. Each dealer, in addition to trading with other dealers, also trades with price sensitive costumers. In equilibrium prices and quantities have to be consistent with the set of generalized demand functions and the market clearing conditions for each link. We refer to this structure as the *OTC game*. The OTC game is, essentially, a generalization of the Vives (2011) variant of Kyle (1989) to networks. Most of our results apply to *any* network.

While finding an equilibrium in the space of generalized demand functions is a complex problem, we simplify this by first solving for the map between signals and posterior beliefs. For this, we specify a simpler, auxiliary game in which dealers, connected in the same network and acting in the same informational environment as in the OTC game, do not trade. Instead, their aim is to make a best guess of their own value conditional on their signals and the guesses of the other dealers they are connected to. We label this structure the conditional-guessing game. We then establish an equivalence between the posterior beliefs in the OTC game and the equilibrium beliefs in the conditional-guessing game. As each dealer's equilibrium guess depends on her neighbors' guesses, and through those, depends on her neighbors' neighbors' guesses etc., each equilibrium guess will partially incorporate the private information of all the dealers in a connected network. However, dealers do not internalize how the informativeness of their guess affects others' decision, and the equilibrium will typically not be informationally efficient. That is, dealers tend to put too much weight on their own signal, making their guess inefficiently informative about the common component.

In the OTC game, we show that the equilibrium price in a given transaction is a weighted sum of the posterior beliefs of the counterparties, and hence it inherits the main properties of beliefs. In addition, each dealer's equilibrium position is proportional to the difference between her expectation and the price. Therefore, a dealer sells at a price higher than her belief to relatively optimist counterparties and buys at a price lower than her belief from pessimists. This gives rise to dispersed prices and profitable intermediation for dealers with many counterparties, as it is characteristic of real-world OTC markets. The proportionality coefficient of a dealer's positions depends on her exposure to adverse selection. When a dealer cares less about the private information of her counterparty (for instance, when she participates in multiple transactions, she can use prices from other transactions as additional sources of information), this coefficient is larger.

An attractive feature of our model is that it yields a rich set of empirical predictions. Indeed, given the equilibrium demand curves, we can calculate the joint distribution of several financial indicators as a function of the underlying network and our parameters. The main implications stem from observing that trading with connected dealers is less costly in terms of the price impact. Therefore, more central dealers trade larger quantities and intermediate more. As they learn more from all the prices they observe, they earn more profit per trade, even if they trade at less dispersed prices. Even when the econometrician does not observe the underlying network structure, our observations imply that large transactions should be associated with smaller cost per unit of traded asset, larger profitability, more price volatility across time and less price dispersion across transactions. We contrast these findings with existing empirical work of and find substantial support.[1]

As a second application, we revisit narratives behind the episodes of OTC market distress around the recent financial crisis. The stylized facts that have been identified by various studies[2] are that price dispersion and price impacts tend to increase, and volume tends to decrease. In our model, we can capture these effects when we remove links from the network, which suggests an explanation based on counterparty risk. In contrast, changes in the informational structure tend to move price dispersion and volume in the same direction inconsistently with the evidence. The intuition is follows. When a pair of dealers is not willing to trade, information

[1] The negative association of unit cost and transaction size is a robust pattern in various markets (see Green, Hollifield and Schurhoff (2007), Edwards, Harris and Piwowar (2007) and Li and Schürhoff (2012)). Also Hollifield, Neklyudov and Spatt (2012), consistently with our prediction, finds that central dealers trade at lower mark-up. However, in a different context, Li and Schürhoff (2012) finds the opposite.

[2] See Afonso, Kovner and Schoar (2011), Agarwal, Chang and Yavas (2012), Friewald, Jankowitsch and Subrahmanyam (2012) and Gorton and Metrick (2012).

flows are disrupted, increasing price dispersion, and, at the same time, trading is limited, decreasing volume. In contrast, changes in the informational structure affects how relevant a dealer finds the information of her counterparties. Thus, if dealers care more (less) about their counterparties private information, this decreases (increases) price dispersion and volume and increases (decreases) price impact.

In our model, trade takes place in one shot. This is an abstraction, that can be interpreted as a reduced form of the complex dynamic bargaining process which leads to price determination in real work OTC markets. Dealers in our game can find the optimal demand functions just by understanding the set-up. However, an important question whether they can also find the equilibrium price vector without invoking an auctioneer, a concept which would be counterintuitive in our decentralized environment. In the final part of the paper, we justify our approach by constructing an explicit, decentralized protocol for the price-discovery process. This exercise also highlights the advantages and limitations of our static approach compared to a full dynamic treatment.

Related literature

Most models of OTC markets are based on search (e.g. Duffie, Garleanu and Pedersen (2005), Duffie, Gârleanu and Pedersen (2007), Lagos, Rocheteau and Weill (2008), Vayanos and Weill (2008), Lagos and Rocheteau (2009), Afonso and Lagos (2012), and Atkeson, Eisfeldt and Weill (2012)). The majority of these models do not analyze learning through trade. Important exceptions are Duffie, Malamud and Manso (2009) and Golosov, Lorenzoni and Tsyvinski (2009). Their main focus is the time-dimension of information diffusion either between differentially informed agents, or from homogeneously informed to uninformed agents. A key assumption in these models is that there exists a continuum of atomistic agents on the market. Therefore, it is a zero probability event that two agents meet repeatedly or any agent meet with counterparties of their counterparties.[3] This implies that agents are willing to reveal their private information and do not have to asses whether the information of their counterparties is determined by their connectedness and mutual counterparties. In contrast, it is critical to our analysis that each dealer understands that her counterparties have over-

[3] An interesting example of a search model where repeated transactions play a role is Zhu (2012) who analyzes the price formation in a bilateral relationship where a seller can ask quotes from a set of buyers repeatedly. In contrast to our model, Zhu (2012) considers a pure private value set-up. Thus, the issue of information aggregation through trade, which is the focus of our analysis, cannot be addressed in his model.

lapping information as they themselves have common counterparties, or their counterparties have common counterparties, etc. Thus, we provide novel insights for OTC markets in which a small number of sophisticated financial institutions are responsible for the bulk of the trading volume. At the same time, we collapse trade to a single period losing implications on the dynamic dimension. Therefore, search models and our approach offer a complementary view of trade and information diffusion in OTC markets.

There is a growing literature studying trading in a network (e.g. Kranton and Minehart (2001), Rahi and Zigrand (2006), Gale and Kariv (2007), Gofman (2011), Condorelli and Galeotti (2012), Choi, Galeotti and Goyal (2013), Malamud and Rostek (2013), Manea (2013), Nava (2013)). These papers typically consider either the sequential trade of a single unit of the asset or a Cournot-type quantity competition.[4] In contrast, we allow agents to form (generalized) demand schedules conditioning the quantities for each of their transactions on the vector equilibrium prices in these transactions. This emphasizes that the terms of the various transactions of a dealer are interconnected in an OTC market. Also, to our knowledge, none of the papers within this class addresses the issue of information aggregation which is the focus of our analysis.[5]

A separate literature studies Bayesian (Acemoglu et al. (2011)) and non-Bayesian (Bala and Goyal (1998), DeMarzo, Vayanos and Zwiebel (2003), Golub and Jackson (2010)) learning in the context of arbitrary connected social networks. In these papers, agents update their beliefs about a payoff-relevant state after observing the actions of their neighbors in the network. Our model complements these works by considering that (Bayesian) learning takes place through trading.

The paper is organized as follows. The following section introduces the model set-up and the equilibrium concept. In Section 3, we describe the conditional-guessing game, and we show the existence of the equilibrium in the OTC game. We characterize the informational content of prices in Section 4. In Section 5 we illustrate the properties of the OTC game with some simple examples and in Section 6 we discuss potential applications. Section 7 provides dynamic foundations for our main specification. Section 8 concludes.

[4] As an exception, Malamud and Rostek (2013) also use a multi-unit double-auction setup to model a decentralized market. However, they do not consider the problem of learning through trade.

[5] While there is another stream of papers (e.g. Ozsoylev and Walden (2011), Colla and Mele (2010), Walden (2013)) which consider that traders have access to the information of their neighbors in a network, in these models trade takes place in a centralized market.

2 A General Model of Trading in OTC Markets

2.1 The model set-up

We consider an economy with n risk-neutral dealers that trade bilaterally a divisible risky asset in zero net supply. All trades take place at the same time. Dealers, apart from trading with each other, also serve a price sensitive customer-base. Each dealer is uncertain about the value of the asset. This uncertainty is captured by θ_i, referred to as dealer i's value. We assume that θ_i is normally distributed with mean 0 and variance σ_θ^2. Moreover, we consider that values are interdependent across dealers. In particular, $\mathcal{V}(\theta_i, \theta_j) = \rho\sigma_\theta^2$ for any two agents i and j, where $\mathcal{V}(\cdot, \cdot)$ represents the variance-covariance operator, and $\rho \in [0, 1]$. Differences in dealers' values reflect, for instance, differences in usage of the asset as collateral, in technologies to repackage and resell cash-flows, in risk-management constraints.[6]

We assume that each dealer receives a private signal, $s_i = \theta_i + \varepsilon_i$, where $\varepsilon_i \sim IID\, N(0, \sigma_\varepsilon^2)$ and $\mathcal{V}(\theta_j, \varepsilon_i) = 0$, for all i and j.

Dealers are organized into a trading network, g where g_i denote the set of $i's$ links and $m_i \equiv |g_i|$ the number of $i's$ links. A link ij implies that i and j are potential trading partners. Intuitively, agent i and j know and sufficiently trust each other to trade in case they find mutually agreeable terms. Each dealer i seeks to maximize her final wealth

$$\sum_{j \in g_i} q_i^j \left(\theta_i - p_{ij}\right),$$

where q_i^j is the quantity traded in a transaction with dealer j at a price p_{ij}. A network is characterized by an adjacency matrix, which is a $n \times n$ matrix

$$A = (a_{ij})_{ij \in \{1,\ldots,n\}}$$

where $a_{ij} = 1$ if i and j have a link and $a_{ij} = 0$ otherwise. While our main results hold for any network, throughout the paper, we illustrate the results using two main types of networks as examples.

[6] As we show in the Appendix, our formalization of the information structure is equivalent with setting $\theta_i = \hat{\theta} + \eta_i$, where $\hat{\theta}$ is the common value element, while η_i is the private value element of $i's$ valuation.

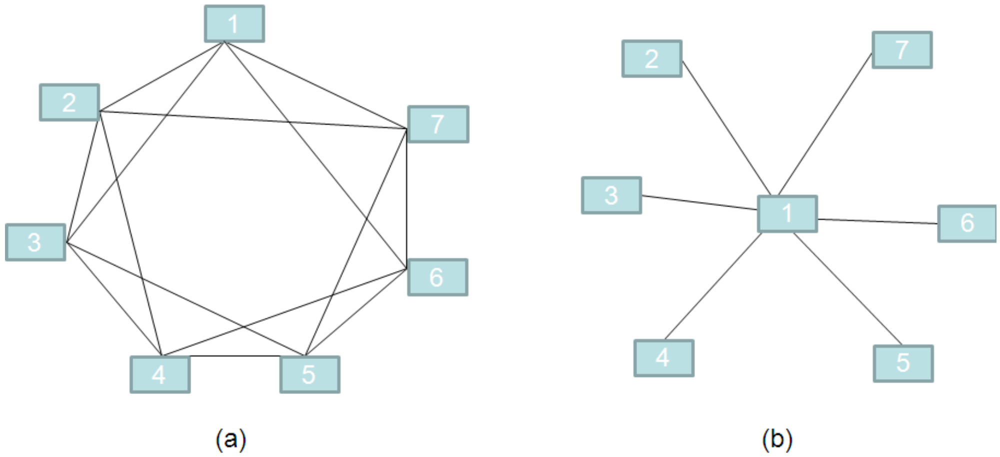

Figure 1: This figures shows two examples of networks. Panel (a) shows a $(7,4)$ circulant network. Panel (b) shows a star network.

Example 1 *In an (n, m)* ***circulant network*** *each dealer is connected with $m/2$ other dealers on her left and $m/2$ on her right. For instance, the $(n, 2)$ circulant network is the circle. A special case of a circulant network is the complete network, where $m = n-1$. (A $(7,4)$ circulant network is shown panel (a) of Figure 1.)*

Example 2 *In an $n-$***star network** *one dealer is connected with $n-1$ other dealers, and no other links exist. (A star network is shown in panel (b) of Figure 1.)*

We define a one shot game where each dealer chooses an optimal trading strategy, provided she takes as given others' strategies but she understands that her trade has a price effect. In particular, the strategy of a dealer i is a map from the signal space to the space of generalized demand functions. For each dealer i with signal s_i, a generalized demand function is a continuous function $Q_i : R^{m_i} \to R^{m_i}$ which maps the vector of prices[7], $\mathbf{p}_{g_i} = (p_{ij})_{j \in g_i}$, that prevail in the transactions that dealer i participates in network g into vector of quantities she wishes to trade with each of her counterparties. The j-th element of this correspondence, $Q_i^j(s_i; \mathbf{p}_{g_i})$, represents her demand function when her counterparty is dealer j, such that

$$\mathbf{Q}_i(s_i; \mathbf{p}_{g_i}) = \left(Q_i^j(s_i; \mathbf{p}_{g_i}) \right)_{j \in g_i}.$$

Note that our specification of generalized demand functions allows for a rich set of strategies.

[7] A vector is always considered to be a column vector, unless explicitly stated otherwise.

First, a dealer can buy a given quantity at a given price from one counterparty and sell a different quantity at a different price to another at the same time. Second, the fact that the quantity that dealer i trades with dealer j, $q_i^j = Q_i^j(s_i; \mathbf{p}_{g_i})$, depends on all the prices $\mathbf{p}_{g_i}$, captures the potential interdependence across all the bilateral transactions of dealer i. For example, if k is linked to i who is linked to j, a high demand from dealer k might raise the bilateral price p_{ki}. This might make dealer i to revise her estimation of her value upwards and adjust her supplied quantity both to k and to j accordingly. Third, the fact that $Q_i^j(s_i; \mathbf{p}_{g_i})$ depends only on $\mathbf{p}_{g_i}$ but not on the full price vector emphasizes the critical feature of OTC markets that the price and the quantity traded in a bilateral transaction are known only by the two counterparties involved in the trade and are not revealed to all market participants. Fourth, the fact that dealers choose demand schedules implies that dealers effectively bargain both over prices and quantities. It is in contrast with most other models of OTC markets where the traded quantity is fixed and agents bargain only over the price.

Apart from trading with each other, dealers also serve a price-sensitive customer base. In particular, we assume that for each transaction between i and j the customer base generates a downward sloping demand

$$D_{ij}(p_{ij}) = \beta_{ij} p_{ij}, \tag{1}$$

with an arbitrary constant $\beta_{ij} < 0$. In our analysis costumers play a pure technical role: the exogenous demand (1) ensures the existence of the equilibrium.[8]

The expected payoff for dealer i corresponding to the strategy profile $\{\mathbf{Q}_i(s_i; \mathbf{p}_{g_i})\}_{i \in \{1,\dots,n\}}$ is

$$E\left[\sum_{j \in g_i} Q_i^j(s_i; \mathbf{p}_{g_i})(\theta_i - p_{ij})\,|s_i\right]$$

where p_{ij} are the elements of the bilateral clearing price vector $\mathbf{p}$ defined by the smallest element of the set

$$\widetilde{\mathbf{P}}\left(\{\mathbf{Q}_i(s_i; \mathbf{p}_{g_i})\}_i, \mathbf{s}\right) \equiv \left\{\mathbf{p} \,\middle|\, Q_i^j(s_i; \mathbf{p}_{g_i}) + Q_i^j(s_j; \mathbf{p}_{g_j}) + \beta_{ij} p_{ij} = 0, \,\forall\, ij \in g\right\}$$

[8] It has been known since Kyle (1989) that with only two trading agents there is no linear equilibrium in a demand submission game. This is not different under our formulation. We follow Vives (2011) and introduce an exogenous demand curve to overcome this problem. This assumption has a minimal effect on our analysis. As we show in Corollary 1, prices and beliefs do not depend on β_{ij} and quantities scale linearly even when $\beta_{ij} \to 0$.

by lexicographical ordering[9], if $\widetilde{\mathbf{P}}$ is non-empty. If $\widetilde{\mathbf{P}}$ is empty, we pick $\mathbf{p}$ to be the infinity vector and say that the market brakes down and define all dealers' payoff to be zero. We refer to the collection of these rules defining a unique $\mathbf{p}$ for any given signal and strategy profile as $\mathbf{p} = \mathbf{P}\left(\{\mathbf{Q}_i\left(s_i; \mathbf{p}_{g_i}\right)\}_i, \mathbf{s}\right)$.

2.2 Equilibrium concept

The environment described above represents a Bayesian game, henceforth the OTC game. The risk-neutrality of dealers and the normal information structure allows us to search for a linear equilibrium of this game defined as follows.

Definition 1 *A Linear Bayesian Nash equilibrium of the OTC game is a vector of linear generalized demand functions* $\{\mathbf{Q}_1(s_1; \mathbf{p}_{g_1}), \mathbf{Q}_2(s_2; \mathbf{p}_{g_2}), ..., \mathbf{Q}_n(s_n; \mathbf{p}_{g_n})\}$ *such that* $\mathbf{Q}_i(s_i; \mathbf{p}_{g_i})$ *solves the problem*

$$\max_{(Q_i^j)_{j \in g_i}} E\left\{\left[\sum_{j \in g_i} Q_i^j(s_i; \mathbf{p}_{g_i})\left(\theta_i - p_{ij}\right)\right] | s_i\right\}, \tag{2}$$

for each dealer i, where $\mathbf{p} = \mathbf{P}(\cdot, \mathbf{s})$.

A dealer i chooses a demand function for each transaction ij, in order to maximize her expected profits, given her information, s_i, and given the demand functions chosen by the other dealers. Then, an equilibrium of the OTC game is a fixed point in demand functions.

3 The Equilibrium

In this section, we derive the equilibrium in the OTC game. We proceed in steps. First, we derive the equilibrium strategies as a function of posterior beliefs. This step is standard. Second, we solve for posterior beliefs. For this, we introduce an auxiliary game in which dealers, connected in the same network and acting in the same informational environment as in the OTC game, do not trade. Instead, they make a best guess of their own value conditional on their signals and the guesses of the other dealers they are connected to. We label this structure the *conditional-guessing game.* Third, we establish equivalence between the posterior beliefs

[9] The specific algorithm we choose to select a unique price vector is immaterial. To ensure that our game is well defined, we need to specify dealers' payoffs as they depend on their strategies both on and off the equilibrium path.

in the OTC game and those in the conditional-guessing game and provide sufficient conditions for existence of the equilibrium in the OTC game for any network.

3.1 Derivation of demand functions

Our derivation follows Kyle (1989) and Vives (2011) with the necessary adjustments. We conjecture an equilibrium in demand functions, where the demand function of dealer i in the transaction with dealer j is given by

$$Q_i^j(s_i; \mathbf{p}_{g_i}) = b_i^j s_i + \left(\mathbf{c_i^j}\right)^T \mathbf{p}_{g_i} \tag{3}$$

for any i and j, where $(\mathbf{c_i^j}) = (c_{ik}^j)_{k \in g_i}$.

As it is standard in similar models, we simplify the optimization problem (2) which is defined over a function space, to finding the functions $Q_i^j(s_i; \mathbf{p}_{g_i})$ point-by-point. That is, for each realization of the vector of signals, $\mathbf{s}$, we solve for the optimal quantity q_i^j that each dealer i demands when trading with a counterparty j. The idea is as follows. Given the conjecture (3) and market clearing

$$Q_i^j(s_i; \mathbf{p}_{g_i}) + Q_j^i(s_j; \mathbf{p}_{g_j}) + \beta_{ij} p_{ij} = 0, \tag{4}$$

the residual inverse demand function of dealer i in a transaction with dealer j is

$$p_{ij} = -\frac{b_j^i s_j + \sum_{k \in g_j, k \neq i} c_{jk}^i p_{jk} + q_i^j}{c_{ji}^i + \beta_{ij}}. \tag{5}$$

Denote

$$I_i^j \equiv -\left(b_j^i s_j + \sum_{k \in g_j, k \neq i} c_{jk}^i p_{jk} \right) / \left(c_{ji}^i + \beta_{ij} \right) \tag{6}$$

and rewrite (5) as

$$p_{ij} = I_i^j - \frac{1}{c_{ji}^i + \beta_{ij}} q_i^j. \tag{7}$$

The uncertainty that dealer i faces about the signals of others is reflected in the random intercept of the residual inverse demand, I_i^j, while her capacity to affect the price is reflected in the slope $-1/\left(c_{ji}^i + \beta_{ij}\right)$. Thus, the price p_{ij} is informationally equivalent to the intercept I_i^j. This implies that finding the vector of quantities $\mathbf{q}_i = \mathbf{Q}_i(s_i; \mathbf{p}_{g_i})$ for one particular realization of the signals, $\mathbf{s}$, is equivalent to solving

$$\max_{(q_i^j)_{j\in g_i}} E\left[\sum_{j\in g_i} q_i^j \left(\theta_i + \frac{1}{c_{ji}^i + \beta_{ij}} q_i^j - I_i^j\right) |s_i, \mathbf{p}_{g_i}\right],$$

or

$$\max_{(q_i^j)_{j\in g_i}} \sum_{j\in g_i} q_i^j \left(E\left(\theta_i | s_i, \mathbf{p}_{g_i}\right) + \frac{1}{c_{ji}^i + \beta_{ij}} q_i^j - I_i^j\right).$$

From the first order conditions we derive the quantities q_i^j for each link of i and for each realization of $\mathbf{s}$ as

$$2\frac{1}{c_{ji}^i + \beta_{ij}} q_i^j = I_i^j - E\left(\theta_i | s_i, \mathbf{p}_{g_i}\right).$$

Then, using (7), we can find the optimal demand function

$$Q_i^j(s_i; \mathbf{p}_{g_i}) = -\left(c_{ji}^i + \beta_{ij}\right)\left(E(\theta_i \,| s_i, \mathbf{p}_{g_i}) - p_{ij}\right) \tag{8}$$

for each dealer i when trading with dealer j.

If we further substitute this into the bilateral market clearing condition (4) we obtain the price between any pair of dealers i and j as a linear combination of the posteriors of i and j, $E(\theta_i \,| s_i, \mathbf{p}_{g_i})$ and $E(\theta_j \left| s_j, \mathbf{p}_{g_j}\right.)$

$$p_{ij} = \frac{\left(c_{ji}^i + \beta_{ij}\right) E(\theta_i \,| s_i, \mathbf{p}_{g_i}) + \left(c_{ij}^j + \beta_{ij}\right) E(\theta_j \left| s_i, \mathbf{p}_{g_j}\right.)}{c_{ji}^i + c_{ij}^j + 3\beta_{ij}}. \tag{9}$$

At this point we depart from the standard derivation. The standard approach is to determine the coefficients in the demand function (3) as a fixed point of (8), given that $E(\theta_i \,| s_i, \mathbf{p}_{g_i})$ can be expressed as a function of coefficients b_i^j and $\mathbf{c_i^j}$. This procedure is virtually intractable for general networks. Instead, our approach is to solve directly for the beliefs in the OTC game. In fact, we can find the equilibrium beliefs in the OTC game without considering the profit motives and the corresponding trading strategies of agents. For this, in the next section we introduce an auxiliary game labeled the *conditional-guessing game.*

3.2 The conditional-guessing game

The conditional guessing game is the non-competitive counterpart of the OTC game. The main difference is that instead of choosing quantities and prices to maximize trading profits, each agent aims to guess her value as precisely as she can. Importantly, agents are not constrained to choose a scalar as their guess. In fact, each dealer is allowed to choose a conditional-guess function which maps the guess of each of her neighbors into her guess.

Formally, we define the game as follows. Consider a set of n agents that are connected in the same network g as in the corresponding OTC game. The information structure is also the same as in the OTC game. Before the uncertainty is resolved, each agent i makes a guess, e_i, about her value of the asset, θ_i. Her guess is the outcome of a function that has as arguments the guesses of other dealers she is connected to in the network g. In particular, given her signal, dealer i chooses a guess function, $\mathcal{E}_i\left(s_i; \mathbf{e}_{g_i}\right)$, which maps the vector of guesses of her neighbors, $\mathbf{e}_{g_i}$, into a guess e_i. When the uncertainty is resolved, agent i receives a payoff

$$-\left(\theta_i - e_i\right)^2,$$

where e_i is an element of the guess vector $\mathbf{e}$ defined by the smallest element of the set

$$\Xi\left(\left\{\mathcal{E}_i\left(s_i; \mathbf{e}_{g_i}\right)\right\}_i, \mathbf{s}\right) \equiv \left\{\mathbf{e} \mid e_i = \mathcal{E}_i\left(s_i; \mathbf{e}_{g_i}\right), \forall\ i\right\}, \tag{10}$$

by lexicographical ordering. We assume that if a fixed point in (10) did not exist, then dealers would not make any guesses and their payoffs would be set to minus infinity. Essentially, the set of conditions (10) is the counterpart in the conditional-guessing game of the market clearing conditions in the OTC game.

Definition 2 *An equilibrium of the conditional guessing game is given by a strategy profile* $(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n)$ *such that each agent* i *chooses strategy* $\mathcal{E}_i : R \times R^{m_i} \rightarrow R$ *in order to maximize her expected payoff*

$$\max_{\mathcal{E}_i}\left\{-E\left(\left(\theta_i - \mathcal{E}_i\left(s_i; \mathbf{e}_{g_i}\right)\right)^2 | s_i\right)\right\},$$

where $\mathbf{e} = \Xi\left(\cdot, \mathbf{s}\right)$.

As in the OTC game, we simplify this optimization problem and find the guess functions

$\mathcal{E}_i(s_i;\mathbf{e}_{g_i})$ point-by-point. That is, for each realization of the signals, $\mathbf{s}$, an agent i chooses a guess that maximizes her expected profits, given her information, s_i, and given the guess functions chosen by the other agents. Her optimal guess function is then given by

$$\mathcal{E}_i(s_i;\mathbf{e}_{g_i}) = E(\theta_i|s_i,\mathbf{e}_{g_i}). \tag{11}$$

In the next proposition, we state that the guessing game has an equilibrium in any network.

Proposition 1 *In the conditional-guessing game, for any network g, there exists an equilibrium in linear guess functions, such that*

$$\mathcal{E}_i(s_i;\mathbf{e}_{g_i}) = \bar{y}_i s_i + \bar{\mathbf{z}}_{g_i}\mathbf{e}_{g_i}$$

for any i, where $\bar{y}_i$ is a scalar and $\bar{\mathbf{z}}_{g_i} = (\bar{z}_{ij})_{j\in g_i}$ is a row vector of length m_i.

We derive the equilibrium in the conditional guessing game as a fixed point problem in the space of $n\times n$ matrices, in which the guess of each agent i is a linear combination of signals and guess functions $\mathcal{E}_i(s_i;\mathbf{e}_{g_i})$ satisfy (11) for all i. In particular, consider an arbitrary $n\times n$ matrix $\overset{'}{V} = \left[\overset{'}{\mathbf{v}}_i\right]_{i=1,..n}$ and let the guess of each agent i be

$$\overset{'}{e}_i = \overset{'}{\mathbf{v}}_i\mathbf{s}, \tag{12}$$

given a realization of the signals $\mathbf{s}$. It follows that, when dealer j takes as given the choices of her neighbors, $\overset{'}{\mathbf{e}}_{g_j}$, her best response guess is

$$\overset{''}{e}_j = E\left(\theta_j|s_j,\overset{'}{\mathbf{e}}_{g_j}\right). \tag{13}$$

Since each element of $\overset{'}{\mathbf{e}}_{g_j}$ is a linear function of the signals and the conditional expectation is a linear operator for jointly normally distributed variables, equation (13) implies that there is a unique vector $\overset{''}{\mathbf{v}}_j$ such that

$$\overset{''}{e}_j = \overset{''}{\mathbf{v}}_j\mathbf{s}. \tag{14}$$

In other words, the conditional expectation operator defines a mapping from the $n\times n$ matrix $\overset{'}{V} = \left[\overset{'}{\mathbf{v}}_i\right]_{i=1,..n}$ to a new matrix of the same size $\overset{''}{V} = \left[\overset{''}{\mathbf{v}}_i\right]_{i=1,..n}$. An equilibrium of the conditional guessing game exists if this mapping has a fixed point. Proposition 1 shows the existence of a fixed point and describes the equilibrium as given by the coefficients of s_i and $\mathbf{e}_{g_i}$ in $E(\theta_i|s_i, \mathbf{e}_{g_i})$ at this fixed point.

In the next section we establish an equivalence between the equilibria of the OTC game and the conditional guessing game. In Section 4 we rely on this equivalence and use the properties of the conditional guessing game to characterize beliefs in the OTC game.

3.3 Equivalence and existence

In this part we prove the main results of this section. First, we show that if there exists a linear equilibrium in the OTC game then the posterior expectations represent an equilibrium expectation vector in the corresponding conditional guessing game. Second, we provide sufficient conditions under which we can construct an equilibrium of the OTC game building on an equilibrium of the conditional guessing game.

Proposition 2 *In any Linear Bayesian Nash equilibrium of the OTC game the vector with elements e_i defined as*

$$e_i = E(\theta_i \,|s_i, \mathbf{p}_{g_i})$$

is an equilibrium expectation vector in the conditional guessing game.

The idea behind this proposition is as follows. We have already showed that in a linear equilibrium each bilateral price p_{ij} is a linear combination of the posteriors of i and j, $E(\theta_i \,|s_i, \mathbf{p}_{g_i})$ and $E(\theta_j \,\left|s_j, \mathbf{p}_{g_j}\right.)$, as described in (9). Therefore, a dealer can infer the belief of her counterparty from the price, given that she knows her own belief. When choosing her generalized demand function, she essentially conditions her expectation about the asset value on the expectations of the other dealers she is trading with. Consequently, the set of posteriors implied in the OTC game works also as an equilibrium in the conditional guessing game.

Proposition 3 *Let $\bar{y}_i$ and $\bar{\mathbf{z}}_{g_i} = (\bar{z}_{ij})_{j\in g_i}$ the coefficients that support an equilibrium in the conditional-guessing game and let $e_i = E(\theta_i \,|s_i, \mathbf{e}_{g_i})$ the corresponding equilibrium expectation of agent i. Then, there exists a Linear Bayesian Nash equilibrium in the OTC game, whenever*

$\rho < 1$ *and the following system*

$$\frac{y_i}{\left(1 - \sum_{k \in g_i} z_{ik} \frac{2 - z_{ki}}{4 - z_{ik} z_{ki}}\right)} = \bar{y}_i \tag{15}$$

$$z_{ij} \frac{\frac{2 - z_{ij}}{4 - z_{ij} z_{ji}}}{\left(1 - \sum_{k \in g_i} z_{ik} \frac{2 - z_{ki}}{4 - z_{ik} z_{ki}}\right)} = \bar{z}_{ij}, \forall j \in g_i$$

has a solution $\{y_i, z_{ij}\}_{i=1,..n, j \in g_i}$ *such that* $z_{ij} \in (0, 2)$. *The equilibrium demand functions are given by (3) with*

$$\begin{aligned} b_i^j &= -\beta_{ij} \frac{2 - z_{ji}}{z_{ij} + z_{ji} - z_{ij} z_{ji}} y_i \\ c_{ij}^j &= -\beta_{ij} \frac{2 - z_{ji}}{z_{ij} + z_{ji} - z_{ij} z_{ji}} (z_{ij} - 1) \ , \\ c_{ik}^j &= -\beta_{ij} \frac{2 - z_{ji}}{z_{ij} + z_{ji} - z_{ij} z_{ji}} z_{ik}. \end{aligned} \tag{16}$$

and the equilibrium prices and quantities are

$$p_{ij} = \frac{\left(c_{ji}^i + \beta_{ij}\right) e_i + \left(c_{ij}^j + \beta_{ij}\right) e_j}{c_{ij}^j + c_{ji}^i + 3\beta_{ij}} \tag{17}$$

$$q_i^j = -\left(c_{ji}^i + \beta_{ij}\right)\left(e_i - p_{ij}\right). \tag{18}$$

These two propositions prove the equivalence between the two games. Proposition 2 shows that one can construct an equilibrium of the conditional guessing game from an equilibrium of the OTC game. Proposition 3 shows that, under some conditions, the reverse also holds. The extra conditions are a consequence of the fact that in the reverse direction we are transforming n expectations, e_i, from the conditional guessing game into $M \geq n$ prices in the OTC game. The conditions make sure that we can do it in a consistent way. While we do not have a general proof that the system (15) has a solution for each network, we have no reason to suspect that it does not.[10]

The conceptual advantage of our way of constructing the equilibrium over the standard approach is that it is based on a much simpler and, as we will see in the next section, much

[10] Our numerical algorithm gives a well behaving solution in all our experiments including a wide range of randomly generated networks. The Matlab code runs in a fraction of a second for any network we experimented. The code for the algorithm along with a detailed explanation are available on the authors' websites. In addition, we provide analytical expressions for the equilibrium objects for the star in the proof of Proposition 4 and for the complete network in Appendix B.

more intuitive fixed point problem. Note also that Proposition 3 also describes a simple numerical algorithm to find the equilibrium of the OTC game for any network. In particular, the conditional guessing game gives parameters $\bar{y}_i$ and $\bar{z}_{ij}$, conditions (15) imply parameters y_i and z_{ij}, then (16) give parameters of the demand function implying prices and quantities by (17)-(18).

The next proposition strengthen the existence result for our specific examples.

Proposition 4 *1. In any network in the circulant family, the equilibrium of the OTC game exists.*

2. In a star network, the equilibrium of the OTC game exists.

Before proceeding to the detailed analysis of the features of the equilibrium in the next section, we make three simple observations.

First, the equivalence of beliefs on the two games implies that any feature of the beliefs in the OTC game must be unrelated in any way to price manipulation, imperfect competition or other profit related motives. It is so, because these considerations are not present in the conditional guessing game.

Second, the equilibrium coefficients both in the conditional guessing game and in the OTC game depend only on the ratio $\frac{\sigma_\theta^2}{\sigma_\varepsilon^2}$ and not on the individual parameters σ_ε^2 and σ_θ^2. We state this result in the following lemma.[11]

Lemma 1 *The coefficients $v_{ij}, \bar{z}_{ij}, \bar{y}_i, z_{ij}, y_i, b_i^j, c_{ij}^j$ and c_{ik}^j do not change in σ_ε^2 and σ_θ^2 if $\frac{\sigma_\theta^2}{\sigma_\varepsilon^2}$ remains constant.*

Finally, consumers' demand has a pure technical role in our analysis. While there is no equilibrium for $\beta_{ij} = 0$, for any $\beta_{ij} < 0$ prices and beliefs are the same and $\frac{q_i^j}{\beta_{ij}}, \frac{q_j^i}{\beta_{ij}}$ remain constant. This is evident by simple observation of expressions (17)- (16). We summarize this in the following Corollary.

[11] Note that the Lemma does not imply that any economic objects depend only on the ratio of $\frac{\sigma_\theta^2}{\sigma_\varepsilon^2}$. For example, it is easy to see that for expected profit this is not the case.

Corollary 1 *For any collection of non-zero* $\{\beta_{ij}\}_{ij\in g}$ *including the limit where all* $\beta_{ij} \to 0$, *prices,* p_{ij} *and beliefs do not change and quantities scale linearly. That is*

$$\frac{q_i^j}{\beta_{ij}}, \frac{q_j^i}{\beta_{ij}}$$

do not change with β_{ij}.

4 Prices and Information Diffusion

In this part we focus on the characteristics of equilibrium beliefs in OTC games and its implications on the informational efficiency of prices. We start with two results on the equilibrium of the conditional guessing game.

Lemma 2 *In the conditional guessing game the following properties hold.*

1. *In any connected network* g *each dealer's equilibrium guess is a linear combination of all signals*

$$e_i = \mathbf{v}_i \mathbf{s},$$

where $\mathbf{v}_i$ *is a row vector of length* n *and* $\mathbf{v}_i > 0$.

2. *In any connected network* g *when* $\rho = 1$, *there exists an equilibrium where each element of the vectors* $\mathbf{v}_i$ *is equal to* $\frac{\sigma_\theta^2}{n\sigma_\theta^2+\sigma_\varepsilon^2}$. *In this equilibrium each expectation efficiently aggregates all the private information in the economy.*

In the OTC game the following corresponding claims also hold.

Proposition 5 *Suppose that there exists an equilibrium in the OTC game. Then,*

1. *in any connected network* g *each bilateral price is a linear combination of all signals in the economy, with a positive weight on each signal;*

2. *in any connected network* g *prices are privately fully revealing when* $\rho \to 1$, *as*

$$\lim_{\rho\to 1} \left(\mathcal{V}\left(\theta_i | s_i, \mathbf{p}_{g_i}\right) - \mathcal{V}\left(\theta_i | \mathbf{s}\right)\right) = 0.$$

These results suggest that a decentralized trading structure can be surprisingly effective in transmitting information. Consider first result 1 of the Proposition 5. This shows that although we consider only a single round of transactions, each price partially incorporates all the private signals in the economy. A simple way to see this is to consider the residual demand curve and its intercept, I_i^j, defined in (6)-(7). This intercept is stochastic and informationally equivalent with the price p_{ij}. The chain structure embedded in the definition of I_i^j is critical. The price p_{ij} gives information on I_i^j which gives some information on the prices agent j trade at in equilibrium. For example, if agent j trades with agent k then p_{jk} affects p_{ij}. By the same logic, p_{jk} in turn is affected by the prices agent k trades at with her counterparties, etc. Therefore, p_{ij} aggregates the private information of signals of every agent, dealer i is indirectly connected to, even if this connection is through several intermediaries.

This property of the equilibrium does not imply that dealers learn as much as in a centralized market. When trade takes place in a centralized market, our environment collapses to the risk-neutral case in Vives (2011). In particular, in a centralized market agents submit simple demand functions to a market maker and the market clears at a single price. As Vives (2011) shows, each dealer i learns all the relevant information in the economy, and her posterior belief is given by

$$E(\theta_i|\mathbf{s}).$$

However, in a network g, a dealer i can use only m_i linear combinations of the vector of signals, $\mathbf{s}$, to infer the informational content of the other $(n-1)$ signals. Except in two special cases, this is generally not sufficient for the dealer to learn all the relevant information in the economy. One trivial special case is when each agent has $m_i = n-1$ neighbors, that is, when the network is complete. The second special case is described by result 2 in Proposition 5. It claims that in the common value limit, the decentralized structure does not impose any friction on the information transmission process in any network. To shed more light on the intuition behind the latter result, we have to understand better the learning process in the conditional guessing game.

Consider the case when $\rho = 1$. The expressions (12)-(14) can be seen as an iterated algorithm to find the equilibrium of the conditional guessing game in an arbitrary network. That is, in the first round, each agent i receives an initial vector of messages, $\mathbf{e}'_{g_i}$, from her neighbors. Given that, each of agent i chooses her best guess, $\overset{\prime\prime}{e}_i$, as in (13). The vector of messages $\overset{\prime\prime}{\mathbf{e}}_{g_i}$, with elements given by (14), is the starting point for i in the following round. By definition, if this algorithm converges to a fixed point, then this is an equilibrium of the conditional guessing game. According to result 2 in Lemma 2, when $\rho = 1$, the equal-weighted sum of signals, $\frac{\sigma_\theta^2}{n\sigma_\theta^2+\sigma_\varepsilon^2}1^\top s$, is a fixed point. The reason is simple. With common values, $\frac{\sigma_\theta^2}{n\sigma_\theta^2+\sigma_\varepsilon^2}1^\top s$ is the best possible guess for each agent given the information in the system. In addition, as the sum of signals is a sufficient statistic, the expectation operator (13) keeps this guess unchanged. Since the equilibrium of the conditional guessing game is continuos in ρ, information is aggregated efficiently also in the OTC game in the common value limit. Clearly though, exactly at $\rho = 1$ there is no equilibrium by the Grossman paradox.

Now we depart from the common value limit case. In this case, information transmission is only partial. In particular, if agent k is located further from agent i, her signal is incorporated to a smaller extent into agent i's belief. To see the intuition, we apply the iterated algorithm defined by (12)-(14) for the example of a circle-network of 11 dealers. We illustrate the steps of the iteration in Figure 2 from the point of view of dealer 6. We plot the weights with which signals are incorporated in the guess of dealer 5, 6 and 7,i.e. $\mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7$. In each figure the dashed lines show messages sent by dealer 5 and 7 in a given round, and the solid line shows the guess of agent 6 given the messages she receives. In round 0, we start the algorithm from the common value limit, $\frac{\sigma_\theta^2}{n\sigma_\theta^2+\sigma_\varepsilon^2}1^T s$, illustrated by the straight dashed lines that overlap in panel A. When $\rho < 1$, in contrast to the common value limit, $\frac{\sigma_\theta^2}{n\sigma_\theta^2+\sigma_\varepsilon^2}1^T s$ is no longer a best guess of θ_6. The reason is that dealer 6's own signal, s_6, is more correlated to her value, θ_6, than the rest of the signals are. Therefore, the best guess of dealer 6 is a weighted sum of the two equal-weighted messages and her own signal. This is shown by the solid line peaking at s_6 in Panel A. Clearly, this is not a fixed point as all other agents choose their guesses in the same way. Thus, in round 2, agent 6 receives messages that are represented by the dashed lines shown on Panel B; these are the mirror images of the round-1 guess of dealer 6. Note that these new messages are less informative for dealer 6 than the equal-weighted messages $\frac{\sigma_\theta^2}{n\sigma_\theta^2+\sigma_\varepsilon^2}1^T s$. The reason is that even when $\rho < 1$, for dealer 6 the average of the other 10 dealers' signals is a sufficient statistic for her about all the information which is in the system apart from her own signal, which she observes anyway. So from the round-0 messages, she could learn everything she wanted to learn. From the round-1 messages she cannot. The extra weight that dealer

5 and 7 place on their own private signals “jams” the information content of the messages for dealer 6. Nevertheless, the round−2 messages are informative, and dealer 6 puts some positive weight on those, and a larger weight on her own signal as the solid line on Panel B shows. This guess has a “kink” at s_5 and s_7, because in this round dealer 6 conditions on messages which overweight these two signals. Since all other agents choose their guess in a similar way in round 2, the messages that dealer 6 gets in round 3 are a mirror image of her own guess, as shown by the dashed lines in Panel C. The solid line in Panel C represents dealer’s 6 guess in round 4. On Panel D, we depict the guess of dealer 6 in each round until round 5, where we reach the fixed point. Note that it has all the properties we suggested: positive weight on each agents’ signals, but decreasing in the distance from dealer 6.

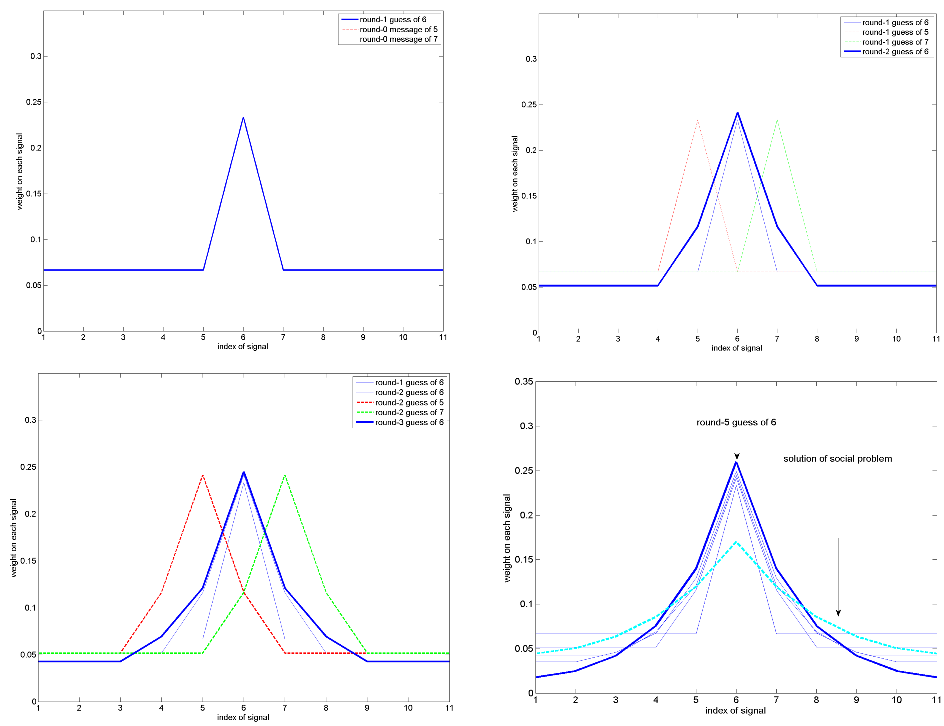

Figure 2: An iterated algorithm to find the equilibrium of the conditional guessing game in a 11-circle. Each line shows weights on a given signal in a given message or guess. Dashed lines denote messages dealer 6 recevies from her contacts of dealer 5 and 7, and the solid line denotes her best response guess of her value. Panel A, B, C illustrates round 1, 2 and 3 of iteration, respectively, while panel D illustrates all rounds until convergence. Parameters are $n = 11$, $\rho = -.8$, $\sigma_\theta^2 = \sigma_\epsilon^2 = 1$, $\beta_{ij} = -\frac{10}{11}$.

From this example, it is clear that each agent’s conditional guessing function affects how much her neighbors can learn from her guess. This, in turn, affects the learning of her neighbors’

neighbors, etc. This is a learning externality which agents do not internalize. Hence, an interesting question is how individual guessing functions should be altered to help the learning process. To be more precise, we define a measure of informational efficiency as

$$U\left(\{\bar{y}_i, \bar{\mathbf{z}}_{g_i}\}_{i\in\{1,\dots n\}}\right) \equiv -E\left[\sum_i \left(\theta_i - \mathcal{E}_i\left(s_i; \mathbf{e}_{g_i}\right)\right)^2\right]. \tag{19}$$

This is the sum of pay-off in the conditional guessing game. Given the equivalence of beliefs in Proposition 2, it captures how close dealers' beliefs are to their value in equilibrium, both for the OTC game as well as for the conditional guessing game. Therefore, this is an intuitive measure of the informativeness of market prices in the OTC game. Then, we ask which conditional guessing functions $\{\mathcal{E}_i\left(s_i; \mathbf{e}_{g_i}\right)\}_{i=1\dots n}$ would maximize informational efficiency (19) subject to $\mathbf{e} = \Xi\left(\cdot, \mathbf{s}\right)$ and (10). We refer to this exercise as solving the informational efficiency problem. To continue our example, we show the solution of the informational efficiency problem in the last panel of Figure 2 (thick dashed curve). As it is apparent, dealers put too much weight on their own signal than what is informationally efficient. The reason is clear from the above explanation. When dealers' distort messages towards their own signals, they do not internalize that they reduce the information content of these guesses for others.

In the following proposition, we show that this observation is not unique to the example. Indeed, in any star network the sum of payoffs would increase if, starting from the decentralized equilibrium, each dealer would put less weight on her own signal and more weight on her neighbors' guesses.

Proposition 6 *Let $U\left(\{\bar{y}_i, \bar{\mathbf{z}}_{g_i}\}_{i\in\{1,\dots n\}}\right)$ be the sum of payoffs in an n-star network for any given strategy profile $\{\bar{y}_i, \bar{\mathbf{z}}_{g_i}\}_{i\in\{1,\dots n\}}$. Then, if $\left\{\bar{y}_i^*, \bar{\mathbf{z}}_{g_i}^*\right\}_{i\in\{1,\dots n\}}$ is the decentralized equilibrium, then*

$$\lim_{\delta\to 0} \frac{\partial U\left(\left\{\bar{y}_i^* - \delta, \bar{\mathbf{z}}_{g_i}^* + \delta\mathbf{1}\right\}_{i\in\{1,\dots n\}}\right)}{\partial\delta} > 0.$$

That is, starting from the decentralized solution, marginally increasing weights on others' guesses and marginally decreasing weights on own signal increases the sum of payoffs.

We check whether our observation that dealers overweight their own signal is robust to different network structures and parameter values. For this, we have generated 1000 random networks in which each link is formed with probability half and with a randomly drawn information precision ratio of $\frac{\sigma_\theta^2}{\sigma_\varepsilon^2}$ for each network.[12] We keep ρ constant at 0.5. The simulations suggest that regardless of the shape of the network each dealer's weight on her private signal, $\bar{y}_i$, in the equilibrium of the conditional guessing game is larger than the weight which solves the informational efficiency problem, $\bar{y}_i^S$. [13] Interestingly, we also observe that ceteris paribus the percentage overweight, defined as $\frac{\bar{y}_i^* - \bar{y}_i^S}{\bar{y}_i^S}$, for the central dealer in a star network is larger than for any dealer in our randomly generated networks. The first column of Table 1 in Section 6.1 shows the coefficients that result from a regression of $\frac{\bar{y}_i^* - \bar{y}_i^S}{\bar{y}_i^S}$ on dealer i's number of links (degree), standard deviation of the degree distribution in the given network (asymmetry), the total number of links in network (density) and information precision $\frac{\sigma_\theta^2}{\sigma_\varepsilon^2}$. It appears that the overweighting is larger for more connected dealers, in asymmetric networks with small density, and when the information precision is large. This is also consistent with our observation that the central dealer in a star overweights the most. Our intuition is that overweighting is most severe when a dealer is in the position to reveal a lot of information to her counterparties. The potential is maximal for the central dealer in a star, who knows all the relevant information (as she can use $n-1$ guesses to learn about $n-1$ signals) and whose guess could be informative for all other, much less informed, dealers in the network.

By Proposition 3, the properties of the equilibrium in the conditional guessing game imply the properties of prices in the OTC game. That is, the correlation between prices p_{ij} and p_{kl} tends to be lower if the link ij is further away from kl. Also, prices could transmit more information even within the constraint imposed by our network structure. From the intuition gathered from the conditional guessing game, it is clear that this distortion is not a result of imperfect competition, or strategic trading motives that agents have. Instead, it is a consequence of the learning externality arising from the interaction between the interdependent value environment and the network structure.[14]

[12] For simplicity, we keep $\sigma_\theta^2 = 1$ and generate σ_ε^2 as the square of a random variable from $N(1, 0.1)$.

[13] We conjecture that a general property might be behind this observation. However, due to the lack of an analytical proof and because simulations use particular parameters and are subject small numerical errors, we cannot be sure that this is the case.

[14] Note that even if in our equilibrium dealers do not manupulate beleifs, in principle their might exist equilibria where they do as we have not proven uniqueness. It is well understood that searching for non-linear equilibria would be a notoriously hard problem even in the centralized version of our set-up. (See Breon-Drish (2012) and Pálvölgyi and Venter (2011)).

5 Discussion and Examples

In this section, we discuss how the decentralized trading environment affects prices, volume and intermediation. We start with a simple example of a pair of dealers that trade and then discuss the effects that arise because of the network structure in the context of an $n-$star network.

5.1 Bilateral trade

The analysis of the trade between two dealers can be seen both as a particular case of our OTC game as well as a special case of a centralized market. This case provides a useful benchmark to highlight some important insights which are partially or fully inherited by a more complex OTC network structure. The equilibrium in the bilateral case can be characterized either following the approach that we described in Section 3.1 or using Proposition 1 in Vives (2011), for $n = 2$.

Consider two dealers, indexed, for simplicity, by their position: $L(eft)$ and $R(ight)$. Then, there exists an equilibrium in which the demand function of dealer $i \in \{L, R\}$ is given by

$$Q_i(s_i; p) = t\left(E\left(\theta_i | s_i, p\right) - p\right), \tag{20}$$

where $t \equiv -\left(c + \beta\right)$ represents dealer's trading intensity and captures the inverse of her price effect. This corresponds to the interpretation that agent L (R) trades t units with counterparty R (L) for every unit of perceived gain, $E\left(\theta_i | s_i, p\right) - p$. Using that the belief of a dealer $i \in \{L, R\}$ is a linear combination between her signal and the price

$$E\left(\theta_i | s_i, p\right) = y s_i + z p,$$

and that the price is a linear combination of their posterior beliefs

$$p = \frac{t\left[E\left(\theta_L | s_L, p\right) + E\left(\theta_R | s_R, p\right)\right]}{2t + (-\beta)}, \tag{21}$$

we can solve for the equilibrium as a fixed point problem as described in Sections (3.2) and

(3.3) and show that

$$
\begin{aligned}
y &= \frac{1-\rho}{1-\rho+\sigma_{\varepsilon}^{2}/\sigma_{\theta}^{2}} \\
z &= \frac{2\rho\sigma_{\varepsilon}^{2}/\sigma_{\theta}^{2}}{1-\rho^{2}+\sigma_{\varepsilon}^{2}/\sigma_{\theta}^{2}}
\end{aligned}
$$

and

$$
t = -\beta\frac{1-\rho^{2}+\sigma_{\varepsilon}^{2}/\sigma_{\theta}^{2}}{2\rho\sigma_{\varepsilon}^{2}/\sigma_{\theta}^{2}}.
$$

As the price in (21) is close to the average valuation of agents, equation (20) implies that the agent with higher than average signal (optimist) tends to buy and the agents with lower than average signal (pessimist) tends to sell the asset. Ultimately, the position that each dealer holds in equilibrium depends on the interaction between two forces. First, the dealers can learn from the price information about the common value component of their values. As dealer L (R) finds the signal of her counterparty R (L) more informative for the estimation of her own value, or as ρ and $\sigma_{\varepsilon}^{2}/\sigma_{\theta}^{2}$ increase, her trading intensity, t, decreases. To see why this is the case, suppose that the dealer with a higher initial belief considers to buy. If the other dealer desires to sell more at the same time, pushing the price downwards, she typically increases her demand as a response. However, this adjustment will be weaker, if she is worried that the large supply indicates a low value for the asset. The stronger the information content of her counterparty's signal, the smaller the quantity response, t, is. As ρ and σ_{ε}^{2} decrease or σ_{θ}^{2} increases, the informational content in others' signals decreases and t increases without bounds. That is, if dealers did not care about the information content of prices (or if they were price takers), there would be no equilibrium with finite quantities as a consequence of risk-neutrality.

Second, dealers adjust their trade depending on how large they perceive the gains from trade to be. In particular, gains from trade, measured as

$$
E\left[\left(E\left(\theta_{L}|s_{L},p\right)-E\left(\theta_{R}|s_{R},p\right)\right)^{2}\right]=\frac{2\sigma_{\theta}^{2}\left(1-\rho\right)^{2}}{1+\sigma_{\varepsilon}^{2}/\sigma_{\theta}^{2}-\rho}
$$

decrease in ρ and $\sigma_{\varepsilon}^{2}/\sigma_{\theta}^{2}$. For instance, as ρ increases, the relative importance of the common component increases. Similarly, as σ_{ε}^{2} increases and the precision of dealers' information decreases, their expectation of the value of θ_i converges to their prior, 0. Both lead to smaller expected difference in posteriors.

In the next part, we discuss to what extent the mechanism of price formation and trading volume changes in a decentralized market.

5.2 Trade in a star network

Just as in the bilateral trade benchmark, dealers' equilibrium trades in a network are driven by both the relevance of others' information, as well as by how large the gains from trade are. However, there are also important differences implied by the network structure.

As an illustration, we provide a numerical example for the simplest possible network. We consider that three dealers are organized in a $3-$star network. They are depicted by the connected squares in Figure 3. As above, we index them by their position: *Periphery* $L(eft)$, $C(entral)$, *Periphery* $R(ight)$. The number in each square is the realization of their signal: $s_L = -2$, $s_C = 0$, $s_R = 1$. We picked $\beta_{L,C} = \beta_{R,C} = \beta = -5$. Prices are in the rhombi located on the links and demand curves are at the bottom of the figure in the form

$$Q_i^j = t_i^j \left(E(\theta_i \left| s_i, \mathbf{p}_{g_i} \right.) - p_{ij} \right) \tag{22}$$

where $t_i^j \equiv -\left(c_{ji}^i + \beta_{ij}\right)$ is the trading intensity of trader i when trading with counterparty j. From (7), it is easy to see that $1/t_i^j$ describes how much the price between dealer i and j needs to adjust, if dealer i changed her quantity by one unit. Just for this example, we omit the superscript to simplify the exposition, and use t_P for the trading intensities of the periphery dealers L and R, and t_C for the trading intensity of the central dealer C (since the periphery dealers are symmetric, $t_P = 10.8$ and $t_C = 10.5$). Substituting in the prices and signals gives the traded quantities in the first line of the rectangles. Below the quantities, between brackets, we calculated the profit or loss realized on that particular trade in the case when the realized value, θ_i, is zero for all dealers. All quantities are rounded to the nearest decimal. For example, the central dealer forms the posterior expectation of -0.1, buys 3 units from the left and sells 2.4 units to the right at prices -0.4 and 0.1 respectively, earning 2.9 unit of profit in total from the trades. Note, that while counterparties hold a position of opposite sign and same order of magnitude, bought and sold quantities over a given link does not add up to 0. This is because the net of the two positions is absorbed by the customers.

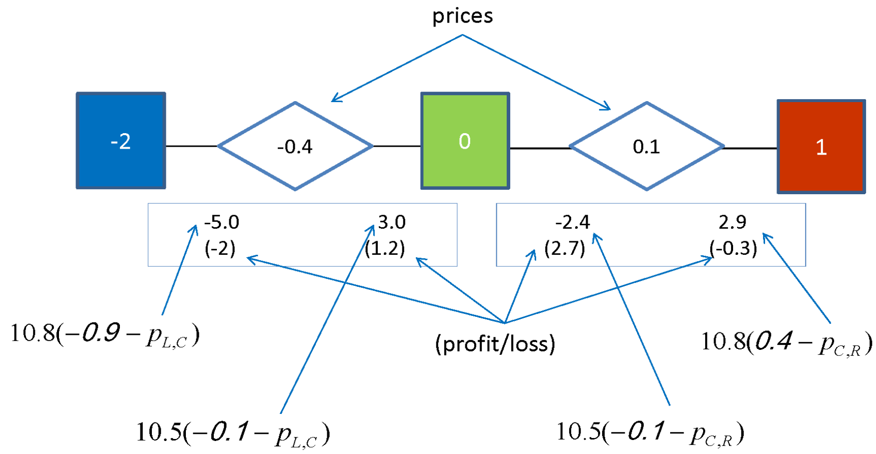

Figure 3: The connected squares depict three dealers organized in a 3−star network. Their realized signals are in the middle of the square. Prices are in rhombi, demand curves are at the bottom of the figure, with the posterior expectations in italic. The traded quantities are in the first line of the rectangles. Below, the profit or loss realized on that particular trade in the case when $\theta_i = 0$ for all i. Parameters are $\rho = 0.5, \sigma_\theta^2 = \sigma_\varepsilon^2 = 1, \beta_{ij} = -5$.

There are a number of observations which generalize to other examples. First, as in the bilateral case, dealers adjust their trades depending on how much information they learn from the prices at which they are trading at. This is reflected in changes in the trading intensity of a dealer in response to changes in ρ and $\sigma_\varepsilon^2/\sigma_\theta^2$. However, the decentralized structure introduces a natural asymmetry in trading. That is, even if all dealers have the same quality of information, trading intensities are different. To see this, consider the best response slope $c_{P,C}^C$ of periphery dealer L (R) to the slope of dealer C, $c_{C,P}^P$, derived from (8) as

$$c_{P,C}^C = \left(c_{C,P}^P + \beta\right)(1 - z_P), \tag{23}$$

where z_P is the coefficient of the price between C and the periphery dealer L (R), $p_{L,C}$ $(p_{R,C})$ in the expectation $E(\theta_L|s_L, p_{L,C})$ $(E(\theta_R|s_R, p_{R,C}))$. We can re-write this expression[15] in terms of trading intensities as

$$\begin{aligned} -t_C &= t_P (z_P - 1) + \beta \\ -t_P &= t_C (z_C - 1) + \beta \end{aligned}$$

[15] In $E(\theta_C|s_C, p_{L,C}, p_{R,C})$ the coefficients of the two prices are equal. Thus, with a slight abuse of notation we denote both z_C in this simple example.

or

$$
\begin{aligned}
t_C &= -\beta \frac{2 - z_P}{z_C + z_P - z_P z_C} \\
t_P &= -\beta \frac{2 - z_C}{z_C + z_P - z_P z_C}.
\end{aligned}
$$

In our example, the trading intensity of the central agent is smaller than that of the other agents. The reason is that the central agent relies on each price as a source of learning less than the periphery agents do, i.e. $z_P > z_C$. Indeed, this is the case, as the central agent can learn from two prices. Therefore, she is less subject to adverse selection. An extra unit sold by the left agent triggers a smaller price-adjustment by the central agent inducing the left agent to trade more aggressively. This explains $t_P > t_C$. These observations generalize when trading takes place in a star of n traders, as described in the following proposition.

Proposition 7 *In an $n-$star network the following statements hold*

1. $z_P > z_C$;
2. $t_P > t_C$;
3. $\frac{\partial t_C}{\partial \sigma^2}, \frac{\partial t_C}{\partial \rho}, \frac{\partial t_P}{\partial \sigma^2}, \frac{\partial t_P}{\partial \rho} < 0$ *for* $n = 3$, *where* $\sigma^2 \equiv \sigma_\varepsilon^2 / \sigma_\theta^2$.

Second, as in the bilateral case, dealers adjust their trade depending on how large they perceive the gains from trade to be. While in a general network gains from trade are pair specific, we characterize how they depend on the underlying parameters in a $n-$star network in the following proposition.

Proposition 8 *In an n-star, let the equilibrium beliefs be given by*

$$e_C = E\left(\theta_C | s_C, \mathbf{p}_{g_C}\right)$$

for the central agent C, and

$$e_i = E\left(\theta_i | s_i, p_{iC}\right)$$

for any periphery agent $i \neq C$. Then, the gains from trade, measured as

$$E\left(\left(e_C - e_i\right)^2\right),$$

decrease in ρ, $\sigma_\varepsilon^2/\sigma_\theta^2$ and in n for any i.

It is interesting to observe how the asymmetry in position simultaneously effects prices and quantities through trading intensities. To see this, using (17) and (22) we re-write the price between L and C as a weighted average of the posterior expectations of L, C and 0 (the bliss point of customers)

$$p_{L,C} = \frac{t_P e_L + t_C e_C + (-\beta)\,0}{t_P + t_C + (-\beta)}.$$

The expression for the price between R and C is analogous. This expression shows that the more aggressive an agent trades, the closer the price is to her expectation, decreasing her perceived per-unit profit $|e_i - p_{ij}|$. Intuitively, in our example, the pessimistic agent prefers to sell assets to the center agent with larger trading intensity than the trading intensity the central agent is willing to buy with. As markets has to clear, the only way they can agree on the terms, if the pessimistic agent, L, gives a price concession to agent C. The reasoning is identical for the trade between R and C.

Third, price dispersion arises naturally in this model. The central dealer is trading the same asset at two different prices, because she is facing two different demand curves. Just as a monopolist does in a standard price-discrimination setting, the central agent sets a higher price in the market where demand is higher. In fact, from (17), we can foresee that the price dispersion in our framework must be closely related to the dispersion of posterior beliefs.

Fourth, profitable intermediation by central agents also arises naturally. That is, the central dealer's net position $(3-2.4)$ is significantly lower than her gross position $(3+2.4)$ as she trades not only to take a speculative bet but also to intermediate between her counterparties.

6 Applications: Theory and Facts

An attractive feature of our model is that it generates a rich set of empirical predictions. Namely, for any given information structure and dealer network, our model generates the full list of demand curves and the joint distribution of bilateral prices and quantities, and measures of price dispersion, intermediation, trading volume etc. Therefore, one can directly compare our results to the stylized facts described by the growing empirical literature using transaction level OTC data. In this way, our model can help to decide whether dealer's asymmetric information can or cannot be behind stylized facts in particular markets, during particular episodes.

To illustrate this feature, we push our model in two directions. In section 6.1, we are interested in the robust implications of our model to the relationship between the standard financial indicators as cost of trading, price dispersion, size of trades and characteristics of the dealer network. In section 6.2, we focus on episodes of distress in OTC markets. In particular, using our model, we confront narratives on potential mechanisms behind these episodes to the observed stylized facts emerging from existing empirical analyses.

6.1 Informational trades in OTC markets

In this part, we highlight some robust empirical predictions of our model. As a main tool, we simulate data sets by solving for our equilibrium in a large number of randomly generated networks, and run regressions on various financial and network characteristics. The underlying thought experiment is that a given network corresponds to a given market where dealers trade repeatedly. In each round of trade there are transactions between all connected dealer-pairs as it is described in our static model. Then, in the next round a new set of signals are realized implying a new round of transactions in the same network. Note that for this thought experiment we do not have to generate realization of signals. Instead, based on the equilibrium coefficients of each demand curve we can form expectations over the signals to get the expected value of any relevant financial indicators. Therefore, running regressions on the expectation of these financial indicators and network characteristics gives us robust connections which should hold across various market structures.

We start by finding the model equivalents of our main financial indicators of interest: trading cost, gross volume, intermediation, price dispersion and price volatility.

The majority of the empirical literature conceptualizes trading cost (also referred to as mark-up or effective spread) as the cost of selling and instantaneously buying back a given quantity as a fraction of the value of the transaction. This is the percentage cost of a round-trip trade. We can construct the theoretical counterpart of the percentage cost of a round-trip trade as follows. Let us consider the difference of the price, p_{ij}^B, at which trader i could buy a quantity q_i^j from trader j, and the price, p_{ij}^S, at which i could sell the same quantity to j and normalize this by the value of the transaction for i given her fundamental valuation θ_i

$$\frac{p_{ij}^B - p_{ij}^S}{q_i^j \theta_i} = \frac{-\frac{b_j^i s_j + \sum_{k \in g_j, k \neq i} c_{jk}^i p_{jk} + q_i^j}{c_{ji}^i + \beta_{ij}} + \frac{b_j^i s_j + \sum_{k \in g_j, k \neq i} c_{jk}^i p_{jk} - q_i^j}{c_{ji}^i + \beta_{ij}}}{q_i^j \theta_i} = \frac{2}{\left(c_{ji}^i + \beta_{ij}\right) \theta_i}, \tag{24}$$

where we used (5). As the distribution of θ_i is the same for each dealer, we will refer to $\frac{-1}{c_{ji}^i + \beta_{ij}} + \frac{-1}{c_{ji}^j + \beta_{ij}}$ and $\frac{1}{|g_i|} \Sigma_{i \in g_i} \frac{2}{c_{ji}^i + \beta_{ij}}$ as our measures of the average cost of trade corresponding to the transaction between i and j, and the average cost of trade corresponding to dealer i, respectively.

For measuring average transaction size or gross volume for a given dealer, we consider the measure $E\left(\Sigma_{i \in g_i} |q_i|\right)$. For intermediation, we consider the absolute ratio of the expected gross trading volume to the expected net trading volume $\left| \frac{E\left(\Sigma_{i \in g_i} |q_i|\right)}{E\left(\Sigma_{i \in g_i} q_i\right)} \right|$ for a given dealer. Clearly, this ratio is always 1 for a dealer who has a single link to trade, but can be very large for dealers who trade a lot with their multiple trading partners, but many of their trades cancel each other. As a measure of profitability, we consider the expected profit per transaction $E\left(q_i^j \left(\theta_i - p_{ij}\right)\right)$.

Note that for a given trading pair, cost of trading is independent of the size of the transaction. This comes from the linear nature of our model. However, in line with the intuition embedded in (23), we expect that more connected dealers trade more at lower percentage cost, as they are less worried about adverse selection. Also as information precision, $\frac{\sigma_\theta^2}{\sigma_\varepsilon^2}$, increases, adverse selection is less severe, decreasing the cost and increasing the size of the transactions. More connected traders should also intermediate the most and have the higher expected profit per trade given the large volume and the more precise posterior due to the larger number of observed prices .

Turning to price dispersion, note that our framework can capture two distinct concepts of price dispersion. Recall, that according to our thought experiment, for each generated network, we can think of a hypothetical panel data-set with both a time-series dimension (across rounds of trades) and a cross-sectional dimension (across dealer-pairs). In this hypothetical panel, we refer to the price variability in the time-series dimension for a given dealer-pair as price volatility. In contrast, we refer to the price variability across the bilateral relationships in a given cross-section in a given round as price dispersion. More formally, consider the covariance matrix of prices in each transaction, Σ_p. The diagonal elements of Σ_p represent price volatility, while

	overweight	profit/trade	intermediation	cost	gross volume	price volatility	dispersion
constant	0.18^{***}	-0.855^{***}	-63.46^{***}	0.105^{***}	-380.3^{***}	37.6^{***}	0.103^{***}
	(34.85)	(-84.172)	(-19.5)	(203.251)	(-190.27)	(477.22)	(12.783)
degree	0.02^{***}	0.15^{***}	19.9^{***}	-0.006^{***}	57.92^{***}	0.99^{***}	-0.024^{***}
	(39.38)	(192.289)	(79.84)	(-171.084)	(378.13)	(164.01)	(-41.164)
density	-0.04^{***}	0.018^{***}	-2.34^{***}	-0.002^{***}	9.62^{***}	0.67^{***}	-0.001
	(-50.12)	(9.602)	(-3.92)	(-34.338)	(26.22)	(46.06)	(-1.585)
asymmetry	0.03^{***}	-0.043^{***}	-23.45^{***}	0.001^{***}	27.24^{***}	-0.22^{***}	0.038^{***}
	(14.55)	(-11.816)	(-19.75)	(3.775)	(37.33)	(-7.93)	(12.939)
information	0.01^{***}	0.059^{***}	1.17^{***}	-0.002^{***}	20.12^{***}	2.27^{***}	0.0004
precision	(-23.31)	(53.318)	(3.28)	(-47.849)	(91.95)	(263.62)	(0.547)
R^2	0.25	0.82	0.42	0.8	0.95	0.91	0.17
N	11000	11000	11000	11000	11000	11000	11000

Table 1: Regressions on a simulated dataset of percentage overweighting, profit per trade, intermediation, cost, gross volume, price volatility and dispersion on dealer and network characteristics, i.e., number of links (degree), standard deviation of the degree distribution (asymmetry), total number of links in network (density) and information precision in 1000 random network where each link is formed with probability half. Standard OLS regressions with t-stats in parenthesis. σ_ϵ^2 is the square of a random variable drawn from $N(1, 0.1)$ and information precision is $\frac{\sigma_\theta^2}{\sigma_\epsilon^2}$. Remaining parameters are fixed at $n = 11$, $\rho = 0.5$ and $\sigma_\theta^2 = 1$.

the off-diagonal elements of Σ_p measure price dispersion. Therefore, a transparent, normalized, single-value measure of the price dispersion across all dealer pairs is the determinant of the correlation matrix of prices. The price dispersion for a given dealer can be measured as the determinant of the sub-matrix of the correlation matrix corresponding to the prices the given dealer trades at.

While empirically price dispersion and price volatility are closely related and might not be separable, in our model these two objects are driven by different forces. Price volatility tends to be large in those transactions in which dealers trade large quantities . This is because demand-schedules are downward sloping implying larger price effects for large trades. Since, as we argued before, dealers with many connections tend to trade a lot, price volatility will be largest for these pairs. In contrast, price dispersion tends to be small across dealers who learn a lot from prices, because their posteriors are close and prices are weighted averages of posteriors. As better connected dealers learn more, price dispersion will be small across dealer-pairs where both counterparties have many connections.

Table 1 shows the output of dealer-level regressions connecting these trading characteristics with network characteristics. We use our simulated data-set of 1000 networks and the dealer and network characteristics we have introduced at the end of Section 4. The table shows standard OLS regressions with t-values in parenthesis. Consistently with our expectations, we see that more connected dealers trade more, earn more profit per trade, intermediate more and trade at smaller mark-up, at less dispersed, but more volatile prices [16] Also, markets with more precise information are more profitable, have larger volumes, more intermediation and smaller mark-ups.

Because of data limitation, studies which connect dealer's network characteristics with economic indicators are rare. As an example Li and Schürhoff (2012), consistently with our predictions, also show that central agents trade more and seem to be better informed than others. Interestingly, the two empirical studies we are aware of directly addressing the relationship between mark-up and network position, Li and Schürhoff (2012), Hollifield, Neklyudov and Spatt (2012), find opposing patterns. While Li and Schürhoff (2012) finds that in the municipal bond market more central dealers trade at higher mark-up, Hollifield, Neklyudov and Spatt (2012), consistently with our prediction, finds that in the collateralized loan market more central dealers trade at lower mark-up. Our model suggests that in municipal bond market pricing of assets is driven by other forces than asymmetric information.

Note that thinking about the underlying trading network structure might be useful even when the econometrician has only limited information on dealers' characteristics. Indeed, Table 1 also gives information on how some financial indicators are connected to others by the underlying network characteristics. In particular, we should expect that larger transaction size is associated to smaller cost of trading, more profitability per transactions, less dispersed prices across simultaneous transactions, but more volatile prices across time-periods. The reason is that each of these characterize transactions of more connected dealers. From this group of predictions, the pattern that percentage cost is decreasing in the size of the transaction is a robust observation found in many different contexts (see Green, Hollifield and Schurhoff (2007), Edwards, Harris and Piwowar (2007) and Li and Schürhoff (2012)). To the extent that information precision is lower for lower rated bonds, our observation on the negative connection between cost of trade and information precision is consistent with the findings of Edwards, Harris and Piwowar (2007) and Bao, Pan and Wang (2011).

[16] Using eigenvalue centrality instead of degree centrality gives very similar results.

	Friewald et al (2012)[17]	Afonso et al. (2012)	Gorton and Metrick (2012)[18]	Agarwal et al. (2012)[19]
Market	Corporate bonds	Fed Funds	Repo	MBS
Event	Subprime & GM/Ford	Lehman	Subprime	Subprime
Price dispersion	$\nearrow$	$\nearrow$	N/A	N/A
Price impact	$\nearrow$	N/A	$\nearrow$	N/A
Volume	$\leftrightarrow$	$\leftrightarrow$	N/A	$\searrow$

Table 2: Stylized facts about financial crises from the empirical literature

6.2 OTC markets in distress

In the previous part, we highlighted few robust implications of our theory for OTC markets where trading is driven by asymmetric information. In this part, as another illustration of the range of potential applications, we confront narratives behind OTC market distress to the stylized facts emerging from existing empirical analyses.

As a starting point, in Table 2 we summarize the findings of four recent empirical papers that investigate the effect of a liquidity event on market indicators. Three main indicators stand out: price dispersion, price impact and volume.

The stylized picture that emerges is that in a financial crisis price dispersion and price impact tend to increase and volume tends to decrease or stay constant. While each paper uses different proxies for these indicators, we consider that the measures introduced in the previous section capture best the economic content of the indicators. In particular, just as before, we use the determinant of the correlation matrix as a measure of price dispersion. For volume, we use the expected gross volume of a each dealer j, $E\left(\Sigma_{i\in g_i}|q_i|\right)$. For price impact, we use the average cost of trades in a given transaction, $\frac{-1}{c^i_{ji}+\beta_{ij}}+\frac{-1}{c^j_{ji}+\beta_{ij}}$.

For the analysis, we use the 2-level core-periphery network depicted on Figure 4. This network has a core of four connected dealers who are linked with one mid-level dealer each. Each mid-level dealer can intermediate between the core group and one other dealer. Given that real-world OTC markets tend to have a core-periphery structure, this network is a simple example to see how transactions within the core group differ from transactions between core and mid-level dealers and mid-level and periphery dealers. For example, the left column on Figure 5 shows dispersion volume and cost of trading as a function of information precision, $\frac{\sigma^2_\theta}{\sigma^2_\varepsilon}$, for each segment of the 2-level core-periphery market. In line with our previous observations, we find that volume is the largest, while cost and price dispersion is the lowest for the transactions

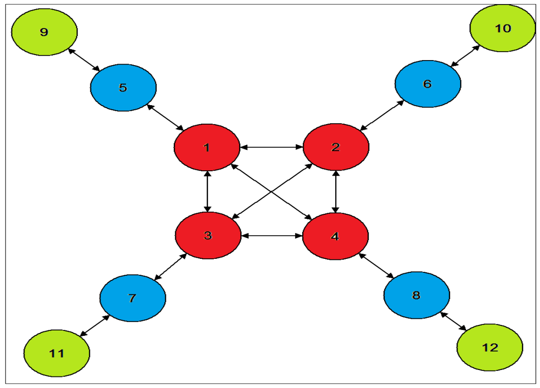

Figure 4: A two-level core-perihpery network. Central dealers are red, mid-level dealers are blue and periphery dealers are green.

within the core group of dealers. Also, price dispersion and volume are increasing, while cost is decreasing in information precision.

Given our measures of the relevant indicators and our 2-level core-periphery example, we confront three potential narratives with the stylized facts.

1. Fundamental uncertainty increases around a crisis. That is, the ratio $\frac{\sigma_\varepsilon^2}{\sigma_\theta^2}$ decreases.

2. Asymmetry of information increases during financial crisis. In our framework, we can illustrate this effect by increasing one of the core dealers (dealer 4's) signal noise σ_ε^2. This captures the idea that some of the dealers can have relative disadvantage of valuating the securities compared to others, as in Dang, Gorton and Holmström (2009).

3. Counterparty risk increases. We capture this by assuming that dealer 4 loses its links to 2 and 3, that is, dealer 4 is no longer a core-dealer. This exercise is in the spirit of Afonso et al. (2012), who find that heightened concerns about counterparty risk restricted the access of weaker banks to interbank credit.

The effects of each of the experiments on price dispersion, trading volume and average cost of trading in a given transaction are illustrated at Figure 5. In particular each column corresponds to an experiment, while each row corresponds to one of the measures. The left column shows the case in which each dealers' information precision increases from 0.35 to 1. The middle column, we show the case in which only dealer's 4 information precision increases

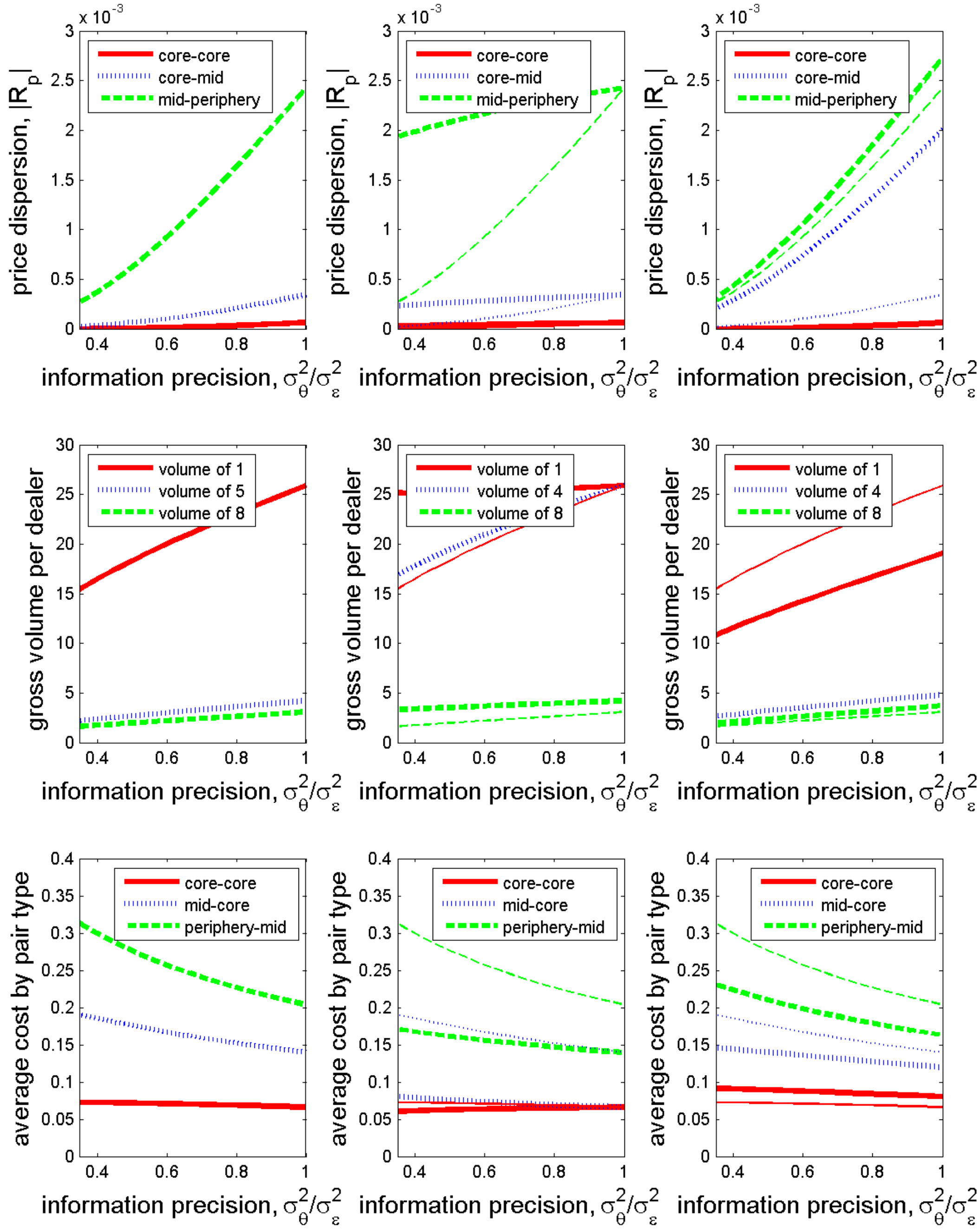

Figure 5: The effects of changing information precision of each dealers (left column), of changing information precision of dealer 4 only (middle column) and of breaking the link between dealers 4 and 2 and between 4 and 3 (right column) on price dispersion (first row), trading volume (middle row) and average cost of trading in a given transaction (bottom row). On each panels in the left and right columns σ_ε^2 of each dealer changes. On each panels in the middle column only dealer 4's σ_ε^2 changes while for others it is fixed at 1. To facilitate the comparison between the scenarios, curves from the left column are copied to each of the other corresponding figures as thin curves. Other paramters are $\rho = 0.5, \sigma_\theta^2 = 1$ and $\beta_{ij} = -\frac{10}{12}$.

from 0.35 to 1 (thick curves), while for all other dealers it is fixed at $\frac{\sigma_\theta^2}{\sigma_\varepsilon^2} = 1$. In the right column, we show the case when there is no link between 4 and 2, nor between 4 and 1 (thick curves), as dealers' information precision increases from 0.35 to 1. To facilitate the comparison between the scenarios, we plot the curves from the left column in each of the other corresponding figures in the middle and right column as thin curves.

The main take-away is that the first two narratives do not seem to be consistent with the stylized facts we observe. In each of these scenarios price dispersion and trading volume move in the same direction. As we explained in Section 5, this is driven by two forces: how much a dealer cares about the private information of her counterparty and how the perceived gains from trade are affected. For example, when $\frac{\sigma_\theta^2}{\sigma_\varepsilon^2}$ decreases, dealers learn more from each other, and, at the same time, they perceive smaller differences in posteriors. Both these effects imply that price dispersion and trading volume decrease. Similarly, when dealer's 4 information precision decreases, she trusts less her private information, and learns more from others, which decreases price-dispersion (not surprisingly, price dispersion decreases even more when all dealers' private information is less precise, as comparing thin and thick curves in the middle panel of the first row shows). Trading volume goes down as well because as information becomes less precise, dealers posterior beliefs are closer, and the expected gains from trade are smaller.[20]

However, the last experiment, in which we break two of the links of dealer 4, gives a different pattern. Comparing thick curves with the corresponding thin curves in the right column we see that breaking links pushes price dispersion up and trading volume down for each group of transactions (core-core, mid-core, and mid-periphery), consistent with the empirical evidence. By assuming away the possibility that dealer 4 can trade with dealer 2 and 3, narrative 3 exogenously reduces the possible trading opportunities, and, at the same, reduces the possible learning opportunities. The former decreases volume, the latter increases the heterogeneity of posterior expectations, and, therefore, increases price dispersion.[21]

[20]Note that comparing thin and thick curves in the middle panel that larger information asymmetry does not imply less trade in our set-up. Indeed, we see that both the volume of 1 and 4 is larger when only 1 has a less precise information (thick curves) compared when each agent has less precise information (thin curves). Increasing asymmetry of information between two dealers tends to affect trading intensities of the two traders into the opposite direction, leading to a small net effect on trading volume. Therefore, the gains-from-trade effect dominates which leads to more trade whenever the information of any of the dealers is more precise.

[21]Along the same lines, one can analyze the responses of each group of traders in each of our experiments. Such analysis would lead to similar insights, so it is omitted for brevity.

Finally, let us emphasize that although we have found more support for (this specific interpretation of) counterparty risk than for information based narratives, our comparative statics exercises are simplistic interpretations of these narratives. While we believe that the intuition behind our conclusions are relevant in a more general context, undoubtedly, more research is needed to conclude that information based stories cannot be the reason behind market breakdowns in OTC markets.

7 Dynamic Foundations

In our model, all trades take place simultaneously. This poses the conceptually complex problem of finding the equilibrium price and quantity vectors at which transactions take place. One solution, commonly used in the literature, is to employ a central auctioneer. In particular, each dealer submits her signal and vector of demand functions to the auctioneer. The auctioneer then informs each pair of dealers what the trading price is, as well as their respective positions. However, in our decentralized environment, a central auctioneer is an unattractive concept.

As an alternative, we consider the following two-step process for finding the equilibrium. In the first step, each dealer can find her best response demand function, given the choices of the other $(n-1)$ dealers. Implicitly, each dealer is able to compute the equilibrium demand functions for herself and all the other dealers, for any realization of the vector of signals. In the second step, realized signals are mapped into traded quantities and prices through a dynamic (non-strategic) protocol in which dealers exchange a series of quotes with their counterparties. The protocol specifies an updating rule for beliefs, a price rule and a quantity rule based on the coefficients of the demand curves determined at the first step. We describe this protocol in detail below and show that it leads to the same outcome as our OTC game.

Suppose that time is discrete, and in each period there are two stages: the morning stage and the evening stage. In the evening, each dealer i sends a message, $h_{i,t}$, to all counterparties she has in the network g. In the morning, each dealer receives these messages. In the following evening, a dealer updates the message she sends by the updating-rule using her signal and the information received in the morning. Messages can be interpreted, for instance, as quotes that dealers exchange with their counterparties. Upon receiving a set of quotes, a dealer might decide to contact her counterparties with other quotes, before trade actually occurs. The protocol stops if there exists an arbitrarily small scalar $\delta_i > 0$, such that $|h_{i,t} - h_{i,t_{\boldsymbol{\delta}}}| \leq \delta_i$ for each i, in any subsequent period $t \geq t_{\boldsymbol{\delta}}$. That, is the protocol stops when no dealer wants to significantly revise her message in the evening after receiving information in the morning.

When the protocol stops, messages are mapped to into prices and quantities by price and quantity rules for each pair of dealers that have a link in the network g. These rules are common knowledge for all dealers. No transactions take place before the protocol stops.

Suppose that there exists an equilibrium in the one-shot OTC game. Let dealers use their equilibrium strategy in the conditional guessing game as the updating rule, such that

$$h_{i,t} = \bar{y}_i s_i + \bar{\mathbf{z}}_{g_i}^T \mathbf{h}_{g_i,t-1},\ \forall i$$

where $\mathbf{h}_{g_i,t} = (h_{j,t})_{j \in g_i}$, and $\bar{y}_i$ and $\bar{z}_{ij}$ have been characterized, for any i and $j \in g_i$, in Proposition 1. Further, consider a price rule based on (17) that determines the price between a pair of agents ij that have a link in the network g as follows

$$p_{ij,t} = \frac{(2 - z_{ji})}{4 - z_{ij} z_{ji}} h_{i,t} + \frac{(2 - z_{ij})}{4 - z_{ij} z_{ji}} h_{j,t},$$

where the relationship between z_{ij} and $\bar{z}_{ij}$ has been characterized in Proposition 3. Given the prices, the quantity rule allocates to agent i, in the transaction with j, $q_{i,t}^{j}(s_i, \mathbf{p}_{g_i,t})$, where the function has been characterized in Proposition 3. Then the following statements hold.

Proposition 9 *Let $\mathbf{h}_t = (h_{i,t})_{i \in \{1,2,\ldots,n\}}$ be the vector of messages sent at time t, and $\boldsymbol{\mu} = (\mu_i)_{i \in \{1,2,\ldots,n\}}$ be a vector of IID $N(0, \sigma_\mu^2)$ random normal variables. Suppose that $\rho < 1$. Then*

1. *If $\mathbf{h}_t = \left(I - \bar{Z}\right)^{-1} \bar{Y} \mathbf{s}$, then $\mathbf{h}_{t+1} = \left(I - \bar{Z}\right)^{-1} \bar{Y} \mathbf{s}$, for any t.*
2. *If $\mathbf{h}_{t_0} = \left(I - \bar{Z}\right)^{-1} \bar{Y} \left(\mathbf{s} + \boldsymbol{\mu}\right)$, then there exists a vector of arbitrarily small scalars $\boldsymbol{\delta} = (\delta_i)_{i \in \{1,2,\ldots,n\}}$ such that trading takes place in period $t_{\boldsymbol{\delta}}$ and*

$$\left| \mathbf{h}_{t_{\boldsymbol{\delta}}} - \left(I - \bar{Z}\right)^{-1} \bar{Y} \mathbf{s} \right| < \frac{1}{2} \boldsymbol{\delta}.$$

3. *If $\mathbf{h}_{t_0} = \left(I - \bar{Z}\right)^{-1} \bar{Y} \left(\mathbf{s} + \boldsymbol{\mu}\right)$, then there exists a vector of arbitrarily small scalars $\boldsymbol{\delta} =$*

$(\delta_i)_{i\in\{1,2,\dots,n\}}$ *such that trading takes place in period* $t_{\boldsymbol{\delta}}$ *and*

$$\left|E\left(\theta_i|s_i,\mathbf{h}_{g_i,t_0},\mathbf{h}_{g_i,t_0+1},...,\mathbf{h}_{g_i,t_\delta}\right)-E\left(\theta_i|s_i,\mathbf{p}_{g_i}\right)\right|<\frac{1}{2}\boldsymbol{\delta},$$

where $\mathbf{p}_{g_i}$ *are the equilibrium prices in the one-short OTC game.*

The first part states that vector of equilibrium beliefs $\mathbf{e}$ defined in Section 3.2, is a steady state of the protocol. The second and third parts state that the dynamic protocol leads to the same beliefs (even if the trader uses the full history of past messages) and, consequently, to the same traded prices and quantities as the corresponding OTC game, independently from which vector of messages we start the protocol at.

Note that in this part we focused on the existence of a decentralized mechanism for finding the equilibrium prices and quantities. We do not claim that the updating-, price- and quantity-rules are optimal for the dealer. For example, it is very likely that the dealer would prefer to use a dynamic strategy where the updating rule has time varying coefficients. Indeed, this highlights the main limitation of our static approach to model the OTC market: we cannot consider such dynamic strategies.

8 Conclusions

In this paper we present a model of strategic information diffusion in over-the-counter markets. In our set-up a dealer can trade any quantity of the asset she finds desirable, and understands that her trade may affect transaction prices. Moreover, she can decide to buy a certain quantity at a given price from one counterparty and sell a different quantity at a different price to another.

We show that the equilibrium price in each transaction partially aggregates the private information of all agents in the economy. The informational efficiency of prices is the highest in networks where each agent trades with every other agent, or in the common value limit, regardless of the network structure. Otherwise, agents tend to overweight their own signal compared to the outcome which would maximize information efficiency.

An attractive feature of our model is that it gives a rich set of empirical predictions. As an illustration for the possible range of applications, we compare the model generated economic indicators under different scenarios of changing economic environment with the stylized facts in recent empirical papers. We find more support for the arguments that the increased counterparty risk was the main determinant of the distress of OTC markets in the recent financial crisis, as opposed to narratives based on the deterioration of the informational environment.

For future research, one could use our model as the main building block for the analysis of endogenous network formation given that our model gives the payoffs for any given network. Alternatively, with a sufficiently detailed data-set in hand, one could use our framework for a structural estimation of the parameters in different states of the economy.

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A Appendix: Proofs

Throughout several of the following proofs we often decompose θ_i into a common value component, $\hat{\theta}$, and a private value component, η_i, such that

$$\theta_i = \hat{\theta} + \eta_i$$

and

$$s_i = \hat{\theta} + \eta_i + \varepsilon_i$$

with $\hat{\theta} \sim N(0, \sigma_{\hat{\theta}}^2)$, $\eta_i \sim IID\ N(0, \sigma_\eta^2)$ and $\mathcal{V}\left(\eta_i, \eta_j\right) = 0$. This implies that

$$(1-\rho)\,\sigma_\theta^2 = \sigma_\eta^2$$

Further, we generalize the notation $\mathcal{V}$ to be the variance-covariance operator applied to vectors of random variables. For instance, $\mathcal{V}(\mathbf{x})$ represents that variance-covariance matrix of vector $\mathbf{x}$, and $\mathcal{V}(\mathbf{x}, \mathbf{y})$ represents the covariance matrix between vector $\mathbf{x}$ and $\mathbf{y}$.

Proof of Proposition 1

We need the following Lemmas for the construction of our proof.

Lemma A.1 *Consider the jointly normally distributed variables $(\theta_i, \mathbf{s})$. Let an arbitrary weighting vector $\boldsymbol{\omega} > \mathbf{0}$. Consider the coefficient of s_i in the projection of $E\left(\theta|s_i\right)$. Adding $\boldsymbol{\omega}^T\mathbf{s}$ as a conditioning variable, additional to s_i, decreases the coefficient of s_i, that is,*

$$\frac{\partial E\left(\theta_i|s_i, \boldsymbol{\omega}^T\mathbf{s}\right)}{\partial s_i} < \frac{\partial E\left(\theta_i|s_i\right)}{\partial s_i}$$

Proof. From the projection theorem

$$E\left(\theta_i|s_i, \boldsymbol{\omega}^T\mathbf{s}\right) = E\left(\theta_i|s_i\right) + \frac{\mathcal{V}\left(\theta_i, \boldsymbol{\omega}^T\mathbf{s}|s_i\right)}{\mathcal{V}\left(\boldsymbol{\omega}^T\mathbf{s}|s_i\right)}\left(\boldsymbol{\omega}^T\mathbf{s} - E\left(\boldsymbol{\omega}^T\mathbf{s}|s_i\right)\right)$$

consequently

$$\frac{\partial E\left(\theta_i|s_i, \boldsymbol{\omega}^T\mathbf{s}\right)}{\partial s_i} = \frac{\partial E\left(\theta_i|s_i\right)}{\partial s_i} - \frac{\mathcal{V}\left(\theta_i, \boldsymbol{\omega}^T\mathbf{s}|s_i\right)}{\mathcal{V}\left(\boldsymbol{\omega}^T\mathbf{s}|s_i\right)}\frac{\mathcal{V}\left(s_i, \boldsymbol{\omega}^T\mathbf{s}\right)}{\mathcal{V}\left(s_i\right)}.$$

Thus, it is sufficient to show that $\mathcal{V}\left(\boldsymbol{\omega}^T\mathbf{s}, s_i\right) > 0$ and $\mathcal{V}\left(\theta_i, \boldsymbol{\omega}^T\mathbf{s}|s_i\right) > 0$. For the former, we know that

$$\mathcal{V}\left(\boldsymbol{\omega}^T\mathbf{s}, s_i\right) = \omega_i\left(\sigma_\varepsilon^2 + (1-\rho)\,\sigma_\theta^2\right) + \rho\sigma_\theta^2\boldsymbol{\omega}^T\mathbf{1} > 0$$

Then, we use the projection theorem to show that

$$\begin{aligned}\mathcal{V}\left(\theta_i, \boldsymbol{\omega}^T\mathbf{s}|s_i\right) &= \begin{pmatrix} \sigma_\theta^2 & \sigma_\theta^2\boldsymbol{\omega}^T\mathbf{1} \\ \sigma_\theta^2\boldsymbol{\omega}^T\mathbf{1} & \mathcal{V}\left(\boldsymbol{\omega}^T s\right)\end{pmatrix} \\ &\quad -\frac{1}{\sigma_\theta^2 + \sigma_\varepsilon^2}\begin{pmatrix} \sigma_\theta^2 \\ \left(\omega_i\left(\sigma_\varepsilon^2 + (1-\rho)\,\sigma_\theta^2\right) + \rho\sigma_\theta^2\boldsymbol{\omega}^T\mathbf{1}\right)\end{pmatrix}\begin{pmatrix} \sigma_\theta^2 & \left(\omega_i\left(\sigma_\varepsilon^2 + (1-\rho)\,\sigma_\theta^2\right) + \rho\sigma_\theta^2\boldsymbol{\omega}^T\mathbf{1}\right)\end{pmatrix}\end{aligned}$$

implying that

$$\mathcal{V}\left(\theta_i, \boldsymbol{\omega}^T\mathbf{s}|s_i\right) = \sigma_\theta^2\boldsymbol{\omega}^T\mathbf{1} - \frac{\left(\omega_i\left(\sigma_\varepsilon^2 + (1-\rho)\,\sigma_\theta^2\right) + \rho\sigma_\theta^2\boldsymbol{\omega}^T\mathbf{1}\right)\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\varepsilon^2} = \sigma_\theta^2\frac{\sigma_\varepsilon^2 + (1-\rho)\,\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\varepsilon^2}\left(\boldsymbol{\omega}^T\mathbf{1} - \omega_i\right) > 0.$$

■

Lemma A.2 *Take the jointly normally distributed system* $\begin{pmatrix}\hat{\theta}\\ \mathbf{x}\end{pmatrix}$ *where* $\mathbf{x} = \hat{\theta}\mathbf{1} + \boldsymbol{\varepsilon}' + \boldsymbol{\varepsilon}''$, *with the following properties*

- $E\begin{pmatrix}\hat{\theta}\\ \mathbf{x}\end{pmatrix} = 0$, $\mathcal{V}\left(\hat{\theta}, \boldsymbol{\varepsilon}'\right) = 0$ *and* $\mathcal{V}\left(\hat{\theta}, \boldsymbol{\varepsilon}''\right) = 0$;
- $\mathcal{V}(\boldsymbol{\varepsilon}')$ *is diagonal, and* $\mathcal{V}(\mathbf{x}) \geq \mathcal{V}\left(\hat{\theta}\mathbf{1} + \boldsymbol{\varepsilon}'\right)$.

Then the vector $\boldsymbol{\omega}$ *defined by*

$$E\left(\hat{\theta}|\mathbf{x}\right) = \boldsymbol{\omega}^T\mathbf{x}$$

has the properties that $\boldsymbol{\omega}^T\mathbf{1} < 1$ *and* $\boldsymbol{\omega} \in (0,1)^n$.

Proof. By the projection theorem, we have that

$$\boldsymbol{\omega}^T = \mathcal{V}\left(\hat{\theta}, \mathbf{x}\right)\left(\mathcal{V}(\mathbf{x})\right)^{-1}.$$

Then

$$\mathcal{V}\left(\hat{\theta}, \mathbf{x}\right)\left(\mathcal{V}(\mathbf{x})\right)^{-1} \leq \mathcal{V}\left(\hat{\theta}, \hat{\theta}\mathbf{1} + \boldsymbol{\varepsilon}' + \boldsymbol{\varepsilon}''\right)\left(\mathcal{V}\left(\hat{\theta}\mathbf{1} + \boldsymbol{\varepsilon}'\right)\right)^{-1} = \mathcal{V}\left(\hat{\theta}, \hat{\theta}\mathbf{1}\right)\left(\mathcal{V}\left(\hat{\theta}\mathbf{1} + \boldsymbol{\varepsilon}'\right)\right)^{-1}.$$

The inequality comes from the fact that both $\mathcal{V}(\mathbf{x})$ and $\mathcal{V}\left(\hat{\theta}\mathbf{1} + \hat{\boldsymbol{\varepsilon}}\right)$ are positive definite matrixes and that $\mathcal{V}(\mathbf{x}) \geq \mathcal{V}\left(\hat{\theta}\mathbf{1} + \hat{\boldsymbol{\varepsilon}}\right)$. (See Horn and Johnson (1985), Corollary 7.7.4(a)).

Since

$$\mathcal{V}\left(\hat{\theta}, \hat{\theta}\mathbf{1}\right)\left(\mathcal{V}\left(\hat{\theta}\mathbf{1} + \boldsymbol{\varepsilon}'\right)\right)^{-1}\mathbf{1} = 1 - \frac{\frac{1}{\sigma_{\hat{\theta}}^2}}{\frac{1}{\mathcal{V}(\boldsymbol{\varepsilon}')}\mathbf{1} + \frac{1}{\sigma_{\hat{\theta}}^2}} < 1,$$

then

$$\boldsymbol{\omega}^T\mathbf{1} < 1$$

which implies that

$$\boldsymbol{\omega} \in (0,1)^n.$$

■

Lemma A.3 *For any network* g, *define a mapping* $F : R^{n\times n} \to R^{n\times n}$ *as follows. Let* $\mathbf{V}$ *be an* $n \times n$ *matrix with rows* $\mathbf{v}_j$ *and*

$$e_j = \mathbf{v}_j\mathbf{s}$$

for each $j = 1, ...n$. The mapping $F(\mathbf{V})$ is given by

$$\begin{pmatrix} E(\theta_1|s_1, \mathbf{e}_{g_1}) \\ E(\theta_2|s_2, \mathbf{e}_{g_2}) \\ ... \\ E(\theta_n|s_n, \mathbf{e}_{g_n}) \end{pmatrix} = F(\mathbf{V})\,\mathbf{s}.$$

Then, the mapping F is a continuos self-map on the space $[0,1]^{n\times n}$.

Proof. Let

$$\mathbf{v}_j^0 = \begin{pmatrix} v_{j1}^0 & v_{j2}^0 & ... & v_{jn}^0 \end{pmatrix}$$

and consider that

$$\begin{aligned} e_j &= \mathbf{v}_j^0 \mathbf{s} \\ &= \mathbf{v}_j^0 \left(\hat{\theta}\mathbf{1} + \boldsymbol{\varepsilon} + \boldsymbol{\eta}\right) \end{aligned}$$

where $\mathbf{1}$ is a column vector of ones and

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 & \eta_2 & ... & \eta_n \end{pmatrix}^T$$

and

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & ... & \varepsilon_n \end{pmatrix}^T$$

Let

$$\hat{e}_j = \frac{e_j}{\mathbf{v}_j^0 \mathbf{1}} = \hat{\theta} + \frac{\mathbf{v}_j^0}{\mathbf{v}_j^0 \mathbf{1}}(\boldsymbol{\varepsilon} + \boldsymbol{\eta})$$

and

$$\hat{\mathbf{e}}_{g_i} = (\hat{e}_j)_{j\in g_i}$$

To prove the result, we apply Lemma A.2 for each $E(\theta_i|s_i, \mathbf{e}_{g_i})$. In particular, for each i, we construct a vector $\boldsymbol{\varepsilon}'_{g_i}$ with the first element $(\varepsilon_i + \eta_i)$ and the j-th element equal to $\frac{v_{jj}^0}{\mathbf{v}_j^0 \mathbf{1}}(\varepsilon_j + \eta_j)$ with $j \in g_i$, and a vector $\boldsymbol{\varepsilon}''_{g_i}$ with the first element 0 and the j-th element equal to $\frac{\mathbf{v}_j^0}{\mathbf{v}_j^0 \mathbf{1}}\left(\boldsymbol{\varepsilon} + \boldsymbol{\eta} - (\varepsilon_j + \eta_j)\mathbf{1}_j\right)$ with $j \in g_i$ ($\mathbf{1}_j$ is a column vector of 0 and 1 at position j). Then, we have that

$$\begin{pmatrix} s_i \\ \hat{\mathbf{e}}_{g_i} \end{pmatrix} = \hat{\theta}\mathbf{1} + \boldsymbol{\varepsilon}'_{g_i} + \boldsymbol{\varepsilon}''_{g_i}$$

Below, we show that the conditions in Lemma A.2 apply.

First, by construction, $\boldsymbol{\varepsilon}'_{g_i}$ has a diagonal variance-covariance matrix. Next, we also show that $\mathcal{V}\begin{pmatrix} s_i \\ \hat{\mathbf{e}}_{g_i} \end{pmatrix} \geq \mathcal{V}\left(\hat{\theta}\mathbf{1} + \boldsymbol{\varepsilon}'_{g_i}\right)$ element by element. Indeed

$$\begin{aligned}
\mathcal{V}(\hat{e}_j) &= \sigma_{\hat{\theta}}^2 + \left(\frac{v_{jj}^0}{\mathbf{v}_j^0\mathbf{1}}\right)^2 \left(\sigma_\varepsilon^2 + \sigma_\eta^2\right) + \left(\frac{\mathbf{v}_j^0}{\mathbf{v}_j^0\mathbf{1}}\right) \mathcal{V}\left(\boldsymbol{\varepsilon} + \boldsymbol{\eta} - \left(\varepsilon_j + \eta_j\right)\mathbf{1}_j\right) \left(\frac{\mathbf{v}_j^0}{\mathbf{v}_j^0\mathbf{1}}\right)^T \\
&\quad + \frac{v_{jj}^0}{\mathbf{v}_j^0\mathbf{1}} \mathcal{V}\left(\left(\varepsilon_j + \eta_j\right), \left(\boldsymbol{\varepsilon} + \boldsymbol{\eta} - \left(\varepsilon_j + \eta_j\right)\mathbf{1}_j\right)\right) \left(\frac{\mathbf{v}_j^0}{\mathbf{v}_j^0\mathbf{1}}\right)^T
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{V}(\hat{e}_j, \hat{e}_k) &= \sigma_{\hat{\theta}}^2 + \frac{v_{kk}^0}{\mathbf{v}_k^0\mathbf{1}} \mathcal{V}\left(\left(\varepsilon_k + \eta_k\right), \left(\boldsymbol{\varepsilon} + \boldsymbol{\eta} - \left(\varepsilon_j + \eta_j\right)\mathbf{1}_j\right)\right) \left(\frac{\mathbf{v}_j^0}{\mathbf{v}_j^0\mathbf{1}}\right)^T \\
&\quad + \frac{v_{jj}^0}{\mathbf{v}_j^0\mathbf{1}} \mathcal{V}\left(\left(\varepsilon_j + \eta_j\right), \left(\boldsymbol{\varepsilon} + \boldsymbol{\eta} - \left(\varepsilon_k + \eta_k\right)\mathbf{1}_j\right)\right) \left(\frac{\mathbf{v}_k^0}{\mathbf{v}_k^0\mathbf{1}}\right)^T \\
&\quad + \frac{\mathbf{v}_j^0}{\mathbf{v}_j^0\mathbf{1}} \mathcal{V}\left(\left(\boldsymbol{\varepsilon} + \boldsymbol{\eta} - \left(\varepsilon_j + \eta_j\right)\mathbf{1}_j\right), \left(\boldsymbol{\varepsilon} + \boldsymbol{\eta} - \left(\varepsilon_k + \eta_k\right)\mathbf{1}_j\right)\right) \left(\frac{\mathbf{v}_k^0}{\mathbf{v}_k^0\mathbf{1}}\right)^T,
\end{aligned}$$

which implies that

$$\mathcal{V}(\hat{e}_j) > \sigma_{\hat{\theta}}^2 + \left(\frac{v_{jj}^0}{\mathbf{v}_j^0\mathbf{1}}\right)^2 \left(\sigma_\varepsilon^2 + \sigma_\eta^2\right) \tag{A.1}$$

and

$$\mathcal{V}(\hat{e}_j, \hat{e}_k) > \sigma_{\hat{\theta}}^2. \tag{A.2}$$

This is because

$$\mathcal{V}\left(\left(\varepsilon_j + \eta_j\right), \left(\boldsymbol{\varepsilon} + \boldsymbol{\eta} - \left(\varepsilon_j + \eta_j\right)\mathbf{1}_j\right)\right) = 0$$

and

$$\mathcal{V}\left(\eta_i, \eta_j\right) = 0 \text{ and } \mathcal{V}(\varepsilon_i, \varepsilon_j) = 0 \ \forall i, j.$$

Moreover,

$$\mathcal{V}(s_i, \hat{e}_j) = \sigma_{\hat{\theta}}^2 + \frac{v_{ji}^0}{\left(\mathbf{v}_j^0\right)^T \mathbf{1}} \left(\sigma_\varepsilon^2 + \sigma_\eta^2\right) > \sigma_{\hat{\theta}}^2. \tag{A.3}$$

From (A.1), (A.2), and (A.3), it follows that $\mathcal{V}\begin{pmatrix} s_i \\ \hat{\mathbf{e}}_{g_i} \end{pmatrix} \geq \mathcal{V}\left(\hat{\theta}\mathbf{1}^T + \boldsymbol{\varepsilon}'_{g_i}\right)$. Then, for each i there exists a column vector $\boldsymbol{\omega}_{g_i} = (\omega_{ij})_{j\in\{i\cup g_i\}}$with the properties that $\boldsymbol{\omega}_{g_i}^T\mathbf{1} < 1$ and $\boldsymbol{\omega}_{g_i} \in (0,1)^{m_i+1}$, such that

$$E\left(\hat{\theta}|s_i, \hat{\mathbf{e}}_{g_i}\right) = \boldsymbol{\omega}_{g_i}^T \begin{pmatrix} s_i \\ \hat{\mathbf{e}}_{g_i} \end{pmatrix}.$$

It is immediate that

$$E\left(\theta_i|s_i, \mathbf{e}_{g_i}\right) = E\left(\theta_i|s_i, \hat{\mathbf{e}}_{g_i}\right) = E\left(\hat{\theta}|s_i, \hat{\mathbf{e}}_{g_i}\right) + E\left(\eta_i|s_i\right)$$

where

$$E\left(\eta_i|s_i\right)=\frac{\sigma_\eta^2}{\sigma_{\hat{\theta}}^2+\sigma_\eta^2+\sigma_\varepsilon^2}s_i.$$

Then, from Lemma A.1,

$$v_{ii}=\omega_{ii}+\sum_{k\in g_i}\omega_{ik}\frac{v_{ki}^0}{\mathbf{v}_k^0\mathbf{1}}+\frac{\sigma_\eta^2}{\sigma_{\hat{\theta}}^2+\sigma_\eta^2+\sigma_\varepsilon^2}=\frac{\partial E\left(\theta_i|s_i,\mathbf{e}_{g_i}\right)}{\partial s_i}<\frac{\partial E\left(\theta_i|s_i\right)}{\partial s_i}<1 \tag{A.4}$$

and

$$v_{ij}=\sum_{k\in g_i}\omega_{ik}\frac{v_{kj}^0}{\mathbf{v}_k^0\mathbf{1}}<\sum_{k\in g_i}\omega_{ik}<1,\forall j\in g_i. \tag{A.5}$$

Thus far, we have used only that $v_{ij}^0\geq 0$ (and not that $v_{ij}^0\in[0,1]$). This implies that if $v_{ij}^0\geq 0$, then $v_{ij}\in[0,1]$. Then it must be true also that $\forall v_{ij}^0\in[0,1]$, then $v_{ij}\in[0,1]$. This concludes the proof. ■

Now we are ready to prove the statement.

An equilibrium exists if there exists a matrix V and matrices $\bar{Y}$ and $\bar{Z}$ such that

$$V\mathbf{s}=\left(\bar{Y}+\bar{Z}V\right)\mathbf{s},$$

and

$$F\left(V\right)\mathbf{s}=\left(\bar{Y}+\bar{Z}V\right)\mathbf{s},$$

where $F\left(\cdot\right)$ is the mapping introduced in Lemma A.3. The first condition insures that

$$\mathbf{e}=V\mathbf{s}$$

is a fixed point in (10), and the second condition insures that first order conditions (11) are satisfied.

We construct an equilibrium for $\rho<1$ and for $\rho=1$ as follows.

Case 1: $\rho<1$

By Brower's fixed point theorem, the mapping $F\left(\cdot\right)$ admits a fixed point on $[0,1]^{n\times n}$. Let $V^*\in[0,1]^{n\times n}$ be a matrix such that

$$F(V^*)=V^*.$$

Let $\bar{Y}$ be a diagonal matrix with elements

$$\bar{y}_i=\omega_{ii}+\frac{\sigma_\eta^2}{\sigma_{\hat{\theta}}^2+\sigma_\eta^2+\sigma_\varepsilon^2}$$

and let $\bar{Z}$ have elements

$$\bar{z}_{ij}=\begin{cases}\frac{\omega_{ij}}{\left(\mathbf{v}_j^*\right)^T\mathbf{1}}, & \text{if } ij\in g\\ 0, & \text{otherwise}\end{cases}$$

where ω_{ii} and ω_{ij} have been introduced the proof above. Both matrices $\bar{Y}\geq 0$ and $\bar{Z}\geq 0$.

Substituting V^* in (A.4) and (A.5), it follows that

$$V^* = \bar{Y} + \bar{Z}V^*,$$

and since $F(V^*) = V^*$, then

$$F(V^*) = \bar{Y} + \bar{Z}V^*.$$

Case 2: $\rho = 1$

Let $\bar{Y} = 0$ and $\bar{Z}$ have elements $\bar{z}_{ij}$ if $ij \in g$, and 0 otherwise, with $\sum_{j \in g_i} \bar{z}_{ij} = 1$. Let V^* be a matrix with $v^*_{ij} = \frac{\sigma^2_{\hat{\theta}}}{n\sigma^2_{\hat{\theta}} + \sigma^2_{\varepsilon}}$ for any i and j.

It is straightforward to see that

$$V^* = \bar{Z}V^*.$$

Next we show that

$$F(V^*) = V^*.$$

Next, we show that the matrix V^* with $v^*_{ij} = \frac{\sigma^2_{\hat{\theta}}}{n\sigma^2_{\hat{\theta}} + \sigma^2_{\varepsilon}}$ for any i and j satisfies

$$F(V^*) = V^*.$$

By definition, matrix V^* with $v^*_{ij} = \frac{\sigma^2_{\hat{\theta}}}{n\sigma^2_{\hat{\theta}} + \sigma^2_{\varepsilon}}$ is a fixed point of the mapping $F(\cdot)$, if

$$e_i = v^* \sum_{i=1}^{n} s_j, \ \forall i \in \{1, 2, ..., n\}$$

then

$$E\left(\theta_i | s_i, \mathbf{e}_{g_i}\right) = v^* \sum_{j=1}^{n} s_j, \forall i \in \{1, 2, ..., n\}.$$

As $\rho = 1$, then $\theta_i = \hat{\theta}$ for any i and

$$\begin{aligned} E\left(\theta_i | s_i, \mathbf{e}_{g_i}\right) &= E\left(\hat{\theta} | v^* \sum_{i=1}^{n} s_i\right) \\ &= \frac{1}{v^*} \frac{\sigma^2_{\hat{\theta}}}{n\sigma^2_{\hat{\theta}} + \sigma^2_{\varepsilon}} v^* \sum_{i=1}^{n} s_i \\ &= v^* \sum_{i=1}^{n} s_i. \end{aligned}$$

It follows that

$$F\left(V^*\right) = \bar{Z}V^*.$$

This concludes the proof.

In the remaining proofs, we use the results in the following Lemma

Lemma A.4 *In the conditional guessing game, the following properties hold whenever $\rho < 1$.*

1. *$\bar{Y} \in (0,1]^{n\times n}$,*

2. *$\lim_{n\to\infty} \bar{Z}^n = 0$ and $\left(I - \bar{Z}\right)$ is invertible, and*

3. *$\mathbf{e} = \left(I - \bar{Z}\right)^{-1} \bar{Y}\mathbf{s}$, where $\mathbf{e} = (e_i)_{i\in\{1,2,\ldots,n\}}$, $\bar{Y}$ is a matrix with elements $\bar{y}_i$ on the diagonal and 0 otherwise, and $\bar{Z}$ is a matrix with elements $\bar{z}_{ij}$, when i and $j (\neq i)$ have a link and 0 otherwise.*

4. *The coefficients in matrices $V, \bar{Y}$ and $\bar{Z}$ depend on σ_ε^2 and σ_θ^2 only through the ratio $\frac{\sigma_\varepsilon^2}{\sigma_\theta^2}$*

Proof.

1. From (A.4) it also follows that

$$\bar{y}_i < \frac{\partial E\left(\theta_i | s_i, \mathbf{e}_{g_i}\right)}{\partial s_i} < 1.$$

Moreover, as $\rho < 1$, then $\sigma_\eta^2 > 0$, which implies that $\bar{y}_i > 0$. It follows that $\bar{Y}$ is invertible.

2. We first show that matrix V^* is nonsingular. For this we construct a matrix $W^* = \bar{Y}^{-1}\left(I - \bar{Z}\right)$ and show that $W^*V^* = I$. Indeed, the element on the position (i,i) on the diagonal of W^*V^* is equal to

$$\begin{aligned} \frac{1}{\bar{y}_i}\left(v_{ii}^* - \sum_{k\in g_i} \bar{z}_{ik} v_{ki}^*\right) &= \frac{1}{\bar{y}_i}\left(\omega_{ii} + \sum_{k\in g_i} \omega_{ik} \frac{v_{ki}^*}{\left(\mathbf{v}_k^*\right)^T \mathbf{1}} + \frac{\sigma_\eta^2}{\sigma_{\hat{\theta}}^2 + \sigma_\eta^2 + \sigma_\varepsilon^2} - \sum_{k\in g_i} \frac{\omega_{ik}}{\left(\mathbf{v}_k^*\right)^T \mathbf{1}} v_{ki}^*\right) \\ &= \frac{1}{\bar{y}_i}\left(\omega_{ii} + \frac{\sigma_\eta^2}{\sigma_{\hat{\theta}}^2 + \sigma_\eta^2 + \sigma_\varepsilon^2}\right) \\ &= 1 \end{aligned}$$

while the element on the position (i,j) off the diagonal of W^*V^* is equal to

$$\frac{1}{\bar{y}_i}\left(v_{ij}^* - \sum_{k\in g_i} \bar{z}_{ik} v_{kj}^*\right) = \frac{1}{\bar{y}_i}\left(\sum_{k\in g_i} \omega_{ik} \frac{v_{kj}^*}{\left(\mathbf{v}_k^*\right)^T \mathbf{1}} - \sum_{k\in g_i} \frac{\omega_{ik}}{\left(\mathbf{v}_k^*\right)^T \mathbf{1}} v_{kj}^*\right) = 0$$

where we used again the fact that V^* is a fixed point in (A.4) and (A.5). Since V^* is nonsingular, then $\left(I - \bar{Z}\right)$ is also nonsingular as

$$\left(I - \bar{Z}\right) = \bar{Y}\left(V^*\right)^{-1}.$$

Given that $\bar{Z} \geq 0$ and $\left(I - \bar{Z}\right)^{-1} \geq 0$ (which follows from above) this implies, as shown in Meyer (2000), that the largest eigenvalue of $\bar{Z}$ is strictly smaller than 1. This is a useful result, as it is sufficient to show that

$$\lim_{n\to\infty} \bar{Z}^n = 0_{n\times n}$$

and that

$$\left(I-\bar{Z}\right)^{-1}=\sum_{n=1}^{\infty}\bar{Z}^{n}.$$

(For both claims see Meyer (2000) pp. 620 & 618.)

3. The equilibrium outcome guess vector is, by construction

$$\mathbf{e}=V^{*}\mathbf{s}$$

which implies that

$$\mathbf{e}=\left(I-\bar{Z}\right)^{-1}\bar{Y}.$$

4. To see the result, observe that the mapping $F(V)$ is invariant in $\frac{\sigma_\theta^2}{\sigma_\varepsilon^2}$. To see this, consider the part of the mapping $F(V)$ which determines the first column of $F(V)$ as an example. If V^1 is a matrix with n rows and m_1 columns containing the columns of V corresponding to g_1, then the transformation which determines the first column of $F(V)$ is

$$\begin{aligned}
& \left[\begin{array}{cccc}\sigma_\theta^2 & \rho\sigma_\theta^2\times 1_{1\times n-1} & i_1 & V^1\end{array}\right]\left(\left[\begin{array}{c}i_1\\ V^1\end{array}\right]\left(\rho\sigma_\theta^2\times 1_{n\times n}+diag\left(\left(1-\rho\right)\sigma_\theta^2+\sigma_\varepsilon^2\right)\times 1_{1\times n}\right)\left[\begin{array}{cc}i_1 & V^1\end{array}\right]\right)^{-1} \\
= & \left[\begin{array}{cccc}1 & \rho\times 1_{1\times n-1} & i_1 & V^1\end{array}\right]\left(\left[\begin{array}{c}i_1\\ V^1\end{array}\right]\left(\rho\times 1_{n\times n}+diag\left(\left(1-\rho\right)+\frac{\sigma_\varepsilon^2}{\sigma_\theta^2}\right)\times 1_{1\times n}\right)\left[\begin{array}{cc}i_1 & V^1\end{array}\right]\right)^{-1}
\end{aligned}$$

where i_1 is the first column of the identity matrix and the operator $diag(\cdot)$ forms a diagonal matrix with the given vector in the diagonal. By the same argument, each columns of $F(V)$ depend only on $\frac{\sigma_\varepsilon^2}{\sigma_\theta^2}$ and not on the σ_ε^2 and σ_θ^2 separately.

■

Proof of Proposition 2

In an equilibrium of the OTC game, prices and quantities satisfy the first order conditions (8) and must be such that all bilateral trades clear.

Since market clearing conditions (4) are linear in prices and signals, we know that each price (if an equilibrium price vector exists) must be a certain linear combination of signals. Thus, each price is normally distributed.

From the first order conditions we have that

$$q_i^j(s_i,\mathbf{p}_{g_i})=-\left(c_{ji}^i+\beta_{ij}\right)\left(E(\theta_i\left|s_i,\mathbf{p}_{g_i}\right.)-p_{ij}\right).$$

The bilateral clearing condition between a trader i and trader j that have a link in network g implies that

$$-\left(c_{ji}^i+\beta_{ij}\right)\left(E(\theta_i\left|s_i,\mathbf{p}_{g_i}\right.)-p_{ij}\right)-\left(c_{ij}^j+\beta_{ij}\right)\left(E(\theta_j\left|s_j,\mathbf{p}_{g_j}\right.)-p_{ij}\right)+\beta_{ij}p_{ij}=0$$

and solving for the price p_{ij} we have that

$$p_{ij} = \frac{\left(c^i_{ji} + \beta_{ij}\right) E(\theta_i \left| s_i, \mathbf{p}_{g_i}\right.) + \left(c^j_{ij} + \beta_{ij}\right) E(\theta_j \left| s_j, \mathbf{p}_{g_j}\right.)}{c^i_{ji} + c^j_{ij} + 3\beta_{ij}}$$

Since agent i knows $E(\theta_i \left| s_i, \mathbf{p}_{g_i}\right.)$, by definition, the vector of prices $\mathbf{p}_{g_i}$ is informationally equivalent for her with the vector of posteriors of her neighbors $\mathbf{E}_{g_i} = \left\{E(\theta_j \left| s_j, \mathbf{p}_{g_j}\right.)\right\}_{j \in g_i}$. This implies that

$$E(\theta_i \left| s_i, \mathbf{p}_{g_i}\right.) = E(\theta_i \left| s_i, \mathbf{E}_{g_i}\right.).$$

Note also that as each price is a linear combination of signals and $E\left(\theta_j|\cdot\right)$ is a linear operator on jointly normal variables, there must be a vector $\mathbf{w}_i$ that $E(\theta_i \left| s_i, \mathbf{p}_{g_i}\right.) = E(\theta_i \left| s_i, \mathbf{E}_{g_i}\right.) = \mathbf{w}_i\mathbf{s}$. That is, the collection of $\{\mathbf{w}_i\}_{i=1,...n}$ has to satisfy the system of n equations given by

$$\mathbf{w}_i\mathbf{s} = E(\theta_i \left| s_i, \{\mathbf{w}_j\mathbf{s}\}_{j \in g_i}\right.)$$

for every i. However, the collection $\{\mathbf{w}_i\}_{i=1,...n}$ that is a solution of this system, is also an equilibrium of the conditional guessing game by construction.

Lemma A.5 *Suppose that there exists a linear equilibrium in the OTC game in which the belief of each dealer i is given by*

$$E(\theta_i|s_i, \mathbf{p}_{g_i}) = y_i s_i + \sum_{j \in g_i} z_{ij} p_{ij}.$$

Then the equilibrium demand functions

$$Q^j_i(s_i; \mathbf{p}_{g_i}) = b^j_i s_i + \left(\mathbf{c}^\mathbf{j}_\mathbf{i}\right)^T \mathbf{p}_{g_i}$$

must be such that

$$\begin{aligned} b^j_i &= -\beta_{ij} \tfrac{2 - z_{ji}}{z_{ij} + z_{ji} - z_{ij} z_{ji}} y_i \\ c^j_{ij} &= -\beta_{ij} \tfrac{2 - z_{ji}}{z_{ij} + z_{ji} - z_{ij} z_{ji}} \left(z_{ij} - 1\right) \\ c^j_{ik} &= -\beta_{ij} \tfrac{2 - z_{ji}}{z_{ij} + z_{ji} - z_{ij} z_{ji}} z_{ik}. \end{aligned}$$

Proof. Taking into account that agents' beliefs have an affine structure and identifying coefficients in (8) we obtain that

$$\begin{aligned} b^j_i &= -\left(c^i_{ji} + \beta_{ij}\right) y_i \\ c^j_{ij} &= -\left(c^i_{ji} + \beta_{ij}\right) (z_{ij} - 1) \\ c^j_{ik} &= -\left(c^i_{ji} + \beta_{ij}\right) z_{ik} \end{aligned}$$

for any i and $j \in g_i$. Therefore, for any pair ij that has a link in the network g, the following two equations must hold at the same time

$$\begin{aligned} c_{ij}^{j} &= -\left(c_{ji}^{i}+\beta_{ij}\right)\left(z_{ij}-1\right) \\ c_{ji}^{i} &= -\left(c_{ij}^{j}+\beta_{ij}\right)\left(z_{ji}-1\right). \end{aligned}$$

which implies that

$$\begin{aligned} c_{ij}^{j} &= \frac{\left(z_{ij}-1\right)\left(z_{ji}-2\right)}{z_{ij}+z_{ji}-z_{ij}z_{ji}}\beta_{ij} \\ c_{ji}^{i} &= \frac{\left(z_{ji}-1\right)\left(z_{ij}-2\right)}{z_{ij}+z_{ji}-z_{ij}z_{ji}}\beta_{ij}. \end{aligned}$$

A simple manipulations shows that

$$c_{ji}^{i}+\beta_{ij}=\frac{2-z_{ji}}{z_{ij}+z_{ji}-z_{ij}z_{ji}}\beta_{ij}$$

and

$$\frac{\left(c_{ji}^{i}+\beta_{ij}\right)}{c_{ji}^{i}+c_{ij}^{j}+3\beta_{ij}}=\frac{2-z_{ji}}{4-z_{ij}z_{ji}}.$$

■

Proof of Proposition 3, Lemma 1 and Corollary 1

We show that given an equilibrium of the conditional-guessing game and the conditions of the proposition, we can always construct an equilibrium for the OTC game with beliefs given by

$$E(\theta_i\,|s_i,\mathbf{p}_{g_i})=E(\theta_i\,|s_i,\mathbf{e}_{g_i}).$$

To see this, consider an equilibrium of the conditional-guessing game in which

$$E(\theta_i\,|s_i,\mathbf{e}_{g_i})=\bar{y}_i s_i+\sum_{k\in g_i}\bar{z}_{ik}E(\theta_k\,|s_k,\mathbf{e}_{g_k})$$

for every i. If the system (15) has a solution, then

$$E(\theta_i\,|s_i,\mathbf{e}_{g_i})=\frac{y_i}{\left(1-\sum_{k\in g_i}z_{ik}\frac{2-z_{ki}}{4-z_{ik}z_{ki}}\right)}s_i+\sum_{k\in g_i}z_{ij}\frac{\frac{2-z_{ij}}{4-z_{ij}z_{ji}}}{\left(1-\sum_{k\in g_i}z_{ik}\frac{2-z_{ki}}{4-z_{ik}z_{ki}}\right)}E(\theta_k\,|s_k,\mathbf{e}_{g_k}) \quad \text{(A.6)}$$

holds for every realization of the signals, and for each i. Using that from Lemma A.5

$$\frac{2-z_{ki}}{4-z_{ik}z_{ki}}=\frac{\left(c_{ki}^{i}+\beta_{ik}\right)}{c_{ki}^{i}+c_{ik}^{k}+3\beta_{ik}},$$

we can rewrite (A.6) as

$$E(\theta_i\,|s_i,\mathbf{e}_{g_i}) = y_i s_i + \sum_{k\in g_i} z_{ik}\frac{\left(c_{ki}^i+\beta_{ik}\right)E(\theta_i\,|s_i,\mathbf{e}_{g_i}) + \left(c_{ik}^k+\beta_{ik}\right)E(\theta_k\,|s_k,\mathbf{e}_{g_k})}{c_{ki}^i+c_{ik}^k+3\beta_{ik}}.$$

Now we show that picking the prices and demand functions

$$p_{ij} = \frac{\left(c_{ki}^i+\beta_{ik}\right)E(\theta_i\,|s_i,\mathbf{e}_{g_i}) + \left(c_{ik}^k+\beta_{ik}\right)E(\theta_k\,|s_k,\mathbf{e}_{g_k})}{c_{ki}^i+c_{ik}^k+3\beta_{ik}} \tag{A.7}$$

$$Q_i^j(s_i;\mathbf{p}_{g_i}) = -\left(c_{ji}^i+\beta_{ij}\right)\left(E(\theta_i\,|s_i,\mathbf{e}_{g_i}) - p_{ij}\right) \tag{A.8}$$

is an equilibrium of the OTC game.

First note that this choice implies

$$E(\theta_i\,|s_i,\mathbf{e}_{g_i}) = y_i s_i + \sum_{k\in g_i} z_{ik}p_{ij} = E(\theta_i\,|s_i,\mathbf{p}_{g_i}). \tag{A.9}$$

The second equality comes from the fact that the first equality holds for any realization of signals and the projection theorem determines a unique linear combination with this property for a given set of jointly normally distributed variables. Thus, (A.8) for each ij link is equivalent with the corresponding first order condition (8). Finally, (A.9) also implies that the bilateral clearing condition between a dealer i and dealer j that have a link in network g

$$-\left(c_{ji}^i+\beta_{ij}\right)\left(E(\theta_i\,|s_i,\mathbf{p}_{g_i}) - p_{ij}\right) - \left(c_{ij}^j+\beta_{ij}\right)\left(E(\theta_j\,|s_j,\mathbf{p}_{g_j}) - p_{ij}\right) + \beta_{ij}p_{ij} = 0$$

is equivalent to (A.7). That concludes the statement.

Proof of Lemma 3.

The Lemma is a simple consequence of result 4 in Lemma A.4 and the expressions for all the coefficients in the statement.

■

Proof of Corollary 1.

This follows straightforwardly from Lemma A.5.Substituting b_i^j and c_{ij}^j in (17) and (18) we obtain that

$$p_{ij} = \frac{(2-z_{ji})\,e_i + (2-z_{ij})\,e_j}{4-z_{ij}z_{ji}}$$

$$Q_i^j(s_i;\mathbf{p}_{g_i}) = -\frac{2-z_{ji}}{z_{ij}+z_{ji}-z_{ij}z_{ji}}\beta_{ij}\left(e_i-p_{ij}\right).$$

This implies that prices do not depend on β_{ij} and that equilibrium quantities are linear in β_{ij}.

■

Proof of Proposition 4

Case 1: Circulant networks

In circulant networks, it is easy to see that beliefs must be symmetric in the equilibrium of the conditional guessing game in the sense of

$$\bar{z}_{ij} = \bar{z}_{ji}.$$

The proof develops in two steps.

Step 1: First, we show that each $\bar{z}_{ij} \in [0,1]$. To see this, consider any symmetric matrix X with each of its elements non-negative. Suppose that there exists a $x_{ij} = x_{ji} \geq 1$ (including the case when $i = j$). It is easy to see that in X^2 the diagonal element on the i-th row is

$$\sum_{k=1}^{n} x_{ik}^2 \geq 1.$$

As $\bar{Z}$ is also symmetric and non-negative element by element, a simple induction argument shows that there cannot be a $\bar{z}_{ij} \geq 1$ element, because in this case in all matrices $\bar{Z}^2, \bar{Z}^4, \bar{Z}^8 ... \bar{Z}^{2^k}$ there is at least one diagonal not smaller than 1. This would be in contradiction with the property that $\bar{Z}^n \underset{n\to\infty}{\to} 0$ shown in Lemma A.4.

Step 2: Now, we search for equilibria such that beliefs are symmetric, that is

$$z_{ij} = z_{ji}$$

for any pair ij that has a link in network g.

The system (15) becomes

$$\begin{aligned}
\frac{y_i}{\left(1 - \sum\limits_{k \in g_i} z_{ik} \frac{2 - z_{ik}}{4 - z_{ik}^2}\right)} &= \bar{y}_i \\
z_{ij} \frac{\frac{2 - z_{ij}}{4 - z_{ij}^2}}{\left(1 - \sum\limits_{k \in g_i} z_{ik} \frac{2 - z_{ik}}{4 - z_{ik}^2}\right)} &= \bar{z}_{ij}
\end{aligned}$$

for any $i \in \{1, 2, ..., n\}$. Working out the equation for z_{ij}, we obtain

$$\frac{z_{ij}}{2 + z_{ij}} = \bar{z}_{ij} \left(1 - \sum_{k \in g_i} \frac{z_{ik}}{2 + z_{ik}}\right)$$

and summing up for all $j \in g_i$

$$\sum_{j \in g_i} \frac{z_{ij}}{2 + z_{ij}} = \sum_{j \in g_i} \bar{z}_{ij} \left(1 - \sum_{k \in g_i} \frac{z_{ik}}{2 + z_{ik}}\right).$$

Denote

$$S_i \equiv \sum_{k \in g_i} \frac{z_{ik}}{2 + z_{ik}}.$$

Substituting above and summing again for $j \in g_i$

$$S_i \left(1 + \sum_{j \in g_i} \bar{z}_{ij}\right) = \sum_{j \in g_i} \bar{z}_{ij}$$

or

$$S_i = \frac{\sum_{j \in g_i} \bar{z}_{ij}}{\left(1 + \sum_{j \in g_i} \bar{z}_{ij}\right)}.$$

We can now obtain

$$z_{ij} = \frac{2\bar{z}_{ij}\left(1 - S_i\right)}{1 - \bar{z}_{ij}\left(1 - S_i\right)} \tag{A.10}$$

and

$$y_i = \bar{y}_i \left(1 - S_i\right).$$

Finally, the following logic show that $z_{ij} \leq 2$. As $\bar{z}_{ij} < 1$, $2\bar{z}_{ij} < \left(1 + \sum_{j \in g_i} \bar{z}_{ij}\right)$ implying that $2\bar{z}_{ij}\left(1 - S_i\right) < 1$ or $2\bar{z}_{ij}\left(1 - S_i\right) < 2\left(1 - \bar{z}_{ij}\left(1 - S_i\right)\right)$, which gives the result by A.10.

Case 2: Star networks

Without loss of generality, we characterize a star network with dealer 1 at the centre. There exist at least one equilibrium of the conditional guessing game such that for dealer 1

$$\bar{z}_{1i} = \bar{z}_C \tag{A.11}$$

for any i. Similarly, for any dealer i in the periphery

$$\bar{z}_{i1} = \bar{z}_P.$$

The system (15) becomes

$$\frac{y_C}{1 - (n-1)\, z_C \frac{2 - z_P}{4 - z_C z_P}} = \bar{y}_C \tag{A.12}$$

$$z_C \frac{\frac{2 - z_C}{4 - z_C z_P}}{1 - (n-1)\, z_C \frac{2 - z_P}{4 - z_C z_P}} = \bar{z}_C \tag{A.13}$$

for the central agent and

$$\frac{y_P}{1 - z_P \frac{2 - z_C}{4 - z_P z_C}} = \bar{y}_P \tag{A.14}$$

$$z_P \frac{\frac{2 - z_P}{4 - z_P z_C}}{1 - z_P \frac{2 - z_C}{4 - z_P z_C}} = \bar{z}_P \tag{A.15}$$

for agents in the periphery.

The proof develops in two steps.

Step 1: We first solve for the coefficients $(\bar{y}_C, \bar{z}_C)$ and $(\bar{y}_P, \bar{z}_P)$ and show that they are smaller than 1.

We start with dealer 1, who chooses her demand function conditional on the beliefs of the other $(n-1)$ dealers. Given that she knows s_1, she can invert the signals of all the other dealers. Hence, her belief is given by

$$E(\theta_1|s_1, \mathbf{e}_{g_1}) = E(\theta_1|\mathbf{s}) = \frac{1-\rho}{1+\sigma^2-\rho}\left(s_1 + \frac{\rho\sigma^2}{(1-\rho)(1+\sigma^2-\rho+n\rho)}\sum_{i=1}^{n} s_i\right).$$

where we denote $\sigma^2 \equiv \frac{\sigma_\varepsilon^2}{\sigma_\theta^2}$. Given that

$$E(\theta_1|s_1, \mathbf{e}_{g_1}) = v_{11}s_1 + \sum_{j=2}^{n} v_{1j}s_j,$$

this implies that

$$v_{11} = \frac{1-\rho}{1+\sigma^2-\rho}\left(1 + \frac{\rho\sigma^2}{(1-\rho)(1+\sigma^2-\rho+n\rho)}\right) \tag{A.16}$$

$$v_{1j} = \frac{1-\rho}{1+\sigma^2-\rho}\frac{\rho\sigma^2}{(1-\rho)(1+\sigma^2-\rho+n\rho)} \tag{A.17}$$

for all $j \neq 1$.

Further, the belief of a periphery dealer i is given by

$$E(\theta_i|s_i, e_1) = \begin{pmatrix} 1 \\ \tilde{\mathcal{V}}(\theta_i, e_1) \end{pmatrix}^T \begin{pmatrix} 1+\sigma^2 & \tilde{\mathcal{V}}(s_i, e_1) \\ \tilde{\mathcal{V}}(s_i, e_1) & \mathcal{V}(e_1) \end{pmatrix}^{-1} \begin{pmatrix} s_i \\ e_1 \end{pmatrix},$$

where $\tilde{\mathcal{V}}(\cdot,\cdot) \equiv \frac{\mathcal{V}(\cdot,\cdot)}{\sigma_\theta^2}$ is the scaled covariance operator and

$$\tilde{\mathcal{V}}(e_1) = \frac{(1-\rho)(1+(n-1)\rho) + \sigma^2\left(1+(n-1)\rho^2\right)}{(1+\sigma^2-\rho)(1+\sigma^2+(n-1)\rho)}$$

$$\tilde{\mathcal{V}}(s_i, e_1) = \rho$$

$$\tilde{\mathcal{V}}(\theta_i, e_1) = \rho\frac{(1-\rho)(1+(n-1)\rho) + \sigma^2(2+(n-2)\rho)}{(1+\sigma^2-\rho)(1+\sigma^2+(n-1)\rho)}.$$

Since

$$\begin{aligned} E(\theta_i|s_i, e_1) &= \frac{\tilde{\mathcal{V}}(e_1) - \tilde{\mathcal{V}}(\theta_i, e_1)\rho}{\tilde{\mathcal{V}}(e_1)(1+\sigma^2) - \rho^2}s_i + \frac{\tilde{\mathcal{V}}(\theta_i, e_1)\left(1+\sigma^2\right) - \rho}{\tilde{\mathcal{V}}(e_1)(1+\sigma^2) - \rho^2}e_1 \\ &= v_{ii}s_i + v_{i1}s_i + \sum_{\substack{j=2 \\ j\neq i}}^{n} v_{1j}s_j \end{aligned}$$

for any $i \neq 1$, it follows that

$$
\begin{aligned}
v_{i1} &= \frac{\tilde{\mathcal{V}}(\theta_i, e_1)\left(1+\sigma^2\right)-\rho}{\tilde{\mathcal{V}}(e_1)(1+\sigma^2)-\rho^2} v_{11} && \text{(A.18)} \\
v_{ii} &= \frac{\tilde{\mathcal{V}}(e_1)-\tilde{\mathcal{V}}(\theta_i, e_1)\rho}{\tilde{\mathcal{V}}(e_1)(1+\sigma^2)-\rho^2} + \frac{\tilde{\mathcal{V}}(\theta_i, e_1)\left(1+\sigma^2\right)-\rho}{\tilde{\mathcal{V}}(e_1)(1+\sigma^2)-\rho^2} v_{1j} && \text{(A.19)} \\
v_{ij} &= \frac{\tilde{\mathcal{V}}(\theta_i, e_1)\left(1+\sigma^2\right)-\rho}{\tilde{\mathcal{V}}(e_1)(1+\sigma^2)-\rho^2} v_{1j} && \text{(A.20)}
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{y}_P &= \frac{\tilde{\mathcal{V}}(e_1)-\tilde{\mathcal{V}}(\theta_i, e_1)\rho}{\tilde{\mathcal{V}}(e_1)(1+\sigma^2)-\rho^2} \\
\bar{z}_P &= \frac{\tilde{\mathcal{V}}(\theta_i, e_1)\left(1+\sigma^2\right)-\rho}{\tilde{\mathcal{V}}(e_1)(1+\sigma^2)-\rho^2}
\end{aligned}
$$

Moreover, since

$$e_1 = E(\theta_1|s_1, \mathbf{e}_{g_1}) = \bar{y}_C s_1 + \sum_{j=2}^{n} \bar{z}_C e_j = \bar{y}_C s_1 + \sum_{j=2}^{n} \bar{z}_C \left(\bar{y}_P s_i + \bar{z}_P e_1\right),$$

then

$$E(\theta_1|s_1, \mathbf{e}_{g_1}) = \frac{\bar{y}_C}{1-(n-1)\bar{z}_C\bar{z}_P} s_1 + \sum_{j=2}^{n} \frac{\bar{z}_C\bar{y}_P}{1-(n-1)\bar{z}_C\bar{z}_P} s_i.$$

This implies that

$$\bar{z}_C = \frac{v_{1j}}{\bar{y}_P + (n-1)\bar{z}_P v_{1j}}$$

and

$$\bar{y}_C = \frac{v_{11}\bar{y}_P}{\bar{y}_P + (n-1)\bar{z}_P v_{1j}}.$$

Tedious, but straightforward calculations show that[22]

$$0 < \bar{z}_C < \bar{z}_P < 1.$$

Step 2: We now solve the system (A.12-A.15). From equation (A.15) it is easy to see that

$$z_P = 2\bar{z}_P \tag{A.21}$$

Next, we show that

$$z_C = \bar{z}_C\left(n + 2\bar{z}_P - n\bar{z}_P - 1\right) + 1 - \sqrt{\left(\left(\bar{z}_C\left(n\left(1-\bar{z}_P\right) + 2\bar{z}_P - 1\right) + 1\right)\right)^2 - 4\bar{z}_C}.$$

[22] In addition, since $-\sigma_\theta^4 \frac{(\rho-1)^2}{\sigma_\theta^2+\sigma_\varepsilon^2-\rho\sigma_\theta^2} < 0 \Leftrightarrow \left(\mathcal{V}(\theta_i, e_1) - \mathcal{V}(e_1)\right)\left(\sigma_\theta^2+\sigma_\varepsilon^2\right) < 0$, then $\frac{\mathcal{V}(\theta_i, e_1)\left(\sigma_\theta^2+\sigma_\varepsilon^2\right)-\rho\sigma_\theta^4}{\mathcal{V}(e_1)\left(\sigma_\theta^2+\sigma_\varepsilon^2\right)-\rho^2\sigma_\theta^4} < 1$, or $\bar{z}_p < 1$.

Indeed, substituting the solution for z_P in equation (A.13), we obtain

$$z_C \frac{2 - z_C}{4 - 2\bar{z}_P z_C} = \bar{z}_C \left(1 - (n-1)\, z_C \frac{2 - 2\bar{z}_P}{4 - 2\bar{z}_P z_C}\right)$$

or

$$z_C^2 + z_C \left(\bar{z}_C \left(2\bar{z}_P - 2\right)(n-1) - 2\bar{z}_C \bar{z}_P - 2\right) + 4\bar{z}_C = 0.$$

This equation has two solutions

$$\begin{aligned} z_{C1} &= \bar{z}_C \left(n + 2\bar{z}_P - n\bar{z}_P - 1\right) + 1 - \sqrt{\left(\left(\bar{z}_C \left(n\left(1 - \bar{z}_P\right) + 2\bar{z}_P - 1\right) + 1\right)\right)^2 - 4C} \\ z_{C2} &= \bar{z}_C \left(n + 2\bar{z}_P - n\bar{z}_P - 1\right) + 1 + \sqrt{\left(\left(\bar{z}_C \left(n\left(1 - \bar{z}_P\right) + 2\bar{z}_P - 1\right) + 1\right)\right)^2 - 4C}. \end{aligned}$$

First note that the expression under the square root is always positive. This follows from the fact that it is increasing in n. Since even for $n = 3$, $\left(\bar{z}_C\left(2 - \bar{z}_P\right) + 1\right)^2 - 4C = 0$ does not have a solution for $\bar{z}_P$ in the unit interval, then the expression under the square root must be always positive. Clearly as $\bar{z}_C$ and $\bar{z}_P$ are in the unit interval, then $\left(n + 2\bar{z}_P - n\bar{z}_P - 1\right) > 0$ for any $n \geq 2$, and both z_{C1} and z_{C2} are positive. However, only the z_{C1} is smaller than 2. For this, note that z_{C2} is larger than 2 iff

$$\bar{z}_C \left(n + 2\bar{z}_P - n\bar{z}_P - 1\right) - 1 + \sqrt{\left(\left(\bar{z}_C \left(n\left(1 - \bar{z}_P\right) + 2\bar{z}_P - 1\right) + 1\right)\right)^2 - 4\bar{z}_C} > 0$$

or

$$0 > \left(1 - \bar{z}_C \left(n + 2\bar{z}_P - n\bar{z}_P - 1\right)\right)^2 - \left(\left(\left(\bar{z}_C \left(n\left(1 - \bar{z}_P\right) + 2\bar{z}_P - 1\right) + 1\right)\right)^2 - 4\bar{z}_C\right) = -4\left(1 - \bar{z}_P\right)(n-2)\,\bar{z}_C$$

which always holds. In contrast, z_{C1} is always smaller than 2 as

$$\begin{aligned} \bar{z}_C \left(n + 2\bar{z}_P - n\bar{z}_P - 1\right) + 1 - \sqrt{\left(\left(\bar{z}_C \left(n\left(1 - \bar{z}_P\right) + 2\bar{z}_P - 1\right) + 1\right)\right)^2 - 4\bar{z}_C} &< 2 \\ \left(\left(\left(\bar{z}_C \left(n\left(1 - \bar{z}_P\right) + 2\bar{z}_P - 1\right) + 1\right)\right)^2 - 4\bar{z}_C\right) - \left(\bar{z}_C \left(n + 2\bar{z}_P - n\bar{z}_P - 1\right) - 1\right)^2 &= 4\bar{z}_C \left(1 - \bar{z}_P\right)(n-2) > 0. \end{aligned}$$

Finally, given z_C and z_P, the solutions for (A.12) and (A.14) follow immediately.

Proof of Lemma 2

1. Suppose network g is connected. This implies that between any two agents i and j, there exists a sequence of dealers $\{i_1, i_2, ..., i_r\}$ such that $ii_1 \in g$, $i_k i_{k+1} \in g$, and $i_r j \in g$ for any $k \in \{1, 2, ..., r\}$. The sequence $\{i_1, i_2, ..., i_r\}$ forms a *path* between i and j. The length of this path, r, represents the distance between i and j.

   Let V^* be the fixed point, as introduced in the proof of Proposition 1. Then, the equilibrium guess vector is given by

   $$\mathbf{e} = V^* \mathbf{s}.$$

   Suppose that there exists an equilibrium

   $$v_{ij}^* = 0$$

for some i and j at distance r from each other. Then from (A.5) if follows that

$$\sum_{k\in g_i}\omega_{ik}\frac{v^*_{kj}}{\left(\mathbf{v}^*_k\right)^T\mathbf{1}}=0,$$

and, since $\omega_{ik}>0$ for $\forall i,k\in\{1,2,...,n\}$, then it must be that

$$v^*_{kj}=0,\ \forall k\in g_i.$$

This means that all the neighbors of agent i place 0 weight on j's information. Further, this implies

$$\sum_{l\in g_k}\omega_{il}\frac{v^*_{lj}}{\left(\mathbf{v}^*_l\right)^T\mathbf{1}}=0,$$

and

$$v^*_{lj}=0,\ \forall l\in g_k.$$

Hence, all the neighbors and the neighbors of the neighbors of agent i place 0 weight on j's information. We can iterate the argument for r steps, and show that it must be that any agent at distance at most r from i places 0 weight on j's information. Since the distance between i and j is r, then

$$v^*_{jj}=0,$$

which is a contradiction, since (A.4) must hold and $\rho<1$ ($\sigma^2_\eta>0$). This concludes the first part of the proof.

2. See Case 2 in the proof of Proposition 1.

Proof of Proposition 5

1. From Proposition 2 we know that in any equilibrium of the OTC game

$$E(\theta_i|s_i,\mathbf{p}_{g_i})=E(\theta_i|s_i,\mathbf{e}_{g_i}).$$

Lemma 2 shows that each equilibrium expectation in the conditional guessing game is a linear combination of all signals in the economy

$$E(\theta_i|s_i,\mathbf{e}_{g_i})=\mathbf{v}_i\mathbf{s}$$

where $\mathbf{v}_i>0$ for all i. Since the equilibrium price in any trade between two dealers i and j is a weighted sum of their respective beliefs, as in (17), and the weights are positive, then the result follows immediately.

2. As $\rho\to 1$, we show that there exists an equilibrium such that

$$\lim_{\rho\to 1}E\left(\theta_i|s_i,\mathbf{p}_{g_i}\right)=v^*\sum_{i=1}^n s_i,\ \forall i\in\{1,2,...,n\}$$

where $v^{*}=\frac{\sigma_{\theta}^{2}}{n\sigma_{\theta}^{2}+\sigma_{\varepsilon}^{2}}$.

If there exists an equilibrium in the OTC game, then it follows from the proof of Proposition 1 that

$$E(\theta_i \,|s_i, \mathbf{p}_{g_i}) = \bar{y}_i s_i + \sum_{k\in g_i} \bar{z}_{ik} E(\theta_k \,|s_k, \mathbf{p}_{g_k}).$$

or

$$E(\theta_i \,|s_i, \mathbf{e}_{g_i}) = \bar{y}_i s_i + \sum_{k\in g_i} \bar{z}_{ik} E(\theta_k \,|s_k, \mathbf{e}_{g_k}).$$

Taking the limit as $\rho \to 1$, and using Case 2 in the proof of Proposition 1, we have that

$$\lim_{\rho\to 1} E\left(\theta_i | s_i, \mathbf{p}_{g_i}\right) = \frac{\sigma_\theta^2}{n\sigma_\theta^2 + \sigma_\varepsilon^2} \sum_{i=1}^{n} s_i.$$

Given that

$$\lim_{\rho\to 1} E\left(\theta_i | s_i, \mathbf{p}_{g_i}\right)$$

The conditional variance is

$$\mathcal{V}\left(\theta_i | s_i, \mathbf{p}_{g_i}\right) = \sigma_\theta^2 - \mathcal{V}\left(E\left(\theta_i | s_i, \mathbf{p}_{g_i}\right)\right)$$

and taking the limit $\rho \to 1$, we obtain

$$\lim_{\rho\to 1} \mathcal{V}\left(\theta_i | s_i, \mathbf{p}_{g_i}\right) = \sigma_\theta^2 - \left(\frac{\sigma_\theta^2}{n\sigma_\theta^2 + \sigma_\varepsilon^2}\right)^2 n\left(\sigma_\varepsilon^2 + n\sigma_\theta^2\right).$$

and

$$\begin{aligned}
\lim_{\rho\to 1} \mathcal{V}\left(\theta_i | \mathbf{s}\right) &= \sigma_\theta^2 - \mathcal{V}\left(E\left(\hat{\theta} | \mathbf{s}\right)\right) \\
&= \sigma_\theta^2 - \left(\frac{\sigma_\theta^2}{n\sigma_\theta^2 + \sigma_\varepsilon^2}\right)^2 n\left(\sigma_\varepsilon^2 + n\sigma_\theta^2\right) \\
&= \sigma_\theta^2 \frac{\sigma_\varepsilon^2}{n\sigma_\theta^2 + \sigma_\varepsilon^2}
\end{aligned}$$

Proof of Proposition 6

Observe that $V = (I = Z)^{-1} Y$, for n-star network has the elements of

$$\begin{aligned} v_{11} &= y_C \frac{1}{1-(n-1)\, z_C z_P} \\ v_{i1} &= y_C \frac{z_P}{1-(n-1)\, z_C z_P} \\ v_{ii} &= y_P \frac{1-(n-2)\, z_C z_P}{1-(n-1)\, z_C z_P} \\ v_{1i} &= y_P \frac{z_C}{1-(n-1)\, z_C z_P} \\ v_{ij} &= y_P \frac{z_C z_P}{1-(n-1)\, z_C z_P} \end{aligned}$$

where y_C, y_P are the weights on the private signal and z_C, z_P are the weights on the others' guesses in the central and periphery agents' guessing function respectively. As maximizing $E\left(-\Sigma_i (\theta - e_i)^2\right)$ is equivalent with maximizing

$$2tr\,(V\Sigma_{\theta s}) - tr\left(V\Sigma V^T\right)$$

where $\Sigma_{ii} = 1+\sigma^2, \Sigma_{ij} = \rho, [\Sigma_{\theta s}]_{ii} = 1, [\Sigma_{\theta s}]_{ij} = \rho$, we calculate the expressions for the components of this objective function.

$$\begin{aligned} \left[V\Sigma V^T\right]_{11} &= \left(1+\sigma^2\right) v_{11}^2 + \left(1+\sigma^2\right)(n-1)\, v_{1i}^2 + \rho 2\,(n-1)\, v_{1i} v_{11} + \rho\,(n-1)\,(n-2)\, v_{1i}^2 \\ &= \frac{\left(\left(1+\sigma^2\right) y_C^2 + \left(\left(1+\sigma^2\right)+\rho(n-2)\right)(n-1) y_P^2 z_C^2 + \rho 2 (n-1) y_C y_P z_C\right)}{\left(1-(n-1) z_C z_P\right)^2} \end{aligned}$$

and

$$\begin{aligned} \left[V\Sigma V^T\right]_{ii} &= \left((n-2)\,(n-3)\, v_{ij}^2 + (n-2)\, 2\,(v_{i,1}+v_{i,i})\, v_{ij} + 2 v_{i,1} v_{i,i}\right) \rho + \left(\sigma^2+1\right)\left(v_{ii}^2 + (n-2)\, v_{ij}^2 + v_{i,1}^2\right) = \\ &= \frac{\left(y_C + z_C y_P (n-2)\left(1-\frac{(n-1)}{2} z_C z_P\right)\right) 2 z_P y_P \rho + \left(\sigma^2+1\right)\left((1-(n-2) z_C z_P)^2 y_P^2 + (n-2) y_P^2 z_P^2 z_C^2 + y_C^2 z_P^2\right)}{\left(1-(n-1) z_C z_P\right)^2} \end{aligned}$$

and

$$tr\left(V\Sigma V^T\right) = \left[V\Sigma V^T\right]_{11} + (n-1)\left[V\Sigma V^T\right]_{ii}.$$

Also,

$$\begin{aligned} tr\,(V\Sigma_{\theta s}) &= v_{11} + (n-1)\, v_{ii} + \rho\,(n-1)\,(v_{1i}+v_{i1}) + \rho\,(n-1)\,(n-2)\, v_{ij} = \\ &= \frac{y_C + \rho\,(n-1)\, y_P z_C}{(1-(n-1)\, z_C z_P)} + (n-1)\frac{y_P\,(1-(n-2)\, z_C z_P\,(1-\rho)) + \rho y_C z_P}{(1-(n-1)\, z_C z_P)} \end{aligned}$$

This implies that

$$\lim_{\delta\to 0} \frac{\partial U\,(z_C+\delta, z_P+\delta, y_C-\delta, y_P-\delta)}{\partial \delta} = -\frac{f\,(\bar{z}_P, \bar{z}_C, \bar{y}_C, \bar{y}_P; n, \rho, \sigma)}{\left(-1+(n-1) z_C z_P\right)^3},$$

where $f(\cdot)$ is a polynomial. Then we substitute in the analytical expressions for the decentralized optimum $z_C^*, z_P^*, y_C^*, y_P^*$ given in closed form in Proposition 4 and rewrite $\lim_{\delta\to 0}\frac{\partial U\left(z_C^*+\delta, z_P^*+\delta, y_C^*-\delta, y_P^*-\delta\right)}{\partial\delta}$ as a fraction. Both the numerator and the denominator are polynomials of σ^2.of order 9. A careful inspection reveals that each of the coefficients are positive for any $\rho\in(0,1)$ and $n\geq 3$.(Details on the resulting expressions in these calculations are available from the authors on request.)

Proof of Proposition 7

1. From the proof of Proposition 4, we have that

$$z_C = \bar{z}_C\left(n+2\bar{z}_P - n\bar{z}_P - 1\right) + 1 - \sqrt{\left(\left(\bar{z}_C\left(n\left(1-\bar{z}_P\right)+2\bar{z}_P-1\right)+1\right)\right)^2 - 4\bar{z}_C}.$$

   Then,

$$z_C < 2\bar{z}_P = z_P$$

   iff

$$\bar{z}_C < \frac{\bar{z}_P\left(1-\bar{z}_P\right)}{\bar{z}_P\left(1-\bar{z}_P\right)(n-1)+\bar{z}_P^2+1},$$

   which holds for any $0<\bar{z}_C<\bar{z}_P<1$.

2. From the definition of the trading intensity and from (8) it follows that

$$\frac{t_C}{t_P} = \frac{2-z_P}{2-z_C},$$

   which implies $t_C < t_P$.

3. Tedious, but straightforward calculations show that $\frac{\partial\bar{z}_P}{\partial\rho}, \frac{\partial\bar{z}_P}{\partial\sigma^2} > 0$, and when $n=3$, $\frac{\partial\bar{z}_C}{\partial\rho}, \frac{\partial\bar{z}_C}{\partial\sigma^2} > 0$ (see the proof of Proposition 4 for closed-form solutions for $\bar{z}_P$ and $\bar{z}_C$). The statements follows given the definition of t_C and t_P, after further algebra showing that $\frac{\partial t_C}{\partial z_P}, \frac{\partial t_C}{\partial z_C}, \frac{\partial t_P}{\partial z_P}, \frac{\partial t_P}{\partial z_C} < 0$ and $\frac{\partial z_P}{\partial\bar{z}_P} > 0, \frac{\partial z_P}{\partial\bar{z}_P} = 0, \frac{\partial z_C}{\partial\bar{z}_C} > 0, \frac{\partial z_C}{\partial\bar{z}_P} > 0$.

Proof of Proposition 8

Given (A.18)-(A.20), we find that

$$\begin{aligned}&E\left(\left(e_C-e_P\right)^2\right) = \mathcal{V}\left(e_C-e_P\right)\\ &= \frac{(\rho-1)^2\left((n-2)\rho^2+2\right)\sigma^4+2(1-\rho)(\rho+1)((n-2)\rho+2)\sigma^2+2(\rho+1)(\rho-1)^2(1-\rho+n\rho)}{\left(1-\rho+\sigma^2\right)\left(\left(\rho^2(n-2)+1\right)\sigma^4+(1-\rho)(\rho+1)((n-2)\rho+2)\sigma^2+(\rho+1)(1-\rho)^2(1-\rho+n\rho)\right)},\end{aligned}$$

where, as before $\sigma^2\equiv\frac{\sigma_\varepsilon^2}{\sigma_\theta^2}$.

The monotonicity in n follows from straightforward derivations.

To show the monotonicity in σ^2, we write

$$\left(1-\rho+\sigma^2\right)^2\left(\begin{array}{c}\left(\rho^2\left(n-2\right)+1\right)\sigma^4+\left(1-\rho\right)\left(\rho+1\right)\left(\rho\left(n-2\right)+2\right)\sigma^2\\+\left(\rho+1\right)\left(1-\rho\right)^2\left(\left(n-1\right)\rho+1\right)\end{array}\right)^2\frac{\partial\mathcal{V}\left(e_C-e_P\right)}{\partial\sigma^2}$$

as a polynomial of σ^2 and check that each coefficient in this polynomial is negative.

To show the monotonicity in ρ, we write

$$\left(1-\rho+\sigma^2\right)^2\left(\begin{array}{c}\left(\rho^2\left(n-2\right)+1\right)\sigma^4+\left(1-\rho\right)\left(\rho+1\right)\left(\rho\left(n-2\right)+2\right)\sigma^2\\+\left(\rho+1\right)\left(1-\rho\right)^2\left(\left(n-1\right)\rho+1\right)\end{array}\right)^2\frac{\partial\mathcal{V}\left(e_C-e_P\right)}{\partial\rho}$$

as a polynomial of σ^2. Each coefficient is negative at $n=3$ and decreasing in n.

Proof of Proposition 9

Dealers revise their messages according to the rule that

$$h_{i,t}=\bar{y}_i s_i+\bar{\mathbf{z}}_{g_i}^T\mathbf{h}_{g_i,t-1},\ \forall i.$$

or, in matrix form

$$\mathbf{h}_{t+1}=\bar{Y}\mathbf{s}+\bar{Z}\mathbf{h}_t.$$

1. Since $\mathbf{h}_{t_0}=\left(I-\bar{Z}\right)^{-1}\bar{Y}\mathbf{s}$, then

$$\begin{aligned}\mathbf{h}_{t_0+1} &= \bar{Y}\mathbf{s}+\bar{Z}\mathbf{h}_{t_0}\\ &= \bar{Y}\mathbf{s}+\bar{Z}\left(I-\bar{Z}\right)^{-1}\bar{Y}\mathbf{s}\\ &= \bar{Y}\mathbf{s}+\left(I-\left(I-\bar{Z}\right)\right)\left(I-\bar{Z}\right)^{-1}\bar{Y}\mathbf{s}\\ &= \bar{Y}\mathbf{s}+\left(I-\bar{Z}\right)^{-1}\bar{Y}\mathbf{s}-\bar{Y}\mathbf{s}\\ &= \left(I-\bar{Z}\right)^{-1}\bar{Y}\mathbf{s}\end{aligned}$$

   It follows straightforwardly, from an inductively argument that

$$\mathbf{h}_t=\left(I-\bar{Z}\right)^{-1}\bar{Y}\mathbf{s}.$$

2. From

$$\mathbf{h}_{t_0+1}=\bar{Y}\mathbf{s}+\bar{Z}\mathbf{h}_{t_0}$$

   it follows that

$$\mathbf{h}_{t_0+n}=\left(I+\bar{Z}+...+\bar{Z}^{n-1}\right)\bar{Y}\mathbf{s}+\bar{Z}^n\mathbf{h}_{t_0}$$

   In the limit as $n\to\infty$, from Proposition 1 we know that

$$\lim_{n\to\infty}\mathbf{h}_{t_0+n}=\left(I-\bar{Z}\right)^{-1}\bar{Y}\mathbf{s}.$$

   This implies that for any vector $\boldsymbol{\gamma}\in R_+^n$, there exists an $n_{\boldsymbol{\gamma}}$ such that

$$\left|\mathbf{h}_{t_0+n}-\left(I-\bar{Z}\right)^{-1}\bar{Y}\mathbf{s}\right|<\boldsymbol{\gamma},\forall n\geq n_{\boldsymbol{\gamma}}.$$

Fix an arbitrarily small vector $\boldsymbol{\gamma}$. Then

$$-\boldsymbol{\gamma} < \left(I - \bar{Z}\right)^{-1} \bar{Y}\mathbf{s} - \mathbf{h}_{t_0+n_{\boldsymbol{\gamma}}} < \boldsymbol{\gamma}$$

and

$$-\boldsymbol{\gamma} < \mathbf{h}_{t_0+n} - \left(I - \bar{Z}\right)^{-1} \bar{Y}\mathbf{s} < \boldsymbol{\gamma}, \forall n \geq n_{\boldsymbol{\gamma}}.$$

Adding up these two inequalities we have that

$$-2\boldsymbol{\gamma} < \mathbf{h}_{t_0+n} - \mathbf{h}_{t_0+n_{\boldsymbol{\gamma}}} < 2\boldsymbol{\gamma}, \forall n \geq n_{\boldsymbol{\gamma}}.$$

This shows that there exists $\boldsymbol{\delta} = 2\boldsymbol{\gamma}$ and $t_{\boldsymbol{\delta}} = t_0 + n_{\boldsymbol{\gamma}}$ such that

$$|\mathbf{h}_t - \mathbf{h}_{t_{\boldsymbol{\delta}}}| < \boldsymbol{\delta}, \forall t \geq t_{\boldsymbol{\delta}}.$$

which implies that the protocol stops at $t_{\boldsymbol{\delta}}$.

3. We start by observing that

$$E\left(\theta_i | s_i, \mathbf{h}_{g_i,t_0}, \mathbf{h}_{g_i,t_0+1}, ..., \mathbf{h}_{g_i,t_0+n}\right) = E\left(\theta_i | s_i, \mathbf{h}_{g_i,t_0+n}\right), \forall n \geq 0.$$

Further, in the limit $n \to \infty$, we have that

$$\lim_{n\to\infty} \mathbf{h}_{t_0+n} = \left(I - \bar{Z}\right)^{-1} \bar{Y}\mathbf{s} = \mathbf{e},$$

and subsequently

$$\lim_{n\to\infty} \mathbf{h}_{g_i,t_0+n} = \mathbf{e}_{g_i}, \forall i.$$

Then

$$\lim_{n\to\infty} E\left(\theta_i | s_i, \mathbf{h}_{g_i,t_0+n}\right) = E\left(\theta_i | s_i, \mathbf{e}_{g_i}\right) = E\left(\theta_i | s_i, \mathbf{p}_{g_i}\right).$$

As above, we can construct $t_{\boldsymbol{\delta}}$ such that protocol stops and show that

$$\left|E\left(\theta_i | s_i, \mathbf{h}_{g_i,t_0+n}\right) - E\left(\theta_i | s_i, \mathbf{p}_{g_i}\right)\right| < \frac{1}{2}\boldsymbol{\delta}.$$

B Appendix: Complete network

In the complete network, each dealer i chooses her demand function conditional on the beliefs of the other $(n-1)$ dealers. Given that she knows s_i, she can invert the signals of all the other dealers. Hence, her belief is given by

$$E(\theta_i | s_i, \mathbf{e}_{g_i}) = E(\theta_i | \mathbf{s}) = \sigma_\theta^2 \frac{1-\rho}{\sigma_\theta^2 + \sigma_\varepsilon^2 - \rho\sigma_\theta^2} \left(s_i + \frac{\rho\sigma_\varepsilon^2}{(1-\rho)\left(\sigma_\theta^2 + \sigma_\varepsilon^2 - \rho\sigma_\theta^2 + n\rho\sigma_\theta^2\right)} \sum_{i=1}^{n} s_i \right).$$

Then, following the same procedure as above (for a star), and taking into account that in a star trading strategies are symmetric, we obtain that

$$E(\theta_i|s_i, \mathbf{e}_{g_i}) = \bar{y}s_i + \bar{z}\sum_{\substack{j=1\\j\neq i}}^{n} e_j$$

where

$$e_j = E(\theta_j|s_j, \mathbf{e}_{g_j})$$

and

$$\begin{aligned}
\bar{y} &= \frac{\sigma_\theta^2 (1-\rho)(1+(n-1)\rho)}{\sigma_\theta^2 + \sigma_\varepsilon^2 + \rho^2\sigma_\theta^2 - 2\rho\sigma_\theta^2 - 2\rho\sigma_\varepsilon^2 - n\rho^2\sigma_\theta^2 + n\rho\sigma_\theta^2 + n\rho\sigma_\varepsilon^2} \\
\bar{z} &= \frac{\rho\sigma_\varepsilon^2}{\sigma_\theta^2 + \sigma_\varepsilon^2 + \rho^2\sigma_\theta^2 - 2\rho\sigma_\theta^2 - 2\rho\sigma_\varepsilon^2 - n\rho^2\sigma_\theta^2 + n\rho\sigma_\theta^2 + n\rho\sigma_\varepsilon^2}.
\end{aligned}$$

Solving the system (15), we obtain

$$\begin{aligned}
y_i &= \frac{\sigma_\theta^2 (1-\rho)(1+(n-1)\rho)}{\left(\sigma_\theta^2 (1-\rho)(1+(n-1)\rho) + \sigma_\varepsilon^2 (1+2(n-3)\rho)\right) + 3\rho\sigma_\varepsilon^2}, \forall i \\
z_{ij} &= \frac{2\rho\sigma_\varepsilon^2}{\left(\sigma_\theta^2 (1-\rho)(1+(n-1)\rho) + \sigma_\varepsilon^2 (1+2(n-3)\rho)\right) + 2\rho\sigma_\varepsilon^2}, \forall ij.
\end{aligned}$$

Substituting in the expressions for b_i^j, c_{ij}^j, and c_{ij}^k, respectively, in Proposition (3) we obtain

$$\begin{aligned}
b_i^j &= -\frac{\beta_{ij}}{2}\frac{\sigma_\theta^2 (1-\rho)}{\rho\sigma_\varepsilon^2}\frac{(1+(n-1)\rho)\left(\sigma_\theta^2 (1-\rho)(1+(n-1)\rho) + (1+2n\rho-4\rho)\sigma_\varepsilon^2\right)}{\sigma_\theta^2 (1-\rho)(1+(n-1)\rho) + (1+2n\rho-3\rho)\sigma_\varepsilon^2} \\
c_{ij}^i &= \beta_{ij}\frac{1}{2\rho\sigma_\varepsilon^2}\left(\sigma_\theta^2 (1-\rho)(1+(n-1)\rho) + \sigma_\varepsilon^2 (1+2(n-3)\rho)\right) \\
c_{ij}^k &= -\beta_{ij}.
\end{aligned}$$