Abstract

For how long should central banks keep interest rates low, beyond the end of a liquidity trap? The optimal duration of forward guidance balances its benefits and costs: an expansion today versus future volatility. I solve for it in closed form, as a function of the trap duration, the severity of the recession, and the slopes of demand and supply—which govern the response of activity to news. This analytical solution method relies on modelling guidance as a probability of low rates once the trap ended, and gives similar results to fully specified Ramsey policy. A simple rule approximates optimal commitment well: announce a duration of half of (one plus) the product of trap’s duration and depth. It also alleviates credibility problems, by anchoring expectations of those who mistake commitment for bad news. Lastly, two models of aggregate demand with heterogeneous agents give opposite conclusions regarding optimal FG duration—disconnect which the closed-form solution proposed here helps understand.

JEL Codes: E21, E31, E40, E50

Keywords: forward guidance; liquidity trap; zero lower bound; optimal monetary policy; heterogeneous agents; heterogeneous beliefs; delphic; odyssean; incomplete markets; unemployment risk; hand-to-mouth.

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1 Introduction

What is left for a central bank to do in a liquidity trap, when it is out of standard ammunitin? This once exotic theoretical question has been haunting the developed world for the past decades: Japan since 25 years and counting; the US and Europe since the financial crisis of 2008. One of the remaining policy options is "forward guidance" (FG for short): the promise to keep interest rates low even after normal conditions have resumed, when increasing rates would be the response warranted by optimal policy (an exhaustive discussion of other policy options at the zero lower bound ZLB can be found in Woodford, 2012).

A large and important theoretical literature developed to study optimal interest-rate policy in a liquidity trap, deriving FG as an optimal policy prescription in a variety of environments—an insight by now well-known, originally due to the seminal work of Jung, Teranishi and Watanabe (2005), Eggertsson and Woodford (2003), Adam and Billi (2006, 2007), and Nakov (2008). But FG is more than a theoretical result: several central banks have been doing it, for example the Federal Reserve in various forms since 2003. The literature that I review below keeps expanding, analyzing a variety of refinements of optimal policy and FG.

Yet to the best of my knowledge (and to my surprise when starting this work) no closed-form solution for the optimal duration of FG exists in the literature that I scoured—despite the explosion of studies on the topic. A closed-form answer to the question "how long should central banks keep interest rates at zero beyond the end of the trap" is as valuable as the welfare question is itself important. Beyond the value of closed-form solutions to policy-relevant theoretical problems, in the present context, a closed-form solution for the optimal FG duration unveils in a transparent way how this depends on structural features of the economy, and on deep parameters that can be estimated; it also allows to understand that under some conditions, it is optimal to refrain form FG altogether. Moreover, closed-form solutions can inspire and inform "simple rules" that approximate, for practical and operational purposes, what are often complicated and opaque optimal policies. Examples abound, but to stay in the realm of monetary policy we can cite simple Taylor rules and inflation targeting: the feedback between closed-form solutions, theory, and policy practice

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1 The literature analyzing the consequences of the ZLB in formal New Keynesian models has been pioneered by i.a. Fuhrer and Madigan (1997), Wolman (1998), Orphanides and Wieland (1998), Krugman (1998), and Uhlig (2000). Krugman discusses the virtues of lowering long rates by keeping interest rates low in the future and creating expected inflation; see also the discussion by Rogoff (1998) in the same volume.

2 Campbell et al. (2016) review much of the literature, and provide evidence that Federal Reserve FG was counterproductive before 2011, but had expansionary effects thereafter; the authors associate that with a change in policy strategy, namely to FG becoming "Odyssean": a term explained and discussed in more detail below. Williams (2013) discusses at least three varieties of FG used by the Fed since 2003, and Filardo and Hofman (2014) provide a comprehensive discussion of the international evidence.
based on simple rules is obvious.

The main novel contribution of this paper is thus to provide such a closed-form solution: the expected optimal FG duration as an explicit function of the duration of the liquidity trap, its severity (the size of the shock), and the slopes of the aggregate demand and supply curves. This relies chiefly on a stochastic way to model FG, which delivers an (itself novel, as far as I know) closed-form expression for the equilibrium as a function of expected duration.³ The optimal FG duration strikes the intertemporal balance between two forces. The first is expansionary: future cuts in interest rates are expansionary today, which improves welfare at the ZLB for the usual reasons emphasized by the seminal work of Eggertsson and Woodford (2003). Yet a second force reduces lifetime welfare: when the future becomes the present, low interest rates generate positive consumption gaps (and inflation), which is inefficient. This welfare cost is made of two components: in each state with FG, welfare is decreased because of inefficient (inflation and) consumption volatility (this cost has also been discussed by Eggertsson and Woodford, 2003, p.178); but also across states, more FG implies a larger weight on the future in lifetime welfare. Throughout the paper, I will concentrate mostly on the first component of the cost—this provides an upper bound on the optimal degree of FG (a robustness section studies the effect of adding the second channel).

Inspired by that closed form expression, I then propose an operational policy prescription rooted in the welfare analysis: a simple rule for FG relating the "instrument" (the duration of low interest rates) to the duration of the liquidity trap (LT) and the magnitude of the financial disruption causing the LT. The central bank can announce that, once the trap is over, it will keep interest rates at zero for a duration of

\[
\text{Half of (one period plus LT duration } \times \text{ disruption)}
\]

This is desirable in terms of credibility and communication, for it allows bypassing the commitment problem inherent to FG in certain information environments, such as those emphasized by Andrade et al., 2015; Bassetto, 2016; or Wiederholt, 2015. The simple rule allows getting the best of both (state-contingent and time-dependent) worlds. The duration and depth of the trap can be measured and verified ex-post, once the trap ended—almost tautologically. Thus, announcing "FG starts when the trap ends, and for a duration that is such and such function of the trap duration and depth" is an effective way to communicate what may otherwise be a self-defeating commitment. In particular, if some agents regard

³The analysis is similar in spirit to Woodford’s (2011) analysis of optimal government spending in a LT (see also Bilbié, Monacelli and Perotti, 2014) in that it restrains attention to a constrained optimum. Furthermore, my modelling of FG stochastically is related to Woodford’s analysis of future spending expansions and their impact on the spending multiplier.
such commitment in a "delphic" way, inferring something about fundamentals, "odyssean" FG can have perverse effects; this is a point made numerically by Andrade et al in the context of their model with heterogeneous beliefs, that I reiterate here analytically. But as Bassetto (2016) has shown quite generally, delphic FG can be very useful in this context to anchor the expectations of delphic agents. The simple rule that I propose is one such example of delphic (communication) FG, and is also a good approximation to odyssean (commitment) FG.

One outstanding issue with FG in the baseline NK model is what Del Negro, Giannoni and Patterson called "the FG puzzle": that the power of FG increases, the more it lasts, and the more it is pushed into the future. McKay Nakamura and Steinsson (2015) proposed one solution to this puzzle, based on incomplete markets and uninsurable unemployment risk: in such an economy, FG's effects are dampened mainly because aggregate demand implies a form of discounting of the future that is absent in the baseline NK model. In the last section, I generalize the simple closed-form solution obtained in the baseline model to compare two models of aggregate demand with heterogeneous agents. In the first, constrained agents are unemployed, and the risk of becoming unemployed is uninsurable: an incomplete markets model drawing on McKay, Nakamura and Steinsson (2015, 2016). In the second, constrained agents ("hand-to-mouth") are employed, as in Bilbiie (2004, 2008); Gali, Lopez-Salido and Valles (2004, 2007); and Eggertsson and Krugman (2012), and also in some of the recent "HANK" models, e.g. Kaplan, Violante and Moll (2016). I first confirm McKay et al's result in my simple analytical framework: the power of FG is diminished, and this is due mainly to discounting in the aggregate Euler equation. Moreover, and as a direct consequence of this, optimal FG duration—which I calculate in closed-form for this model too—is decreasing with the share of constrained agents: this result is novel.

The model with employed hand-to-mouth is very different. The power of FG is now increasing with the share of constrained agents, which is a direct consequence of the mechanism unveiled in Bilbiie (2008): the elasticity of aggregate demand is increasing with the share of constrained agents. The consequences for optimal policy are non-trivial: optimal FG duration can even be increasing with the share of constrained agents, before decreasing sharply towards zero, its optimal value when the elasticity of aggregate demand to interest rates becomes very large.

A vast literature studied optimal policy under a ZLB constraint—thus providing an implicit, but not closed-form explicit, answer to the question that I study too; in what follows I review briefly the main contributions. The seminal papers were written in the early 2000s,

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4The distinction between "delphic" and "odyssean" FG was introduced by Campbell, Evans, Fisher, and Justiniano (2012), and will be discussed extensively below.
as Japan was entering its liquidity trap that, more than two decades after, still persists. Jung, Teranishi, and Watanabe (2005) were the first to derive analytical conditions for Ramsey optimal policy and show that it is optimal to keep nominal interest rates at zero for longer than implied by the (known, in their framework) duration of the trap, i.e. forward guidance; they also showed that commitment implies a longer FG duration than discretion—all these findings are contained in their equation (35). Eggertsson and Woodford (2003), extended this optimal policy exercise to the case where the shock causing the LT follows a two-state Markov chain—which thereafter became the norm for tractable analysis of stochastic ZLB equilibria. Among other contributions, they discussed the trade-offs between current and future volatility faced by a central bank contemplating FG in a liquidity trap, and studied the virtues of a simple policy of price level targeting as an approximation to the Ramsey policy. Adam and Billi (2006, 2007) and Nakov (2008) extended this to a fully stochastic setup, where shocks follow autoregressive processes and the ZLB is an occasionally binding constraint. They emphasized the large welfare gains from policy commitment compared to discretion; "forward guidance" is inherently part of the optimal commitment policy in this framework, too. More recently, Nakata (2016) and Schmidt (2013) extended the analysis of Ramsey-optimal policy at the ZLB to the joint analysis of monetary and fiscal policy, under both commitment and discretion: among other results, they find that uncertainty creates more scope for stimulus at the ZLB. Several papers explore the implications of the risk of the ZLB binding again in the future. Under this assumption, the central bank has an incentive to honor its promises, unlike in a setup where the ZLB is a one-off event. FG is thus naturally "sustainable"—if the duration is not too large (Walsh, 2016) and if the shocks are frequent enough (Nakata, 2014). In a similar environment, Nakata and Schmidt (2014) identify a deflationary bias outside the ZLB brought about by the anticipation of future ZLB risk—appointing a conservative central bank à la Rogoff (1985) alleviates this bias and improves welfare. I abstract from all such considerations in this paper, although I believe them to be important for policymaking.

A series of contributions look at forward guidance in environments with information frictions—all of which are relevant for my analysis of the virtues of a simple rule. Andrade, Gaballo, Mengus, and Mojon (2015) provide empirical evidence supporting the view that agents believe the central bank’s announcements, and therefore agree on the interest rate

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5 Jung et al’s first version, as cited in Eggertsson and Woodford (2003), is dated 2001. Note that their results also hold in continuous time, as shown by Werning (2012) in an otherwise identical perfect-foresight setup.

6 Furthermore, Nakov (2008) also studied price-level targeting and simple instrument rules and argued that, when deflationary shocks are large, simple rules fare much worse than optimal policy. Levin, Lopez-Salido, Nelson and Yun (2010) caution against the potential welfare losses that can result from FG in a large-shock environment.
projections. But agents disagree on the implications of those policies on macroeconomic outcomes. The authors build a model of heterogeneous beliefs, in which a fraction of agents take FG to be "delphic", thus inferring from the central bank’s announcement of low interest rates that circumstances will stay gloomy. The other agents take FG to be "odyssean", i.e. believing the central bank. I use the same framework to look at optimal FG and review this paper’s findings in more detail in due course. Wiederholt (2015) builds a model with dispersed information, where a fraction of households update their inflation expectations while the rest do not. The latter group does not respond to FG, but the former does: on the one hand, their expected inflation rises as they believe the FG announcement. On the other, they also infer from today’s commitment that the bad state will persist, which reduces inflation expectations. Both features—some agents do not update, and those who do become in part more pessimistic as a consequence of FG—go in the same direction, making the overall effect of FG unambiguously lower than in a perfect-information setup. Bassetto (2016) provides further foundations for the benefits of (delphic) FG as a communication device, in a strategic game with cheap talk from the central bank. Without private information, odyssean FG (meant purely as a commitment device) is redundant in this model. But with private information, communication can improve welfare and (delphic) FG emerges naturally as such communication strategy; this further interacts with the odyssean FG’s commitment and credibility dimensions. I draw on these contributions to justify the benefits of a simple rule.

My recommendation for a simple rule is related to the analysis of institutional solutions to incentives to deviate from optimal policy, based on delegation: Rogoff (1985) and Walsh (1995) are classic references in the Barro-Gordon setup; Vestin (2006), Svensson and Woodford (2001), Walsh (2003) and Bilbiie (2014) study optimal delegation in the NK model. In the context of simple policies that are beneficial at the ZLB, previous examples already mentioned above include Eggertsson and Woodford’s price level targeting, Nakov’s price level targeting, and more recently Nakata and Schmidt’s prescription for appointing a conservative central banker à la Rogoff in an environment where the ZLB may bind again in the future.

2 Forward Guidance in a Liquidity Trap: a Stochastic Approach

The baseline model is a by now standard New Keynesian model, whose details are skipped since they are readily available e.g. in the textbooks of Woodford (2003) or Gali (2008). The
economy is described by standard aggregate demand and supply equations:

\[ c_t = E_t c_{t+1} - \sigma (i_t - E_t \pi_{t+1} - \rho_t) \] (1)

\[ \pi_t = \beta E_t \pi_{t+1} + \kappa c_t. \] (2)

The first equation is the "IS curve", where \( i_t \) is the nominal interest rate in levels, \( \rho_t \) is an exogenous disturbance that moves the natural interest rate, and \( \sigma \) the elasticity of intertemporal substitution. The second equation is a standard "New Keynesian Phillips curve" coming from a forward-looking pricing model à la Calvo-Yun or Rotemberg model with slope \( \kappa = \psi (\varphi + \sigma^{-1}) \), where \( \psi = (1 - \zeta) (1 - \beta \zeta) / \zeta \) with \( \zeta \) the probability to be unable to change one’s price in the Calvo-Yun model and \( \varphi \) is the inverse constant-consumption elasticity of labor supply.

For most of the paper, in the interest of tractability and of obtaining closed-form expressions, I will focus attention on an even simpler aggregate supply curve. Namely, each period a fraction of firms can re-optimize their price freely while the remaining fraction keep their price fixed; importantly, firms do not take into account the forward-looking nature of their decision, in that they do not recognize that today’s reset price affects prevails with some probability in future periods. Essentially, such a setup reduces to assuming \( \beta = 0 \) in the pricing decision and leads to the Phillips curve

\[ \pi_t = \kappa c_t. \] (3)

where now \( \kappa = (\varphi + \sigma^{-1}) (1 - \zeta) / \zeta \) and \( \zeta \) is now interpreted as the ratio of the shares of flexible to fixed prices. Appendix A.5 extends the supply side to the case \( \beta > 0 \), which makes the algebra becomes more involved without affecting the results. Indeed, I compare the conclusions of the two models for optimal FG below, see Figure 5; we will see that optimal FG varies very little with the discount factor of firms.

2.1 The Liquidity Trap

I follow the seminal paper by Eggertsson and Woodford (2003) to model the zero lower bound. Namely, \( \rho_t \) follows a Markov chain with two states. The first is the steady state denoted by \( S \), with \( \rho_t = \rho \), and is absorbing: once in it, there is a probability of 1 of staying. The other state is transitory and denoted by \( L \): \( \rho_t = \rho_L < 0 \) with persistence probability \( p \) (conditional upon starting in state \( L \), the probability that \( \rho_t = \rho_L \) is \( p \), while the probability

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\(^7\)Below, I study three models with heterogenous agents which have a more complicated aggregate demand side.
that \( \rho_t = \rho \) is \( 1 - p \).

At time \( t \), consider a negative realization of \( \rho_t = \rho_L < 0 \), meant to capture in this reduced-form model an increase in spreads as in Woodford (2011) and Curdia and Woodford (2010). To simplify things even further, I assume that the monetary authority tracks the natural interest rate of this economy, i.e. \( r_n^t = \rho_t \) whenever feasible, meaning \( i_t = \max(\rho_t, 0) \). If follows that the ZLB will bind when \( \rho_t = \rho_L < 0 \), while the flexible-price efficient equilibrium will be achieved whenever \( \rho_t = \rho \).

Since the shock is unexpected, we can solve the model in the ZLB state, denoting by subscript \( L \) the time-invariant equilibrium values of inflation and consumption therein:

\[
\begin{align*}
\pi_L &= \kappa c_L, \\
c_L &= \frac{\sigma}{1 - \rho_L} \rho_L
\end{align*}
\]

where I define the composite parameter:

\[ \nu \equiv 1 + \sigma \kappa \geq 1. \]

The parameter \( \nu \) captures the response of consumption in a liquidity trap to news about future income/consumption. It is larger than 1, for an increase in future income is associated with inflation and, at the lower bound, with a fall in real interest rates which is further expansionary.\(^8\) This effect is stronger, the higher the elasticity of intertemporal substitution, and the larger the slope of the Phillips curve (the more flexible are prices). Naturally, \( \nu = 1 \) when aggregate supply is vertical (prices are fixed \( \kappa = 0 \)).

The LT equilibrium features deflation and a recession, as long as the following condition holds:

\[ p < \frac{1}{\nu}. \]

The restriction is needed to rule out expectations-driven liquidity traps.\(^9\) One can easily apply the apparatus developed below to solve for optimal FG in sunspot equilibria in which (5) is violated, too. However, as Schmitt-Grohe and Uribe (2015) argue, the optimal mone-

\(^8\)In a "regular" equilibrium whereby the zero bound does not bind and monetary policy follows an active interest rate rule, this parameter is obviously less than one (because news about future income bring about higher real rates, through higher inflation); in particular, \( \nu = 1 - \sigma \kappa (\phi - 1) \), where \( \phi > 1 \) is the response of nominal interest rates to expected inflation.

\(^9\)When the condition does not hold, the economy can be subject to persistent self-fulfilling sunspot fluctuations, as pointed out by Benhabib, Schmitt-Grohe and Uribe (2001, 2002). Mertens and Ravn (2015) study the implications for fiscal multipliers, and Christiano and Eichenbaum (2015) argue that such equilibria are not learnable. An additional restriction rules out "starvation" \( p < \frac{1 + \sigma \rho_L}{\nu} < \frac{1}{\nu} \); it is needed to ensure that consumption stays positive, \( C_L > 0 \) in levels or \( c_L > -1 \) in deviations. This restriction is in fact tighter than the no-sunspot one (5), thus ruling out explosive paths.
tary policy in such an equilibrium in fact implies increasing interest rates (policy which, in a sunspot equilibrium, is expansionary). Throughout this paper, I therefore confine attention to economies for which the constraints hold, and so the LT is "fundamental" in nature.

2.2 Forward guidance

In this paper, I advance a (to the best of my knowledge) novel way to model FG which allows for transparent closed-form solutions; namely, I model FG stochastically through a Markov chain, as a state of the world with a probability distribution, as follows.\(^\text{10}\) Recall that the (stochastic) expected duration of the LT is \(T_L = (1 - p)^{-1}\), the stopping time of the Markov chain. I assume that after this time \(T_L\), the central bank commits to keep the interest rate at 0 while \(\rho_t = \rho > 0\), with probability \(q\). Denote this state by \(F\), and let \(T_F = (1 - q)^{-1}\) denote the expected duration of FG. Appendix 1 reviews the main properties properties of the Markov chain implied by this structure: there are three states, liquidity trap \(L\) (\(i_t = 0\) and \(\rho_t = \rho_L\)), forward guidance \(F\) (\(i_t = 0\) and \(\rho_t = \rho\)) and steady state \(S\) (\(i_t = \rho_t = \rho\)), of which the last one is absorbing. The probability to transition from \(L\) to \(L\) is, as before, \(p\), and from \(L\) to \(F\) it is \((1 - p)q\). The persistence of state \(F\) is \(q\), and the probability to move back to steady state from \(F\) is hence \(1 - q\).

Under this stochastic structure, expectations are determined by:

\[
E_t c_{t+1} = pc_L + (1 - p) q c_F \tag{6}
\]
\[
E_t \pi_{t+1} = p \pi_L + (1 - p) q \pi_F
\]

Evaluating the IS (1) and Phillips (3) curves during state \(F\) and solving for the time-invariant equilibrium delivers Proposition 1, which describes the equilibrium once the trap is over and during the FG period.

**Proposition 1** *Equilibrium consumption and inflation during the forward guidance state \(F\) are given by:*

\[
c_F = \frac{\sigma}{1 - q \nu} \rho \tag{7}
\]
\[
\pi_F = \kappa c_F.
\]

As in a deterministic setting and as shown in all papers using FG, keeping the interest rates low creates an expansion in the "future" (in the \(F\) state). This is true as long as the

\(^{10}\)Woodford, 2011 uses a similar framework to look at fiscal stimulus that extends beyond the duration of the liquidity trap, and studies the impact on the spending multiplier.
persistence probability of the FG state satisfies the same restriction as the persistence of the trap itself, namely:

\[ q < \frac{1}{\nu} \]  

(8)

The future expansion is increasing in the degree of forward guidance \( q \):

\[
\frac{dc_F}{dq} = \frac{\sigma \nu}{(1 - q \nu)^2} \rho = \frac{\nu}{1 - q \nu} c_F > 0
\]

In addition to creating a future expansion, FG also mitigates the effect of the LT by bringing about an expansionary force today. In particular, in the \( L \) state we now have (evaluating the model (1)–(3) taking into account the zero lower bound and expectation formation as defined in (6)):

\[
c_L = \frac{(1 - p) q \nu}{1 - p \nu} c_F + \frac{\sigma}{1 - p \nu} \rho_L
\]

(9)

Replacing the equilibrium expression in the \( F \) state found in Proposition 1, we obtain Proposition 2, which describes the equilibrium solution during the liquidity trap, under forward guidance

**Proposition 2** Equilibrium consumption and inflation in the LT state, under FG, are given by:

\[
c_L = \frac{(1 - p) q \nu}{1 - p \nu} \rho + \frac{\sigma}{1 - p \nu} \rho_L
\]

\[
\pi_L = \kappa c_L
\]

(10)

with restrictions (5) and (8).

More FG leads to higher consumption (and hence inflation) during the trap:\(^1\)

\[
\frac{dc_L}{dq} = \frac{\nu (1 - p)}{1 - p \nu} \frac{1}{1 - q \nu} c_F > 0.
\]

Indeed, as the probability approaches the bifurcation point defined by (8), in other words as the FG becomes "long", the effect of FG becomes explosive.\(^2\) This is (our stochastic fram-
work’s version of) one aspect of what Del Negro et al have dubbed the "forward guidance puzzle". Another aspect of that puzzle is that the higher the persistence of the trap \( p \) (the further into the future FG starts), the higher the expansionary effect of FG during the trap \( dc_L/dq \) (it can be easily shown that the cross derivative \( d^2c_L/(dqdp) \) is positive).

Figure 1 plots consumption in the liquidity trap and in the FG state, along with (annualized) inflation in the liquidity trap, as a function of the FG probability \( q \) for the parameter region defined by (8). The illustrative parametrization used in the Figure has \( \beta = 0.99, \psi = 0.01, \sigma = 1, \varphi = 1, p = 0.8 \) and a spread shock of \( \rho_L = -0.01 \), i.e. 4 percent per annum. This delivers a recession of 5.5 percent and annualized inflation of 1 percent in the absence of FG \( (q = 0) \). Notice that FG \( q = 0.815 \) closes both the consumption gap and inflation, illustrating a more general result emphasized in the following Proposition.

**Proposition 3** The duration of FG that stabilizes the economy perfectly (closes the gap and delivers zero inflation) is determined by the persistence probability:

\[
q^0 = \frac{1}{\nu (1 - p + \Delta_L)},
\]

where \( \Delta_L \equiv \frac{(-\rho_L)}{\rho} > 0 \).

\( \Delta_L \) is a new parameter capturing the relative size of the financial disruption causing the
ZLB to bind (and hence the recession);\textsuperscript{13} $\Delta_L$ turns out to be one of the key determinants of the optimal level of FG. Clearly, $q^0$ defined by Proposition 3 is not the optimal horizon of FG. Nor is FG necessarily pushed forward into the future as much as possible (which, in this stochastic case, means "close to the asymptote"). The reason why it is not optimal to use FG to perfectly close the gap in the trap has to do with the welfare costs of FG which are in fact incurred in the future, in the F state.\textsuperscript{14}

Optimal policy consists of FG that strikes a balance these two opposing forces, as we will see next. First, I solve for what I call "optimal forward guidance": taking as constraints the (time-invariant) equilibrium conditions, and given a policy of keeping interest rates low beyond the end of the trap with a certain probability $q$, I solve for the probability $q$ that maximizes welfare—which determines the expected duration of FG. Then, I solve for the full Ramsey-optimal policy: the optimal path of interest rates that maximizes welfare, which in equilibrium will imply FG: interest rates are at zero beyond the end of the trap for a certain duration, as shown in the literature reviewed in the Introduction. As we shall see, the two notions of optimal policy imply FG durations that are very close for standard parameterizations—the advantage of the former being that it delivers a closed-form solution.

\section{Optimal Forward Guidance}

It is well-known (Woodford 2003, Ch. 6; see also Woodford 2011 for a ZLB application) that in this economy the welfare function can be represented as a quadratic loss function under certain conditions that are fulfilled here (an optimal subsidy makes the steady-state efficient); namely, a benevolent central bank will minimize

$$E_0 \sum_{t=0}^{\infty} \beta^t L_t = \frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t \left( \pi_t^2 + \chi c_t^2 \right),$$

where $\chi$ is the welfare-based relative weight attached to stabilizing real activity, equal in the baseline model to $\kappa/\theta$.

Since the equilibrium solution is time-invariant in each of the three states, the per-period loss function is, for any state $j = \{L, F, S\}$: $\pi_j^2 + \chi c_j^2 = (\chi + \kappa^2) c_j^2$. Recall that in state $S$ the economy is back to steady state, so the loss there is zero. Appendix 1 uses the Markov chain structure to derive in detail the appropriate lifetime welfare objective, which is of the

\textsuperscript{13}Curdia and Woodford (2009) outline a model with credit frictions in which this disruption occurs endogenously as a spread shock.

\textsuperscript{14}Eggertsson and Woodford (2003), in the context of optimal policy under a ZLB constraint, discuss both the cost of future distortions and the finite horizon of FG, but without providing a closed-form solution of the optimal FG duration.
form:

\[ W = \frac{1}{1 - \beta p^2} \left[ c_L^2 + \omega(q) c_F^2 \right], \]

where \( \omega(q) \) is the appropriate discount factor for the FG state, given the Markov chain structure,

\[ \omega(q) = \frac{1 - \beta p + \beta (1 - p) q}{1 - \beta q}. \] (11)

In particular, the optimal weight counts for the times the process spends in state \( F \) when starting from \( F \) (given by \((1 - \beta q)^{-1}\)); as well as for all the times spent in time \( F \) when starting from \( L \), before being absorbed into \( S \) (given by \( \beta (1 - p) q/((1 - \beta p) (1 - \beta q)) \)).

Notice that the optimal weight of future consumption variability \( \omega(q) \) is increasing in \( q \), which is intuitive: the longer the economy spends in the \( F \) state, the larger the total welfare cost of consumption variability in that state.

The central bank chooses FG duration (persistence probability \( q \)) by solving the optimization problem

\[ \min_q W = \frac{1}{1 - \beta p^2} \left[ c_L^2 + \omega(q) c_F^2 \right], \] (12)

taking as constraints the equilibrium values \( c_F \) and \( c_L \) given respectively by (7) and (10). The first-order condition of this problem is:

\[ c_L \frac{dc_L}{dq} + \omega(q) c_F \frac{dc_F}{dq} + \frac{1}{2} \frac{d\omega(q)}{dq} c_F^2 = 0 \] (13)

and has a clear intuitive interpretation. The first term is the welfare benefit of more forward guidance, through remedying the LT-caused recession and hence minimizing consumption volatility in the trap; this is proportional to the level of consumption in the trap. The last two terms are the total cost of forward guidance: the former is the direct cost, a future consumption boom being associated with inefficient volatility; the latter is the discounting effect discussed above: the longer the time spent under FG, the larger the cost (which is proportional to consumption volatility in the \( F \) state).

For the sake of simplicity, transparency and analytical tractability, it will be useful to consider a modified welfare function that is simplified even further. In particular, we consider the case whereby the central bank attaches equal weights (in the sense of ignoring the extra discounting costs) to future and present in its intertemporal objective:

\[ \omega(q) = 1, \omega'(q) = 0, \]

hence solving

\[ \min_q \frac{1}{1 - \beta p^2} \left( c_L^2 + c_F^2 \right). \]
While this is not strictly speaking correct, it generally provides an upper bound measure on the amount of optimal FG, because it ignores part of the welfare costs of FG. Furthermore, since this component of the welfare cost is proportional to squared consumption, it is second-order and likely to be small; nevertheless, we will also look at the case of optimal discounting subsequently for robustness and compare it to the closed-form solution found here. In this case of treating present and future symmetrically, the first-order condition simplifies to

\[-c_L \frac{d c_L}{dq} = c_F \frac{d c_F}{dq},\]

which delivers Proposition 4.\(^\text{15}\)

**Proposition 4** The optimal duration of Forward Guidance (FG) is \(q = 0\) if \(\Delta_L < \frac{(1-\mu)^2}{1-p}\) and \(q^* > 0\) otherwise, where:

\[q^* = \frac{\Delta_L - \frac{(1-\mu)^2}{1-p}}{\nu (1-p + \Delta_L)}.

The closed-form solution obtained here (by virtue of the tractable setup used to model FG) allows hitherto unexplored insights into the magnitude of the optimal duration and its economic determinants; furthermore, similar insights carry through to more sophisticated models of aggregate demand studied below.

First, some FG is optimal \((q^* > 0)\) whenever the size of the disruption is larger than the threshold defined by the proposition, which under the baseline calibration is 0.14. Conversely, if the disruption is smaller than this threshold, it is optimal to refrain from FG altogether. The intuition is that when the disruption is small, the welfare cost of the trap, albeit first-order, is also small—and so is the benefit of FG. Therefore, the welfare cost of FG, although second-order, is enough to prevent an optimizing central bank from doing FG; to the best of my knowledge, this finding is novel.

Second, quite evidently, the optimal level of FG is strictly lower than the perfectly stabilizing level of FG \((q^* < q^0)\) since the latter simply ignores the welfare cost of FG in the future.

Optimal FG is shaped by its key determinants as follows. The higher the disruption \(\Delta_L\) and/or its persistence \(p\), the higher the optimal level of FG \((dq^*/d\Delta_L > 0; dq^*/dp > 0)\). The intuition is the same for both parameters: larger shock or larger persistence create a higher recession, a higher welfare cost of the LT and hence more of a welfare scope for FG. With the composite parameter \(\nu = 1 + \kappa \sigma = 1 + \psi (\varphi \sigma + 1)\), things are different. Note that \(\nu\) itself is increasing with price flexibility \(\psi\) and with intertemporal substitution \(\sigma\), while it is

\(^{15}\text{Appendix 2 shows that the sufficient second-order condition also holds.}\)
decreasing with labor elasticity (increasing with $\varphi$). The optimal level of FG is increasing with $\nu \ (dq^*/d\nu > 0)$ if

$$\Delta L < \frac{1 - (p\nu)^2}{1 - p}$$

which under the baseline parameterization is 1.67. More price flexibility calls for more FG, but only when the disruption is not "too large". Given that the steady-state interest rate is around 1 percent, this restriction seems very likely to be verified (it requires that the shock be lower than 6.5 percent per annum).

3.1 Robustness: Optimal Discounting and Forward-Looking Pricing

In order to obtain sharp closed-form solutions, I made two simplifying assumptions: equal discounting of present and future, and a contemporaneous Phillips curve. I now illustrate that the results are robust to relaxing these assumptions; since a closed-form solution is not feasible any longer, we do this numerically. Take first optimal discounting, the optimal duration of FG is determined by solving (13) under (11). Appendix A.4 contains the detailed solution and Figure 7, which illustrates that the optimal $q$ in this case (plotted there as a function of $p$ and $\nu$) is very close to the equal-weights $q^*$ provided in Proposition 4.16 We plot the implied expected FG durations ($1/ (1 - q)$) as a function of the expected trap duration for the two cases in Figure 3 below, and will compare them with the optimal FG duration implied by Ramsey-optimal policy—which I will solve for presently.

Turn now to the more general New Keynesian Phillips curve with discounting (2): Appendix A.5 derives the full solution under FG, which has the same structure as in the previous, simpler case. Figure 2 plots the optimal FG duration as a function of the degree of forward-looking pricing $\beta$, the LT persistence $p$, the slope of the Phillips curve, and the disruption, each for two cases: equal weights $\omega(q) = 1$ and optimal discounting (11).17 As the Figure illustrates, all of our previous conclusions still hold under the more general forward-looking price-setting (2); furthermore, the difference between the equal weights and optimal discounting scenarios is even smaller.

Most remarkably perhaps, the discount factor of firm’s shareholders $\beta$, which dictates the degree of forward-looking pricing, influences very little the degree of optimal FG. This is one facet of a more general insight that FG in this model is mostly about aggregate demand,

---

16 The main difference occurs as a function of the trap persistence. With optimal discounting, the welfare cost of FG now receives a larger weight, and it is only optimal to do FG if things are bad enough (i.e. if the size and/or persistence of the trap are large enough). This is an illustration of our previous insight that $q^*$ represents in fact an upper bound on optimal FG.

17 The figure makes use of a numerical solution; I show in the Appendix that the solution of the optimal FG problem now results in a (sixth-order) polynomial equation in $q$ that cannot be solved in closed-form.
rather than aggregate supply. In the remainder of the paper, I will therefore work with the simplest case (3) which allows obtaining transparent closed-form solutions; but we can now be reassured that the results derived below are robust to considering the more general (2), as can be easily checked numerically in a similar manner.

![Graph](image)

Figure 2: Optimal $q$ under NK Phillips curve, equal weights (red dashed) vs optimal discounting (solid blue), as a function of $\beta$, $p$, $\kappa$ and $\Delta_L$ respectively.

### 3.2 Optimal Forward Guidance and Ramsey-Optimal Policy

How does the simple, closed-form solution compare with the fully optimal Ramsey policy with a lower bound, pioneered by Jung et al (2005), Eggertsson and Woodford (2003), Adam and Billi (2006) and Nakov (2008), and reviewed in the introduction. To answer this question, I calculate optimal policy in this setup with Markov discount factor shocks, as in Eggertsson and Woodford—I also cover the case where the discount factor shock has a known duration,
as in Jung et al, in Appendix B.1 for completion.

The Ramsey-optimal commitment policy is found by using the Lagrangian method introduced by Jung et al to solve the problem:\(^{18}\)

\[
\min_{c_t} E_0 \sum_{t=0}^{\infty} \beta^t \left[ c_t^2 + 2\phi_t (c_t - \nu c_{t+1} - \sigma \rho_t) \right]
\]

\[
\text{s.t. } c_t \leq \nu E_t c_{t+1} + \sigma \rho_t.
\]

For this simpler version of the standard problem, I combined (1), (3) and the lower bound \(i_t \geq 0\) to deliver the inequality constraint on the control \(c_t\). The first-order conditions for this problem are

\[
\begin{align*}
  c_t + \phi_t - \beta^{-1} \nu \phi_{t-1} &= 0 \\
  (c_t - \nu E_t c_{t+1} - \sigma \rho_t) \phi_t &= 0 \text{ and } \phi_t \geq 0,
\end{align*}
\]

where \(\phi_t\) is the co-state, Lagrange multiplier on the constraint, with given initial value the initial value \(\phi_{-1} = 0\). The second line in (14), together with the constraint, summarizes the Kuhn-Tucker conditions.

Optimal commitment policy implies that the ZLB constraint binds \((\phi_t > 0 \text{ and } i_t = 0)\) for longer than the (expected) duration of the financial shock, as already emphasized by all the papers cited above.\(^{19}\) For the purpose of comparison with the notion of "optimal FG" introduced in this paper, I solve for the optimal duration of FG as implied by Ramsey policy, as a function of the (expected) duration of the shock. Unlike the papers mentioned above, and again by virtue of our simplifying assumptions, we can solve for the path of Lagrange multipliers \(\phi_t\) in closed form. Appendix B describes this in detail and outlines the solution.

Since it is so simple yet captures a key mechanism of the literature reviewed above, it is useful to describe transparently the main idea. A policy of zero interest rates is defined implicitly by a binding constraint, i.e. \(\phi_t > 0\); we can therefore combine the second equation (the binding constraint) in (14) with the first, and obtain a second-order stochastic difference equation. With our simple model, this equation is very easy to solve—provided that the roots are on the right sides of the unit circle (which they are: as I show in the appendix, the roots

---

\(^{18}\)Strictly speaking, the quadratic loss is still \(\Gamma c_t^2\), with \(\Gamma = \kappa^2 + \chi\), under the simplified Phillips curve (3). In formulating the Ramsey problem, I rescaled the objective normalizing \(\Gamma = 1\)—this is just a rescaling of the Lagrange multiplier \(\phi_t\).

\(^{19}\)In particular, equation (35) in Jung et al (2001) reads \(0 \leq T^d \leq T^c < \infty\) : with known shock duration, the time of zero interest rates under discretion \(T^d\) is weakly positive and lower than the time of zero interest under commitment \(T^c\), which is itself finite. Hasui et al (2016) suggest, intriguingly, that in an economy with inflation persistence optimal policy is characterized by front-loading, or early-tightening, rather than forward guidance: rates should be increased before the trap ends.
are $\nu^{-1} < 1$ and $\beta^{-1} \nu > 1$) a unique equilibrium requires two boundary conditions. The first is the initial condition, $\phi_{-1} = 0$, and the second is a boundary condition that implicitly defines the stopping time, i.e. the expected duration of zero interest rates implied by Ramsey policy, call it $T^R_F$. This, we find $T^R_F$ by solving the boundary condition:

$$\phi_{T + T^R_F - 1} = 0,$$

where $T$ is the duration of the shock, a random variable with expected value $(1 - p)^{-1}$. In Appendix B, I show that this delivers a (nonlinear) equation linking $T^R_F$ and all the model parameters including the shock duration, of the form $\phi(T^R_F) = \phi(T^R_F, T)$ (the functions are presented in the Appendix). $\hat{\phi}(.)$ is an increasing function of the FG duration and gives the value at of relaxing the constraint contingent upon the shock taking the value $\rho_L$, value calculated at the stopping time of the shock $T$. At the same moment $T$, $\hat{\phi}(.)$ gives the value of relaxing the constraint during FG, i.e. once the shock stopped hitting but interest rates are still at zero. The optimal duration of FG is found at the intersection of the two.

I compare the Ramsey-implied FG duration $T^R_F$ with the optimal FG duration found above in closed-form, $(1 - q^*)^{-1}$, for the baseline parameter values, in Figure 3—the measures are remarkably close to each other. This extends to other calibrations, it also holds for the optimal-discounting optimal FG (see Figure 7.2 in the Appendix), and is also true in a setup where the duration of the shock is known, as in the original Jung et al paper (I cover this case for completion in Appendix B—see also Figure 8). The bottomline is that the simpler notion of optimal FG proposed in this paper gives very similar conclusions to those arising in a full Ramsey-optimal monetary policy analysis, such as those previously used to argue for forward guidance on welfare grounds.
Figure 3: FG duration implied by Ramsey policy (blue dot-dash), along with optimal FG duration under equal weights (red dash).

4 The Best of Both Worlds: a Simple Rule for Forward Guidance

Both of the optimal FG concepts derived above assume—and in fact rely upon—commitment. But adopting an optimal policy rule and explaining its determinants may be very hard to communicate in practice, even in this simple model and even for the simplest notion of optimal policy introduced here and summarized by \( q^* \). This may indeed aggravate the credibility problem inherent to this type of policy commitment. This issue is likely to be amplified in more complex and realistic models, where the determinants of optimal FG are likely to be less transparent.

To address this issue, I propose a "simple rule" for forward guidance.\(^{20}\) The rule, whose virtues I analyze below, combines the advantages of date-based (or time-dependent) and state-contingent FG; in that sense, it allows to achieve the best of both worlds.

4.1 A Simple FG Rule in the Baseline NK Model

Inspired by the optimal FG closed-form solution in Proposition 4, a simple FG rule consist of committing to (and communicating to the public) a slightly different object, denoted by superscript \( S \), from "simple", namely:\(^{21}\)

\[
q^S = \frac{\Delta_L - (1-p)}{1-p + \Delta_L}
\]

A plot of this probability along with the optimal probability \( q^* \), as a function of the trap persistence, reveals that they are virtually indistinguishable for the baseline calibration—I therefore spare the reader this plot. It is more useful and transparent for operational purposes to express this in terms of expected duration,

\[
T^S_F = \frac{1}{1-q^S} = \frac{1}{2} + \frac{1}{2} \Delta_L T_L,
\]

where we recall that \( T_L = (1-p)^{-1} \) is the expected duration of the trap.

\(^{20}\)This is related to the literature analyzing, in a different, "normal-times" context simple interest rate rules and their roles as approximation to optimal policy, see e.g. Schmitt-Grohé and Uribe (2007). In the context of optimal policy with a lower bound, Eggertsson and Woodford (2003) and Nakov (2008) studied the virtues of price-level targeting.

\(^{21}\)This relies on an approximation of \( q^* \) around the point where \( z = 1 \), which is not too far-fetched in practice since prices are very sticky on aggregate (the estimated slope of the Phillips curve is in the 0.01—0.05 range).
The simple rule (16) captures intuitively the welfare-based determinants of optimal FG: the disruption and the duration of the trap, \( \Delta_L \) and \( T_L \). Figure 4 plots this simple-rule duration together with the optimal durations analyzed above (which are the same as in the previous Figure). The simple rule does remarkably well at approximating optimal policy; the maximum difference under this calibration is of almost two quarters when the trap lasts eight to ten years.\(^{22}\)

![Figure 4: FG duration implied by Ramsey policy (blue dot-dash), along with optimal FG duration under equal weights (red dash) and optimal discounting, as a function of trap duration.](image)

Its key advantage, however, comes from its communication virtues. A central bank wanting to implement FG that is close to optimal in this model can communicate \( T_F^S \) to the public even if the it does not know ex ante the length of the liquidity trap \( T_L \). It is enough that the central bank state “whatever the length of the trap, the nominal interest rate will stay at its effective lower bond for \( T_F \) extra periods, where \( T_F \) is defined ex post (once \( T_L \) is observable and \( \Delta_L \) measurable), as half of one plus the product of the trap’s duration and depth”. This alleviates an important communication loophole of FG that we underline below.

The simple rule delivers the best of both worlds, in the following sense. Since it is specified in terms of durations, it is time-dependent, or date-based. But since it specifies FG duration

\[ \text{The welfare cost of the simple rule relative to optimal } q^* \text{ can be computed using a second-order approximation of the welfare function around } q = q^* \text{ and evaluating it at } q = q^S; \quad W(q^S) - W(q^*) = W'(q^*) (q^S - q^*) + \frac{1}{2} W''(q^*) (q^S - q^*)^2. \] 

The first term is zero by the Envelope Theorem; the loss is thus proportional to the square difference of the two probabilities, and the curvature of welfare in \( q \) evaluated at \( q^* \). This object can be evaluated in any quantitative model in order to assess the accuracy of the simple rule.
as a function of trap duration (which can be observed and verified ex-post) it exploits the advantage of state-contingent rule: it is a way for the central bank to communicate, to signal something about an unobservable state. Evidently, in order to substantiate this point we need a model with information imperfections—our next topic.

4.2 Virtues of Simple Rule with Heterogeneous Beliefs: Delphic Guidance for Delphic Agents

The advantages pertaining to credibility and communication of simple rules apply with particular force in models where the commitment problem is modelled explicitly and is related to information frictions. In this section, I extend the previous analysis to one simple example of such a model, based on Andrade, Gaballo, Mengus and Mojon (2015); the analysis can be extended to different environments such as e.g. Wiederholt’s 2015 dispersed-information or Bassetto’s 2016 analysis of cheap talk and reputation.

To anticipate, the main benefit of a simple rule in a model with "delphic" agents (who take FG to mean communication, rather than commitment), is that it makes FG itself "delphic": it anchors agents expectations. Whereas FG in the form of commitment to an optimal rule becomes self-defeating if some agents use the central bank’s announcements to infer a state. As we will see, in such a setup optimal policy may even refrain from FG altogether, if there are enough such "delphic agents". To substantiate these points, I first calculate in closed-form the optimal duration of (odyssean, commitment-) FG in the model with heterogenous beliefs due to Andrade et al (2015). The only original contribution here is to add one layer that simplifies the model and allows for closed-form solution: I model beliefs about probabilities, rather than durations.

Campbell et al (2012) introduced the distinction between to types of FG: odyssean (as commitment) versus delphic (as communicating fundamentals). Andrade et al (2015) model this distinction as follows. A fraction of "optimistic" agents perceive forward guidance to be "odyssean": they believe that the central bank will keep interest rates low for the announced number of periods after the ZLB stops binding. The remainder, "pessimistic" agents perceive FG in a "delphic" way: they believe that the central bank’s commitment in fact signals something about the fundamentals. Namely, pessimistic believe that the ZLB will bind for exactly as many periods as the central bank promises to do FG. The authors then show that the optimal duration of FG depends on the share of pessimistic agents in a nonlinear way: more pessimistic agents call for more FG up to a threshold, beyond which the contractionary effect of FG through pessimistic agents’ beliefs dominates and less FG is called for.

Their framework translates naturally in the analytical setup of this paper. Assume that
a fraction $\alpha$ of agents are pessimistic and perceive FG in a *delphic* way: they think that in state $F$, which still occurs with probability $p(1-q)$, the value of the discount factor shock is $\rho_L$. The remainder fraction of $1-\alpha$ optimistic agents perceive FG in an *odyssean* way and expect that in state $F$, occurring with probability $p(1-q)$, the value of the discount factor shock is $\rho$. This is the only dimension of disagreement, and otherwise the model is identical to the benchmark model studied previously.

When state $F$ materializes, pessimistic agents update their beliefs (ex post, both agents know the true shock); therefore, equilibrium during the F state is identical for both agents and hence the same as under homogenous beliefs: $c_F^o = c_F^m = c_F = \sigma \rho / (1-q\nu)$, denoting type $j = o, m$ (optimists and pessimists, respectively). Where heterogeneous beliefs make a difference is for the L state, through expectations; ex ante pessimists are wrong, and optimists know this. Expectations are therefore (similar equations hold for inflation expectations):

$$
\begin{align*}
E^o_t c^o_{t+1} &= pc^o_L + (1-p) q c_F \\
E^m_t c^m_{t+1} &= pc^m_L + (1-p) q c_F^{mw}
\end{align*}
$$

where $c^{mw}_F = \sigma \rho_t / (1-q\nu)$ is the value (wrongly) expected by pessimists ex ante. Solving the model under these beliefs, we obtain aggregate consumption during the trap for each type

$$
\begin{align*}
c^o_L = q \frac{(1-p)\nu}{1-p\nu} c^o_F + \frac{\sigma}{1-q\nu} \rho_L \\
\end{align*}
$$

where for optimists $c_F^o = c_F = \sigma \rho / (1-q\nu)$ while for pessimists $c_F^{mw} = \sigma \rho_t / (1-q\nu)$.

Aggregate consumption during the trap is therefore:

$$
\begin{align*}
c_L = q \frac{(1-p)\nu}{1-p\nu} \frac{\sigma \rho}{1-q\nu} (1-\alpha (1+\Delta_L)) + \frac{\sigma}{1-p\nu} \rho_L
\end{align*}
$$

where $\Delta_L$ denotes as before the disruption. This expression illustrates that the heterogeneous beliefs channel of the form studied by Andrade et al weakens the positive effect of FG on consumption during the trap $c_L$ (the factor $1-\alpha (1+\Delta_L)$). Indeed, FG can have perverse contractionary effects if there are too many pessimistic agents, namely:

$$
\alpha > \bar{\alpha} = \frac{1}{1+\Delta_L}. \hspace{1cm} (17)
$$

The threshold is lower, the larger is the disruption $\Delta_L$ (and hence, the larger the mispercep-

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23I refer the reader to Andrade et al (2015) for the subtle points regarding the microfoundations of this beliefs structure in terms of risk sharing. The paper also presents convincing empirical evidence that agents agree on interest rates but disagree on macroeconomic aggregates, as assumed by the model.
tion of pessimistic agents). The presence of pessimists reduces the power of FG proportionally: indeed, $dc_L/dq$ is now scaled down by a factor $1 - \alpha \left(1 + \Delta_L \right)$.

Optimal policy consists of balancing two forces. On the one hand, FG being less effective during the trap (because it may be misinterpreted) means more of it is needed to achieve the same outcome; this generates more scope for FG ceteris paribus, up to the point where FG becomes indeed contractionary. On the other hand, the effect of heterogeneous beliefs on the benefits and costs of FG is asymmetric: the welfare cost of inefficient volatility during the F state is independent of $\alpha$. The trade-off is resolved— for the case of equal discounting of the L and F states— as described in the following Proposition (which substantiates transparently through closed-form solutions a mechanism that is already at work in Andrade et al’s numerical simulations).

**Proposition 5** The optimal duration of FG under heterogenous beliefs is determined by the persistence probability $q_{HB} = 0$ if $\alpha > (1 + \Delta_L)^{-1} \left(1 - \frac{(1-p\nu)^2}{(1-p)(1-\alpha(1+\Delta_L))}\right)$ and $q_{HB}^* > 0$ otherwise, where:

$$q_{HB}^* = \frac{1}{\nu (1-p) (1-\alpha (1 + \Delta_L)) + \Delta_L}.$$

Notice that it is optimal to refrain from FG for values of $\alpha$ that are lower than the threshold making FG contractionary during the trap, $(1 + \Delta_L)^{-1}$. How does the optimal duration of FG depend on the share of pessimists? Simple differentiation of $q_{HB}^*$ with respect to $\alpha$ shows that the degree of optimal FG is increasing with the degree of information imperfections $\alpha$ as long as:

$$\alpha < \frac{1}{1 + \Delta_L} \left(1 - \frac{(1-p\nu)^2 + (1-p\nu)\sqrt{(1-p\nu)^2 + \Delta_L^2}}{(1-p)\Delta_L}\right).$$

Before this threshold is reached, the expansionary effect of heterogeneous beliefs dominates, thus increasing the scope for FG; beyond the threshold, the welfare cost of excess volatility once the trap is over, cost which is unaffected by heterogeneous beliefs, dominates. Intuitively, the expansionary channel prevails when the scope for FG is strong to start with: i.e., when prices are flexible enough, when there is enough intertemporal substitution (both of which translate into a higher $\nu$), when the disruption is large and when the trap is persis-

---

24 I am grateful to Gaetano Gaballo for clarifying discussions on this topic.

25 Notice that since $p < \nu^{-1}$ is needed for equilibrium, this threshold is tighter than the cutoff beyond which it is optimal to refrain from FG, $\alpha < (1 + \Delta_L)^{-1}$. 

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22
Figure 5: Optimal FG persistence with heterogeneous beliefs as a function of $\alpha$, for $\nu = 1.02$ (red dashed) and $\nu = 1.2$ (blue dots).

For our baseline parameter values, optimal FG duration is in fact uniformly decreasing with $\alpha$: the expansionary channel (doing more FG to compensate for its being misinterpreted) is not strong enough to overcome the welfare cost of FG. Figure 5 illustrates this by plotting $q_{HB}^*$ as a function of $\alpha$ with red dashed line. For other parameterizations optimal FG is initially increasing with $\alpha$, e.g. for $\nu = 1.2$ (blue dots in the Figure).

The foregoing optimal policy assumes that the central bank chooses an optimal policy within the class of commitment, Odyssean policy options: it merely changes the duration of its commitment as a function of the share of delphic agents. But the central bank can do better by recognizing the information friction (that some agents are delphic), and using instead a communication policy. Bassetto (2016) shows, in a more general context, that odyssean FG is a redundant policy instrument in an environment with informational asymmetries, while delphic FG allows achieving better equilibria.

Our simple rule (16) has precisely such flavor as suggested by Bassetto’s results. As argued above, it blends the advantages of state-contingent and date-based FG. The former, indeed, the derivative of $q_{HB}^*$ at $\alpha = 0$ is positive iff

$$\frac{\Delta_L}{1 - p} > 2 \left( \frac{(1 - p\nu)^2}{(1 - p)^2} - 1 \right)^{-1}.$$  

A similar picture arises for the optimal FG persistence under optimal discounting (found by solving numerically (13) under (11), with the constraints found under heterogeneous beliefs.)
because it refers to something that all agents can agree on (the end of the trap, ex post). The latter, because it is simpler to communicate a date at which interest rates shall turn positive, rather than a complicated mapping into the evolution of some state variables. The simple rule allows achieving a better equilibrium in this model because it resolves the information asymmetry: it implicitly consists of delphic FG, which is what delphic agents need. And it approximates well odyssean FG, which is what odyssean agents need.

5 Aggregate Demand and Optimal Forward Guidance

The analysis of optimal monetary policy in models with heterogeneous agents is notoriously difficult, even abstracting from the lower bound. In this section, I show how the analytical apparatus developed here can be used to derive closed-form results for optimal FG in models with heterogeneous agents. I turn to two frameworks that introduce tractable heterogeneity on the aggregate demand side, both of which have been used to study liquidity traps—but not the optimal degree of forward guidance. In particular, I study a model with limited asset market participation or hand-to-mouth agents drawing on Bilbiie (2008) but applying also to a simplified version of Eggertsson and Krugman (2012), and a model with incomplete markets and unemployment due to McKay, Nakamura and Steinsson (2015, 2016).

5.1 Optimal FG with (Employed) Hand-to-Mouth

The first framework is taken from Bilbiie (2004, 2008), which built on work by Gali, Lopez-Salido and Valles (2004, 2007)—although none of those articles studied liquidity traps. Eggertsson and Krugman (2012) use a model with an essentially identical IS curve, augmented with a particular theory of the natural interest rate: deleveraging by constrained borrowers pushes the interest rate down, which can trigger a liquidity trap if the shock is large enough. Insofar as the model I use here has essentially the same IS curve, one can see these results as pertaining to the properties of optimal FG in a simplified version of their model, too. The models are identical in the key mechanism making interest rates have larger effects on aggregate demand when there are more constrained agents.

The model relies upon two key ingredients: one class of agents of mass $\lambda$ is excluded from asset markets and hence has no Euler equation. However, they participate in labor markets and make an optimal labor supply decision; I label these agents "hand-to-mouth". The rest

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28 Specificaly, it is the model that one would get if one set the debt limit to zero in Eggertsson and Krugman (thus eliminating debt deflation) and taking as the financial disruption a spread shock (instead of a deleveraging shock).

29 This holds as long as we stay in the "standard" region; Bilbiie discusses in great detail the other region in which aggregate demand logic is inverted. This has stark implications for FG too, which we ignore here.
of the agents also work and trade a full set of state-contingent securities, including shares in monopolistically competitive firms (thus receiving their profits). The equilibrium of the aggregate demand side of this economy delivers an aggregate IS curve of the form:\(^{30}\)

\[
c_t = E_t c_{t+1} - \sigma^{HM} (i_t - E_t \pi_{t+1} - \rho_t),
\]

(18)

where \(\sigma^{HM}\) denotes the aggregate elasticity of intertemporal substitution with hand-to-mouth agents, given by

\[
\sigma^{HM} = \frac{\sigma}{1 - \frac{\lambda}{\lambda + \varphi}}.
\]

The introduction of hand-to-mouth agents increases the elasticity of aggregate demand to interest rates \(\frac{\partial \sigma^{HM}}{\partial \lambda} > 0\), up to a threshold level where this elasticity changes in fact sign—see Bilbiie (2008) for a full analysis of that "inverted aggregate demand logic" region which features a version of the paradox of thrift. Here, we restrain attention to the region whereby the elasticity stays positive, and is increasing in \(\lambda\) and \(\varphi\) for the following reason: an interest rate cut leads to intertemporal substitution by the savers, and hence an increase in their consumption. With sticky prices, this leads to an expansion in labor demand, an increase in the real wage, and hence to an even larger increase in demand by hand-to-mouth agents who only have labor income. This is an equilibrium, for savers optimally choose to work more to produce the extra demanded output; they do so because the increase in wage drives their profit income down, creating a negative income effect that makes them want to work more and sustain the expansion—again, as long as there are not too many hand-to-mouth agents, \(\lambda < (1 + \varphi)^{-1}\).

The rest of the model (in particular: the Phillips curve) is unchanged, and so can be analyzed using exactly the same tools used earlier. In particular, let

\[
\nu^{HM} \equiv 1 + \sigma^{HM} \kappa \geq 1,
\]

which is also strictly increasing in the share of hand-to-mouth agents \(\frac{\partial \nu^{HM}}{\partial \lambda} > 0\) as long as \(\sigma^{HM} > 0\).

We obtain that consumption during the F state of forward guidance is given by:

\[
c_F = \frac{\sigma^{HM}}{1 - q \nu^{HM} \rho},
\]

where we now have the condition \(q < (\nu^{HM})^{-1}\). It is immediately shown then that as long

\(^{30}\)The Appendix contains a brief derivation. See Bilbiie, 2008 for details; see also Eggertsson and Krugman, 2012, equations 2 and 3.
as this condition holds, (together with $\sigma^{HM} > 0$) the future expansion is higher when there is a larger mass of hand-to-mouth agents.

During the liquidity trap, we have:

$$c_L = \frac{q (1 - p) \nu^{HM}}{1 - p \nu^{HM}} c_F + \frac{\sigma^{HM}}{1 - p \nu^{HM}} \rho L,$$

or, substituting equilibrium future consumption:

$$c_L = \frac{\sigma^{HM}}{1 - p \nu^{HM}} \rho L + \frac{q (1 - p) \nu^{HM}}{1 - q \nu^{HM}} - \frac{\sigma^{HM}}{1 - p \nu^{HM}} \rho,$$

with the extra condition $p < (\nu^{HM})^{-1}$.

A higher share of hand-to-mouth agents $\lambda$ implies a higher expansionary effect of FG during the liquidity trap, and at the same time a higher expansionary effect during the F state itself. However, it also implies that the recession is higher to start with, by the same token—all these effects are driven by the elasticity of aggregate demand to interest rates being increasing in $\lambda$. These effects combine in order to shape the optimal duration of forward guidance, emphasized by Proposition 5 (derived, again for the case of equal weighting of the two states).

**Proposition 6** The optimal duration of FG with (employed) hand-to-mouth agents is determined by the persistence probability $\Phi_{HM} = 0$ if $\nu^{HM} < \frac{\sqrt{(1-p)\Delta_L} - 1}{p}$ and $\Phi_{HM}$ otherwise, where.\(^{31}\)

$$\Phi_{HM} = \frac{1}{\nu^{HM}} \frac{\Delta_L - (1-\nu^{HM})^2}{1-p+\Delta_L}.$$

The effect of the share of hand-to-mouth agents on the optimal degree of forward guidance is highly nonlinear. If the disruption causing the liquidity trap is higher than a certain threshold, $\Delta_L > (1 - p)^{-1}$, $\Phi_{HM}$ is uniformly decreasing in $\lambda$: the contractionary effect coming from the steeper recession dominates the expansionary effect of increased FG effectiveness. If the disruption is lower than that threshold, which is the more empirically plausible case, then the optimal degree of FG is increasing with $\lambda$ initially until some maximum level and then decreases steeply to reach zero at the threshold described in the Proposition.\(^{32}\)

\(^{31}\)The implied threshold value of $\lambda$ beyond which it is optimal to do no FG is: $\lambda > \left(1 + \varphi + \frac{\rho (\nu - 1)}{1 - \rho \nu - \sqrt{(1-p)\Delta_L}} \right)^{-1}$. It is strictly smaller than the threshold needed for $\sigma^{HM} > 0$, which is $(1 + \varphi)^{-1}$ (as long as labor is not perfectly elastic, prices are not entirely fixed, and the trap has some persistence).

\(^{32}\)The maximum level is reached at $\lambda = \left(1 + \varphi + \frac{\rho \varphi (\nu - 1)}{\sqrt{1 - (1-p)\Delta_L - \rho \nu}} \right)^{-1}$. 

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region, where \( q_{HM}^* \) is decreasing in \( \lambda \), the elasticity of aggregate demand to interest rates is so large that the cost of the recession starts precluding the benefit of forward guidance.

The left panel of Figure 7 illustrates this by plotting \( q_{HM}^* \) as a function of \( \lambda \) (for the admissible range), under our baseline parameter values. The same panel also plots the optimal FG persistence obtained, in this model, under optimal discounting, namely by solving (numerically) (13) under (11), with \( \nu^{HM} \) instead of \( \nu \). The right panel plots optimal FG in the next (and last) model.

Figure 6 Optimal FG persistence in two heterogenous-agent models, for equal weights (red dashed) and optimal discounting (solid blue), as a function of \( \lambda \) (left) and \( u \) (right panel).

5.2 Optimal FG with Incomplete Markets and Unemployment Risk

McKay Nakamura and Steinsson (2015, 2016) have recently shown that the power of forward guidance is reduced in a model of incomplete markets in which agents are subject to unemployment risk that they cannot perfectly insure. The authors show that such a setup does away with the "forward guidance puzzle" which operates in the standard representative-agent model. The purpose of this section is to apply the apparatus developed above to the simplest version of the McKay et al model, in order to derive the optimal degree of forward guidance in that framework and understand how it depends on market incompleteness.

In particular, McKay et al (2016) study a simplified version of their model which delivers as the sole difference with the standard model a "discounted" version of the aggregate Euler equation. In the Appendix, I show that the same model with one additional assumption (the IS curve is written in terms of aggregate consumption not GDP, and home production
is accounted for in aggregate consumption) delivers the aggregate IS curve (19):

\[ c_t = dE_t c_{t+1} - \sigma^{DE} (\pi_t - E_t \pi_{t+1} - \rho_t), \]

with

\[ d = \left( 1 + \frac{u}{1 - u} \left( \frac{1 - ub_c}{(1 - u)b_c} \right) ^{1/2} \right) ^{-1} < 1 \]

\[ \sigma^{DE} = (1 - ub_c) \sigma < \sigma, \]

where \( b_c \) is the steady-state share of exogenous unemployed consumption (home production) in total consumption, \( u \) is the unemployment probability, and superscript \( DE \) is used to refer to the "discounted Euler equation" model.

There are two differences with the baseline model. First, like in McKay et al (2016) \( d < 1 \) is a "discount coefficient" coming from agents' taking into account the uninsurable risk of ending up in the unemployed state tomorrow, state in which they just consume an exogenous home-production stream.\(^{33}\) Second, \( \sigma^{DE} < \sigma \) because the presence of a group of agents who are insensitive to interest rate changes and, importantly, are unemployed, leads to a reduction in the sensitivity of aggregate demand to interest rates.\(^{34}\)

These two features make the model very different from the "employed hand-to-mouth" model studied in the previous section. In particular, the elasticity of aggregate demand to interest rates is now decreasing with the share of unemployed (hand-to-mouth) agents.

The rest of the model (in particular: the Phillips curve) is again unchanged, and so can be analyzed using exactly the same tools used earlier, now defining

\[ \nu^{DE} = d + \sigma^{DE} \kappa < \nu \]

which is monotonically decreasing (towards zero) with the share of unemployed agents, since both \( d \) and \( \sigma^{DE} \) are monotonically decreasing.

We obtain that consumption during the F state of forward guidance is given by:

\[ c_F = \frac{\sigma^{DE}}{1 - qb^{DE} \rho}, \]

\(^{33}\)A similar "discounted" Euler equation appears also in Del Negro et al (2012), based on a Blanchard-Yaari model with finite lives. The parameter \( d \) is a decreasing function of the probability of death, in that framework; but the elasticity of aggregate demand to interest rates is unchanged.

\(^{34}\)This second property is assumed by McKay et al (2016) in order to deliver the lower elasticity found in a complex model by McKay et al (2015); here, I obtain that based on aggregation and relate it to the unemployment probability.
where we now have the condition \( q < (\nu^{DE})^{-1} \). It follows directly that as long as this condition holds, the future expansion is lower when there is a larger number of unemployed agents (more unemployment risk).

During the liquidity trap, we have:

\[
c_L = \frac{q(1-p)\nu^{DE}}{1-p\nu^{DE}}c_F + \frac{\sigma^{DE}}{1-p\nu^{DE}}\rho_L,
\]

or, substituting equilibrium future consumption:

\[
c_L = \frac{\sigma^{DE}}{1-p\nu^{DE}}\rho_L + \frac{q(1-p)\nu^{DE}}{1-q\nu^{DE}}\frac{\sigma^{DE}}{1-p\nu^{DE}}\rho,
\]

with the extra condition \( p < (\nu^{DE})^{-1} \).

The share of unemployed agents shapes optimal FG through two channels. First, a higher \( u \) reduces the size of the LT recession, since demand is less sensitive to the interest rate shock; this, ceteris paribus, will give less scope for FG. Second, by the same token, it reduces the effect of future low interest rates on future demand, thus dampening the future expansion; furthermore, through discounting, this smaller future expansion now matters less today (this last effect is at work in McKay at al). Since all effects work in the same direction, economic intuition suggests that this setup considerably weakens the case for FG. Proposition 6 formalizes optimal FG in the context of the "discounted Euler equation" model of McKay, Nakamura and Steinsson (2016).

**Proposition 7** The optimal duration of FG with incomplete markets and unemployment risk is determined by the persistence probability \( q^{DE} = 0 \) if \( \nu^{DE} < \frac{\sqrt{(1-p)\Delta_L}}{p} \) and \( q^{DE}_* \) otherwise, where:

\[
q^{DE}_* = \frac{1}{\nu^{DE}} \frac{\Delta_L - \left(1 - p\nu^{DE}\right)^2}{1 - p + \Delta_L}.
\]

The conclusions for optimal FG are thus very different from those of the previous, "employed hand-to-mouth" model; here, it is only for a low enough fraction of unemployed agents that forward-guidance is optimal. When it is, the optimal duration is a decreasing function of the share of unemployed, as long as \( \nu^{DE} < \sqrt{1 - (1-p)\Delta_L / p} \). These conclusions reflect what happens at a deeper level with the elasticity of aggregate demand to interest rates, which is monotonically decreasing with the number of unemployed, in this model.\(^{35}\)

\(^{35}\)A useful special case occurs under the extreme assumption that the unemployed and employed consume the same amount in steady state \((b_c = 1)\). The coefficients simplify considerably, namely: \( d = 1 - w \); \( \sigma^{DE} = (1-u)\sigma \); \( \nu^{DE} = (1-u)\nu \). The expression for optimal FG is also particularly simple:

\[
q^{DE}_* = \frac{1}{(1-u)\nu} \frac{\Delta_L - \left(1 - p\nu^{DE}\right)^2}{1 - p + \Delta_L}, \text{ if } u < \frac{1 + \nu - \sqrt{(1-p)\Delta_L}}{\nu} \text{ and } 0 \text{ otherwise.}
\]
The right panel of Figure 6 above illustrates this by plotting $q_{DE}^*$ (in the general case of Proposition 6) as a function of $u$, under our baseline parameter values and for a value of unemployment benefits/home production of half of steady-state consumption $b_c = 0.5$. The same panel also plots the optimal FG persistence obtained, in this model, under optimal discounting (by solving numerically (13) under (11), with $\nu^{DE}$ instead of $\nu$).

A comparison of the left and right panels of Figure 6 illustrates the starkly different conclusions of the two simple and tractable "incomplete markets" models studied here: more hand-to-mouth implies more FG when the hand-to-mouth are employed, but less FG when they are unemployed. The one dimension along which the two frameworks deliver the same conclusion is that beyond a certain threshold share of hand-to-mouth, it is optimal to not do any forward guidance at all. However, the underlying reason is very different. With employed hand-to-mouth, this happens because a high share implies a large elasticity of aggregate demand to interest rates, and hence a high LT recession; even though FG too becomes more powerful, this effect is dwarfed. With unemployed hand-to-mouth, this happens because a higher share of unemployed implies low elasticity of aggregate demand to interest rates, and hence not very powerful FG. Indeed, under our baseline calibration, if the disruption is not very large (a shock of minus one percent per annum when the annual steady-state interest rate is four percent, not pictured), it is optimal to do no FG at all beyond an unemployment rate of merely three percent.

6 Conclusions

This paper solves in closed-form for the optimal duration of forward guidance (FG for short) in a variety of new Keynesian models; FG is modelled stochastically, as a state of the world with an attached persistence probability which fully determines its expected duration. The optimal duration depends in a very intuitive way on the duration of the liquidity trap, its severity, and a composite parameter capturing the effect of news on aggregate activity (this parameter instead is a function of the aggregate demand and aggregate supply curves). This notion of optimal duration is very close to that implied by Ramsey policy—but it is closed-form, whereas Ramsey policy implies numerical solutions even in the simplest model.

A simple rule for FG approximates optimal policy well in the baseline model. In the spirit of interest rate rules, it consists of explicitly announcing an FG duration that is half times (one plus) the trap's duration, times the financial disruption (the interest rate spreads inducing negative natural interest). The simple rule FG duration is very close to that implied by optimal FG and by Ramsey policy—even when the liquidity trap is very long-lived, which reinforces the desirability of this simple rule. Other arguments for adopting such a rule,
pertaining to its being easier to communicate and more transparent, hence increasing its credibility (see Woodford, 2012 for a discussion), apply to any rule-based regime. They apply more forcefully here, especially at long trap durations, for it is conceivably harder to commit ex-ante to a complicated policy that is supposed to happen in (say) ten years’ time. Committing to the simple rule has a further credibility and communication advantage that is specific to FG, and is related to more general arguments put forth by Bassetto (2016): it can allow bypassing the commitment problem inherent to optimal FG with imperfect information. I show this in a simple model of heterogeneous beliefs based on Andrade et al (2015), but conjecture that it will hold in any framework where some agents use a policy announcement to make inference about the future state, as in Wiederholt (2015). The simple rule blends the best of two worlds: state-contingent and time-dependent FG. It is a good approximation to commitment ("odyssean") FG, and serves as communication ("delphic") FG for delphic agents who may otherwise compromise FG policy.

The simple closed-form solution also extends to two models of aggregate demand with heterogeneous agents: one with financially constrained and employed agents (Bilbiie, 2008; Gali, Lopez-Salido and Vallés, 2007; Eggertsson and Krugman, 2012), and one with constrained but unemployed agents (McKay, Nakamura and Steinsson, 2015, 2016). The conclusions of these models are sharply different, and the key is the employment status of those constrained who merely consume their income. When the constrained are employed and their income is endogenous, the optimal degree of FG is increasing in the share of constrained up to a threshold, after which it sharply decreases towards zero. When the constrained are unemployed and their income exogenous, optimal FG is monotonically decreasing with the share of those constrained; at a certain threshold value for the share of unemployed constrained agents, it becomes optimal to do no FG.

"Communication" has been a key word in describing the benefits of the simple rule that approximates the optimal FG commitment. To implement this rule in practice, it is therefore of the essence to communicate clearly its inputs $T_L$ and $\Delta_L$, and how they are measured. This raises empirical issues well beyond the scope of this paper, but we can suggest some possible avenues. The trap duration $T_L$’s data counterpart corresponds to the duration of growth slowdown—defined for instance as output gap deviations with and without a financial crisis (along the lines of Shularick and Taylor, 2012) or as deleveraging periods (along the lines of Koo, 2008). The financial disruption that governs the depth of the trap $\Delta_L$ can be approximated by following recent estimates of the natural interest rate (e.g. Laubach and Williams, 2015), or estimating a structural model where the spread can be estimated directly (à la Curdia and Woodford, 2009, 2010). Either way, it suggests that more research is needed to inform central banks on what would then become two key policy inputs.
References


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Appendix

A Optimal FG in the Representative-Agent Model: Derivations

This Appendix contains the derivations referred to in text for the representative-agent model.

A.1 Markov chain representation

Central bank sets interest rate equal to natural rate, which is given just by \( i_t = r_t^* = r_t \). There are three states for the vector \((\rho_t, i_t)\) \( L = (\rho_L, 0), F = (\rho, 0), S = (\rho, \rho) \). S is an absorbing state. The probability of transiting into F starting from L is \( q(1 - p) \), and starting from F itself is \( q \). The expected duration of FG is hence \( (1 - q)^{-1} \).

The transition matrix is for \( L,F,S \)

\[
P = \begin{pmatrix} p & q(1 - p) & (1 - p) \frac{1}{1 - q} \\ 0 & q & 1 - q \\ 0 & 0 & 1 \end{pmatrix}
\]

Of particular interest is the submatrix corresponding to transient states

\[
P_1 = \begin{pmatrix} p & q(1 - p) \\ 0 & q \end{pmatrix}
\]

whose powers \( t \) indicate the probability of being in a given state after \( t \) periods, starting from each transient state respectively \( \begin{pmatrix} p^t & (1 - p) \frac{q^t - q^{t-1}}{p - q} \\ 0 & q^t \end{pmatrix} \).

The fundamental matrix of our chain is denoted by \( N \) and defined as

\[
N = (I - P_1)^{-1} = \begin{pmatrix} \frac{1}{1 - p} & \frac{q}{1 - q} \\ 0 & \frac{1}{1 - q} \end{pmatrix}
\]

It gives the expected number of periods the chain is in state, given an initial state, where the initial state is counted when staying in the same state.

One can then define time to absorption, starting from each state, as simply \((e = (1, 1))\)

\[
T = Ne = \begin{pmatrix} \frac{1}{1 - p} + \frac{q}{1 - q} \\ \frac{1}{1 - q} \end{pmatrix}
\]

The absorption probabilities are of course 1 and 1 respectively.

Now let’s compute expected discounted lifetime utility, denoting the time-invariant solution for utility in each state as \( U = \begin{pmatrix} U_L & U_F & U_S \end{pmatrix} \). recall now that once the absorbing steady-state is reached, the flexible-price equilibrium is implemented and so the welfare function in the absorbing state is irrelevant. Therefore, we use \( U_1 = \begin{pmatrix} U_L & U_F \end{pmatrix} \) to calculate
lifetime welfare.

Furthermore, notice that the probability of being in state L after $t$ periods is simply $p^t$, while for state $F$ it is the sum of two conditional probabilities (starting from $L$ and from $F$ itself), namely $q^t + (1 - p) q \frac{(p^t - q^t)}{p - q}$.

Therefore, we obtain that the expected present discounted welfare is given by

$$\frac{1}{1 - \beta p} U_L + \left( \frac{1}{1 - \beta q} + \frac{\beta (1 - p) q}{(1 - \beta p)(1 - \beta q)} \right) U_F,$$

where the second term in brackets accounts for time spent in state $F$ when reaching state $F$ conditional upon starting from state $L$

$$\frac{1}{1 - \beta p} \left( U_L + \frac{1 - \beta p + \beta (1 - p) q}{1 - \beta q} U_F \right),$$

where the welfare weight is $< 1$ when $q < p/(2 - p)$ and the derivative of the welfare weight wrt $q$ is

$$\beta \frac{1 - \beta p + (1 - p)}{(1 - \beta q)^2} > 0$$

A.2 Proofs for equal-weights case

Second-order condition is

$$\left( \frac{dc_L}{dq} \right)^2 + c_L \frac{d^2 c_L}{dq^2} + \left( \frac{dc_F}{dq} \right)^2 + c_F \frac{d^2 c_F}{dq^2} > 0$$

The second derivatives are:

$$\frac{d^2 c_F}{dq^2} = \nu \left( \frac{dc_F}{dq} \frac{(1 - q \nu)}{(1 - q \nu)^2} \right) = \frac{2 \nu^2 c_F}{(1 - q \nu)^2} = \frac{2 \nu}{1 - q \nu} \frac{dc_F}{dq}$$

$$\frac{d^2 c_L}{dq^2} = \frac{(1 - p)}{1 - p \nu} \frac{d^2 c_F}{dq^2} = \frac{(1 - p)}{1 - p \nu} \frac{2 \nu^2 c_F}{(1 - q \nu)^2} \frac{dc_F}{dq}$$

Replacing:

$$\frac{(1 - p)}{1 - p \nu} \frac{dc_F}{dq} + c_L \frac{(1 - p)}{1 - p \nu} \frac{d^2 c_F}{dq^2} + \left( \frac{dc_F}{dq} \right)^2 + c_F \frac{d^2 c_F}{dq^2}$$

$$= \left( \frac{(1 - p)}{1 - p \nu} + 1 \right) \left( \frac{dc_F}{dq} \right)^2 + \left( c_L \frac{(1 - p)}{1 - p \nu} + c_F \right) \frac{2 \nu}{1 - q \nu} \frac{dc_F}{dq} > 0$$
Since derivative is positive

\[
\left( \left( \frac{(1-p)}{1-p\nu} \right)^2 + 3 \right) c_F + 2 \frac{(1-p)}{1-p\nu} c_L > 0
\]

For global convexity, we need:

\[
\Delta_L < \frac{1}{2} \left( 1 - p + 3 \frac{(1-p)^2}{1-p} + 2q\nu (1-p) \right) \frac{1}{1-q\nu}
\]

For local convexity, we know that at \( q^* \)

\[
\left( \frac{(1-p)^2}{(1-p\nu)^2} q^* \nu + 1 \right) c_F (q^*) \left[ \frac{(1-p)}{1-p\nu} \sigma \frac{(1-p)}{1-p\nu} (-\rho_L) \right] = \frac{(1-p)}{1-p\nu} \nu \left( \frac{(1-p)}{1-p\nu} \right)^2 \sigma \rho_L > 0
\]

Using in SOC

\[
\left( \left( \frac{(1-p)}{1-p\nu} \right)^2 + 1 \right) c_F (q^*) + 2 \left( 1 + q^* \nu \left( \frac{(1-p)}{1-p\nu} \right)^2 \right) c_F (q^*) + 2 \frac{(1-p)}{(1-p\nu)^2} \sigma \rho_L > 0
\]

QED

### A.3 Derivatives of \( q^* \)

\[
\frac{dq^*}{dp} = -\frac{1}{\nu} \frac{2 (1-p\nu)(1-p) (\nu - 1) + (1-p)^2 \Delta_L + (2\nu (1-p) - 1 + p\nu)(1-p\nu) \Delta_L}{((1-p)^2 + \Delta_L (1-p))^2} > 0
\]

\[
\frac{dq^*}{d\Delta_L} = \frac{(1-p)}{\nu} \frac{(1-p)^2 + (1-p\nu)^2}{((1-p)^2 + \Delta_L (1-p))^2} > 0
\]

\[
\frac{dq^*}{d\nu} = \frac{1}{\nu^2 (1-p + \Delta_L)} \frac{1}{1-p} \left( \frac{1-(p\nu)^2}{1-p} - \Delta_L \right)
\]

### A.4 Optimal FG with Optimal Discounting

With the optimal weight \( \omega(q) \) given by (11), the first-order condition becomes

\[
\frac{\nu (1-p)^2}{(1-p\nu)^2} \frac{q\nu}{1-q\nu} + \frac{\nu (1-\beta p + \beta (1-p) q)}{(1-q\nu) (1-\beta q)} + \frac{1}{2} \frac{\beta (1-\beta p + 1-p)}{(1-\beta q)^2} = \frac{\nu (1-p)}{(1-p\nu)^2} \Delta_L \quad (20)
\]
whose solution under the baseline parameterization is \( q = 0.64 \). An analytical solution of this (cubic) equation is not very easy to obtain nor very informative. The proof of uniqueness in the equal-discounting case, combined with the observation that \( \omega''(q) > 0 \) (\( \omega(q) \) is convex) suggests that there is a unique equilibrium to this problem satisfying the constraints that \( q \) is a probability and \( q < 1/\nu \) (numerical simulations confirm this).

Figure 7.1 plots the optimal \( q \) in the optimal discounting case, as a function of \( p \) and \( \nu \) for the admissible ranges of these parameters, for the otherwise baseline parameterization. Each panel also plots the corresponding optimal \( q^* \) in the equal-weights discounting case. The main difference occurs as a function of the trap persistence. With optimal discounting, since the welfare cost of FG now receives a larger weight, it is only optimal to do FG if things are bad enough, i.e. if the size and/or persistence of the trap are large enough. This is an illustration of our previous insight that \( q^* \) represents in fact an upper bound on optimal FG. Figure 7.2 plots instead the optimal FG duration and compares it to the Ramsey implied duration and to the equal-weights optimal FG duration—the three are very close to each other.

![Figure 7.1: Optimal FG persistence for equal weights (\( q^* \), with red dashed) and optimal discounting (solid blue), as a function of \( p \) (left) and \( \nu \) (right panel)](image_url)

Figure 7.1: Optimal FG persistence for equal weights (\( q^* \), with red dashed) and optimal discounting (solid blue), as a function of \( p \) (left) and \( \nu \) (right panel)
A.5 Optimal FG with Forward-Looking Pricing

When aggregate supply is given by the more general NKPC with discounting (2), the solution at the zero lower bound without FG (under the same IS equation as before) is standard:

$$c_L = \frac{(1 - \beta p) \sigma}{\Gamma_p} \rho_L; \quad \pi_L = \frac{\kappa}{1 - \beta q} c_L$$

where $\Gamma_p = (1 - \beta p)(1 - p) - \sigma \kappa p > 0$ by restriction (this is the equivalent of $p > 1/\nu$ in text, see also footnote on sunspot equilibria).

Modelling FG in exactly the same way, the equilibrium is state F is now given by:

$$c_F = \frac{(1 - \beta q) \sigma}{\Gamma_q} \rho; \quad \pi_F = \frac{\kappa}{1 - \beta q} c_F,$$

where $\Gamma_q = (1 - \beta q)(1 - q) - \sigma \kappa q$.

During the LT state, taking into account the FG equilibrium solved for above, the closed-form solution is

$$c_L = \frac{(1 - p) q (1 - \beta p)(1 - \beta q) + \sigma \kappa}{\Gamma_p} \sigma \rho + \frac{(1 - \beta p) \sigma}{\Gamma_p} \rho_L$$

Notice that FG has very similar effects to the ones found in the simpler case covered in text,
namely:

\[
\frac{dc_F}{dq} = (1 - \beta q)^2 + \sigma \kappa \frac{\sigma}{\Gamma_q} \sigma \rho = \left( \frac{(1 - \beta q) + \frac{\sigma \kappa}{\Gamma_q}}{\Gamma_q} \right) c_F > 0
\]

\[
\frac{dc_L}{dq} = (1 - p) \left( q \left( 1 - \beta p + \frac{\sigma \kappa}{1 - \beta q} \right) + 1 - \beta p + \frac{\sigma \kappa}{1 - \beta q} + \frac{q \beta \sigma \kappa}{2 (1 - \beta q)^2} \right) c_F > 0
\]

Optimal FG consists of the persistence probability \( q \) that solves the first-order condition:

\[
c_L \frac{dc_L}{dq} + \omega (q) c_F \frac{dc_F}{dq} + \frac{1}{2} \omega' (q) c_F^2 = 0,
\]

given the equilibrium \( c_F \) and \( c_L \) solved above. Given those equilibrium values and the expression for \( \omega (q) \) given in text, it can be easily seen that this is a sixth-order polynomial equation in \( q \). We solve this numerically for the baseline calibration, under the restrictions \( \Gamma_p, \Gamma_q > 0 \) and plot the solution as a function of key parameters in the main text.

**B Ramsey-Optimal Policy and Forward Guidance**

The solution to the Ramsey problem is described by (14) combining the two conditions to eliminate consumption \( c_t \) we obtain:

\[
[- \nu E_t \phi_{t+1} + (1 + \beta^{-1} \nu^2) \phi_t - \beta^{-1} \nu \phi_{t-1} + \sigma \rho_t] \phi_t \geq 0
\]

Call \( T \) the stopping time of the exogenous shock (a stochastic variable with expected value \((1 - p)^{-1}\)) and \( T^R_F \) the (unknown, to be determined) number of periods for which the ZLB binds, determined implicitly by the boundary condition \( \phi_{T+F-1} = 0 \). First notice that once ZLB stopped binding (for any \( t \geq T + T^R_F \)) the economy is back at steady state, \( i_t = \rho \), \( c_t = 0 \), \( \phi_t = 0 \). Therefore, we only need to solve for \( \phi_t \) when ZLB binds, for \( t \leq T + T^R_F - 1 \), case in which we have the second-order difference equation:

\[
\nu E_t \phi_{t+1} - (1 + \beta^{-1} \nu^2) \phi_t + \beta^{-1} \nu \phi_{t-1} = \sigma \rho_t,
\]

or written with lag operators (recall \( L^{-j} x_t = E_t x_{t+j} \)):

\[
[L^{-2} - (\nu^{-1} + \beta^{-1} \nu) L + \beta^{-1}] \phi_{t-1} = \sigma \nu^{-1} \rho_t
\]
The eigenvalues being obvious $\beta^{-1}\nu > 1$ and $\nu^{-1} < 1$ we can solve by (e.g.) factorizing:

$$\left(L^{-1} - \beta^{-1}\nu\right)\left(L^{-1} - \nu^{-1}\right)\phi_{t-1} = \sigma\nu^{-1}\rho_t,$$

which delivers the backward-forward solution

$$\phi_t = \nu^{-1}\phi_{t-1} - \sigma\beta\nu^{-2}E_t \sum_{j=0}^{T+T_R^B-1} (\beta\nu^{-1})^j \rho_{t+j}, \quad (21)$$

where the forward summation goes on only as long as the solution applies, i.e. up to $T + T_R^R - 1$, where $T_R^R$ is unknown.

Consider first what happens after the shock has been absorbed, i.e. between the (stochastic) $T$ and $T + T_R^R - 1$, whereby $E_t\rho_{t+j} = \rho$. Solving for $\phi_{T+T_R^B-1}$ we obtain

$$\phi_{T+T_R^B-1} = \nu^{-T_R^B}\phi_{T-1} - \frac{\sigma\beta\nu^{-2}\rho}{1 - \beta\nu^{-1}} \left( \frac{1 - \nu^{-T_R^B} - \beta\nu^{-1}}{1 - \beta\nu^{-2}} \right),$$

which combined with the boundary condition (15a) $\phi_{T+T_R^B-1} = 0$ delivers the Lagrange multiplier in the moment of absorption

$$\frac{1 - \beta\nu^{-1}}{\sigma\beta\nu^{-2}\rho} \phi_{T-1} = \tilde{\phi} \left( T_R^R \right) \quad (22)$$

where $\tilde{\phi} \left( T_R^R \right) \equiv \frac{\nu^{T_R^B} - 1}{1 - \nu^{-1}} - \beta\nu^{-1}\nu^{T_R^B} - (\beta\nu^{-1})^{T_R^B} \frac{1 - \beta\nu^{-1}}{1 - \beta\nu^{-2}}$.

This defines an increasing function $\tilde{\phi} \left( T_R^R \right)$ that is independent of the exogenous random stopping time $T$.

Consider next what happens when uncertainty prevails, i.e. before the shock has been absorbed: between 0 and $T - 1$. We now solve for the aim is solving for $\phi_{T-1}$ "backward", recalling that the starting state between 0 and $T$ is $\rho_L$ and hence $E_t\rho_{t+j} = p^j\rho_L + (1 - p^j)\rho$. For any $t$ between 0 and $T-1$ we have thus, replacing the expectation

$$\phi_t = \nu^{-1}\phi_{t-1} + \sigma\beta\nu^{-2}\rho \left( \Delta_L + 1 \right) \frac{\nu^{T+T_R^B-t} - (\beta\nu^{-1})^{T+T_R^B-t}}{1 - \beta\nu^{-1}} \frac{1 - \beta\nu^{-1}}{1 - \beta\nu^{-1}},$$

recalling that we sum up to the (endogenous, to be solved for) stopping time. Iterate
backwards, denoting $X_t = \sigma \beta \nu^{-2} \rho \left( (\Delta_L + 1) \frac{1 - (\beta \nu^{-1} p)^{T + T_R^F - t}}{1 - \beta \nu^{-1}} - \frac{1 - (\beta \nu^{-1} p)^{T + T_R^F - t}}{1 - \beta \nu^{-1}} \right)$

$$
\phi_t = \nu^{-(t+1)} \phi_{t-1} + \sum_{i=0}^{t} \nu^{-i} X_{t-i}
$$

$$
= \sigma \beta \nu^{-2} \rho \left( \frac{\Delta_L + 1}{1 - \beta \nu^{-1} p} \sum_{i=0}^{t} \nu^{-i} \left[ 1 - (\beta \nu^{-1} p)^{T + T_R^F - t + i} \right] - \frac{1}{1 - \beta \nu^{-1}} \sum_{i=0}^{t} \nu^{-i} \left[ 1 - (\beta \nu^{-1} p)^{T + T_R^F - t + i} \right] \right)
$$

Applying this to the last period $t = T - 1$

$$
\frac{1 - \beta \nu^{-1}}{\sigma \beta \nu^{-2} \rho} \phi_{T-1} = \hat{\phi} (T_R^F, T)
$$

$$
\hat{\phi} (T_R^F, T) \equiv \left( \frac{\Delta_L + 1}{1 - \beta \nu^{-1} p} \left( \frac{1 - \nu^{-T}}{1 - \nu^{-1}} - (\beta \nu^{-1} p)^{1+T_R^F} \frac{1 - (\beta \nu^{-2} p)^T}{1 - \beta \nu^{-2} p} \right) \right. 
\left. - \frac{1 - \nu^{-T}}{1 - \nu^{-1}} + (\beta \nu^{-1})^{1+T_R^F} \frac{1 - (\beta \nu^{-2})^T}{1 - \beta \nu^{-2}} \right)
$$

This defines another schedule for $\phi_{T-1}$ as a function of $T_R^F$ and $T$, call it $\hat{\phi} (T_R^F, T)$.

The optimal stopping time, aka Ramsey duration of forward guidance, is found by requiring the two solutions for $\phi_{T-1}$ (the value of the constraint when the exogenous shock converges) coincide, i.e. the intersection of the two schedules $\phi_{T-1} (T_R^F)$ defined by 23 and 22 namely:

$$
\frac{\nu^{T_R^F} - 1}{1 - \nu^{-1}} - \beta \nu^{-1} \frac{\nu^{T_R^F} - (\beta \nu^{-1} p)^{T_R^F}}{1 - \beta \nu^{-2}} = \left( \frac{\Delta_L + 1}{1 - \beta \nu^{-1} p} \left( \frac{1 - \nu^{-T}}{1 - \nu^{-1}} - (\beta \nu^{-1} p)^{1+T_R^F} \frac{1 - (\beta \nu^{-2} p)^T}{1 - \beta \nu^{-2} p} \right) \right. 
\left. - \frac{1 - \nu^{-T}}{1 - \nu^{-1}} + (\beta \nu^{-1})^{1+T_R^F} \frac{1 - (\beta \nu^{-2})^T}{1 - \beta \nu^{-2}} \right)
$$

This defines a nonlinear equation for the stopping time $T_R^F$ as a function of the model parameters. We plot the solution (solved for numerically) as a function of the expected duration $T_L = E(T) = (1 - p)^{-1}$ by solving this equation under rational expectations, $T = T_L$.

\[36\] The domain is restricted by the requirements for equilibrium uniqueness and no-starvation, translated in durations $T < 50$. This threshold can be relaxed (and the feasible duration increased) by considering even smaller values of $\nu$, closer to 1 (so more sticky prices and/or less intertemporal substitution). Notice that by definition in that case the approximation implied by the simple rule becomes even better.
B.1 Optimal Forward Guidance in a Deterministic Setting

In this section, I extend the results to the deterministic setting studied originally by Jung et al (2005) and revisited thereafter by others. I compare the solution for optimal FG and the simple rule derived above with the solution obtained in a deterministic setting: the discount factor shock lasts a known number of periods $\rho_t = \rho_L < 0$ for $t = 0, T - 1$, and $\rho_t = \rho$ for $t \geq T$.

Ramsey policy in this deterministic setting is a simplified version of Jung et al (2005). It is the same problem as above, except that $T$ is now known; in particular, the different equation governing the Lagrange multiplier is the same. The solution between the known exogenous stopping time $T-1$ and the endogenous stopping time of zero-interest policy $T+T^R_F-1$ is the same as above, namely (22). What changes is the solution between 0 and $T$, for now we know with certainty that the discount factor will be $\rho$ from $T$ onwards and $\rho_L < 0$ before, i.e.:

$$
\phi_t = \nu^{-1} \phi_{t-1} - \sigma \beta \nu^{-2} \sum_{j=t}^{T-1} (\beta \nu^{-1})^{j-t} \rho_L - \sigma \beta \nu^{-2} \sum_{j=T}^{T+T^R_F-1} (\beta \nu^{-1})^{j-t} \rho
$$

$$
= \nu^{-1} \phi_{t-1} - \sigma \beta \nu^{-2} \frac{1 - (\beta \nu^{-1})^{T-t}}{1 - \beta \nu^{-1}} \rho_L - \sigma \beta \nu^{-2} (\beta \nu^{-1})^{T-t} \frac{1 - (\beta \nu^{-1})^{T^R_F}}{1 - \beta \nu^{-1}} \rho
$$

Now solve backwards, letting $X_t = -\sigma \beta \nu^{-2} \rho \left( \frac{1 - (\beta \nu^{-1})^{T-t}}{1 - \beta \nu^{-1}} \Delta_L - (\beta \nu^{-1})^{T-t} \frac{1 - (\beta \nu^{-1})^{T^R_F}}{1 - \beta \nu^{-1}} \right)$

$$
\phi_t = \nu^{-(t+1)} \phi_{-1} + \sum_{i=0}^{t} \nu^{-i} X_{t-i}
$$

$$
= \sigma \beta \nu^{-2} \rho \left( \frac{\Delta_L}{1 - \beta \nu^{-1}} \sum_{i=0}^{t} \nu^{-i} \left( 1 - (\beta \nu^{-1})^{T-t+i} \right) - \frac{1 - (\beta \nu^{-1})^{T^R_F}}{1 - \beta \nu^{-1}} \sum_{i=0}^{t} \nu^{-i} (\beta \nu^{-1})^{T-t+i} \right)
$$

Evaluating at the last period of negative shock $t = T - 1$

$$
\phi_{T-1} = \frac{\sigma \beta \nu^{-2} \rho}{1 - \beta \nu^{-1}} \left( \frac{\Delta_L}{1 - \nu^{-1}} \Delta_L \frac{1 - (\beta \nu^{-2})^{T}}{1 - \beta \nu^{-2}} - \frac{1 - (\beta \nu^{-1})^{T^R_F}}{1 - \beta \nu^{-1}} (\beta \nu^{-1}) \frac{1 - (\beta \nu^{-2})^{T}}{1 - \beta \nu^{-2}} \right)
$$
The optimal stopping time is thus a solution to

\[
\Delta L \frac{1 - \nu^{-T}}{1 - \nu^{-1}} - \Delta L \left( \beta \nu^{-1} \right) \frac{1 - (\beta \nu^{-2})^T}{1 - \beta \nu^{-2}} - \left( \beta \nu^{-1} \right) \frac{1 - (\beta \nu^{-2})^T}{1 - \beta \nu^{-2}}
\]

\[
= \frac{\nu^{T_F} - 1}{1 - \nu^{-1}} - \beta \nu^{-1} \nu^{T_P} \left( \beta \nu^{-1} \right)^{T_F}
\]

\[
= \frac{1 - \beta \nu^{-2}}{1 - \beta \nu^{-2}} (T_F - T_R) (T_F - 1)
\]

\[\text{(24)}\]

We plot the optimal FG duration implied by Ramsey policy below as a function of the LT duration T, along with the duration of "optimal FG" found as follows.

**Optimal FG** can also be solved in this deterministic setup, but only numerically. We solve for the equilibrium given FG, and consider a central bank choosing the duration of FG that maximizes welfare. FG now takes the form of a pre-announced number of periods with zero interest rates: \(i_t = 0 \& \rho_t = \rho\) for \(t = T, ..., T + T^* - 1\) and \(i_t = \rho\) for \(t \geq T + T^*\). Using the same notation as previously, the solution for consumption during the LT in the baseline model under no FG is \(c_t = \sum_{i=0}^{T-1} \nu^i \sigma \rho_L = \frac{\nu^{T-1}}{\nu-1} \sigma \rho_L\)

Under FG, we first solve for the solution during the FG period and then during the LT period; the solution is obtained using standard methods. During FG, i.e. between \(T\) and \(T + T^*\), we have (using the boundary condition \(c_{T+T^*} = 0\)), for \(t\) from 1 to \(T^*\):

\[c_t = (\nu^{T+T^*-t} - 1) \kappa^{-1} \rho.\]

Evidently, more FG (higher \(T^*\)) leads to a higher future expansion. Given this future solution, which gives the boundary condition for \(c_T\), the equilibrium during the LT (for \(t\) such that \(0 \leq t < T\)) is readily obtained as

\[c_t = \nu^{T-t} c_T + (\nu^{T-t} - 1) \kappa^{-1} \rho_L \]

\[= \kappa^{-1} (\nu^{T-t} (\nu^{T^*} - 1) \rho + (\nu^{T-t} - 1) \rho_L).\]

More FG (higher \(T^*\)) also leads to a higher expansion today. The derivative itself is increasing in \(T\), which is just the illustration of one aspect of what Del Negro et al called "the FG puzzle" (the further FG is pushed into the future, the higher its effect).

We can now find numerically the optimal FG by plugging these closed-form solutions into the welfare function and finding the \(T^*\) that minimizes the loss.\(^{37}\) We plot the optimal FG duration \(T^*\) as a function of LT duration \(T\) under the baseline parameterization, in Figure 8 (blue solid line). Alongside, we plot the optimal duration implied by Ramsey policy in the deterministic setup: the two are virtually identical.

\(^{37}\)The equilibrium is unique for all parameterizations I checked–in other words, the loss function is convex in \(T^*\).
Figure 8: Deterministic optimal FG durations $T_F$, Ramsey-implied (red dashed), and optimal FG (blue solid) as a function of LT duration $T_L$.

Notice that the deterministic model implies ceteris paribus less optimal FG (no matter its definition) than the stochastic model, especially at large durations. This illustrates a more general point (emphasized i.a. by Carlstrom, Fuerst and Paustian in a different context) that, close to the bifurcation point of the stochastic model the conclusions of the stochastic and deterministic settings can be quite different quantitatively.

C Optimal FG with Heterogenous Agents

This Appendix briefly reviews the foundations of two of the heterogenous-agent models used. The loss function used in the heterogenous-beliefs model is also justified here.

C.1 Employed hand-to-mouth model

I summarize the algebra leading to the aggregate IS curve for the "employed hand-to-mouth" model used in text; I refer the reader to Bilbiie (2008) and Eggertsson and Krugman (2012) for detailed descriptions of the environments and more microfoundations. There are two types: savers $S$ (share $1 - \lambda$) and hand-to-mouth $H$ (share $\lambda$). Both types work, and the labor supply decision of each type is governed by (where everything is expressed in shares of aggregate steady-state consumption):

$$\varphi m_t^j = w_t - \sigma^{-1} c_t^j, \ j = S, H$$
Assuming that elasticities are identical across agents, the same equation also holds on aggregate with the same elasticity and income effect. All output is consumed and produced only by labor with constant returns, which implies then \( w_t = (\varphi + \sigma^{-1}) c_t \).

Hand-to-mouth agents only have labor income, and they consume all of it, \( c^H_t = w_t + n^H_t \); combining this with their labor supply, we obtain their consumption function in closed-form \( c^H_t = [(1 + \varphi) / (\sigma^{-1} + \varphi)] w_t \). Substituting in aggregate consumption \( c^S_t = \frac{1}{1 - \chi} c_t - \frac{\lambda}{1 - \chi} c^H_t \) we obtain:

\[
c^S_t = \left(1 - \frac{\lambda}{1 - \chi} \varphi \right) c_t
\]

Savers hold all the assets in this economy (notably, receive profits from monopolistically competitive firms), and their consumption obeys a standard Euler equation

\[
c^S_t = E_t c^S_{t+1} - \sigma (i_t - E_t \pi_{t+1} - \rho_t)
\]

Replacing the expression for \( c^S_t \), we obtain the aggregate Euler equation used in text which has an elasticity of aggregate demand to interest rates \( \sigma^{HM} \) that is increasing in the share of hand-to-mouth (and decreasing in labor elasticity), as long as it is positive, namely \( \lambda < (1 + \varphi)^{-1} \). See Bilbiie (2008) for more details.

For this model, I use the derivation of a quadratic welfare function with two types of agents provided in Bilbiie (2008), namely: \( (\theta \text{ is the elasticity of substitution between goods}) \)

\[
W_t = -\frac{UCC}{2} \psi E_t \sum_{i=1}^{\infty} \left\{ \frac{\varphi + \gamma}{1 - \lambda} \left[ 1 - \lambda (1 - \gamma) (1 + \varphi) \right] \frac{\psi}{\theta} c^2_{t+i} + \pi^2_{t+i} \right\}, \quad (25)
\]

Using the expression for the Phillips curve during the trap and the Markov stochastic structure, this reduces to the same objective function like the RA model, the welfare weight \( \chi (\lambda) \) being immaterial for this problem: \( W = \frac{1}{1 - \beta \gamma^2} \left[ c^2_L + \omega (q) c^2_F \right], \) where the optimal weight is the same as in text. For the case of optimal discounting, the first-order condition is still of the form (20), with the difference that the composite parameter \( \nu_{HM} \) defined in text now replaces \( \nu \).

For the heterogenous-beliefs model used in text, the loss function takes exactly the same form (under the same assumptions used in Bilbiie, 2008 to derive it, namely that the two types of agents consume and work the same quantities in steady-state), with \( \alpha \) replacing \( \lambda \). Thus, the first-order condition is still of the form (20), with the difference that the composite disruption \( \Delta^H_L (\alpha) \) replaces \( \Delta_L \) on the right-hand side.

Matters are slightly different for the "unemployment" model, as explained below—but with little impact on the optimality condition.
We plot the optimal transition probability (for the optimal discounting case) as a function of the deep parameters $\lambda$ and $\alpha$ in Figure 6.

C.2 Incomplete markets and unemployment model

This appendix briefly outlines the model with incomplete markets and unemployment risk which essentially builds on McKay, Nakamura and Steinsson (2015, 2016)–I refer the reader to those papers for more details. The Euler equation of an employed agent (superscript $e$) contemplating unemployment risk (superscript $u$) of a specific stochastic structure described in McKay et al is:

$$(C^e_t)^{-\frac{1}{2}} = \beta (1 + r_t) \exp (-\rho_t t) E_t \left[(1 - u) \left(C^e_{t+1}\right)^{-\frac{1}{2}} + ub^{-\frac{1}{2}}\right],$$

where $u$ is the probability of being unemployed next period and also the unemployment rate, and $r_t$ a measure of the ex-ante real interest rate. As in McKay et al, I also assume that the consumption of unemployed consists of an exogenous stream of unemployment benefits or home production, $C^u_t = b$.

Loglinearization of this around the steady-state delivers, for employed agents, the Discounted Euler equation discovered by McKay et al:

$$c^e_t = \left(1 + \frac{u}{1-u} \left(C^e_t \frac{b}{b_c}\right)^{-1} E_t c^e_{t+1} - \sigma (r_t - \rho_t) \right)$$

$$= 1 + \frac{u}{1-u} \left(C^e_t \frac{b}{b_c}\right)^{-1} E_t c^e_{t+1} - \sigma (r_t - \rho_t) \right)$$

(26)

I make one different assumption with respect to McKay et al, which concerns the accounting of unemployment benefits. This assumption will deliver, in this framework, an elasticity of aggregate demand to interest rates that is different than the one of the employed: in particular, it will be lower. McKay et al also use such a calibration with low elasticity in order to reproduce the elasticity of their more complicated model; here, it is derived within the simple model. Specifically, aggregate consumption is

$$C_t = (1 - u) C^e_t + ub$$

which, in steady state reads

$$\frac{C^e}{C} = \frac{1 - ub_c}{1-u},$$

where I take the share of unemployment benefits in total consumption to be a free parameter denoted $b_c = b/C$.
Loglinearizing aggregate consumption using the previous, we have:

\[ c_t = (1 - ub_c) c^e_t. \]

Now, replacing this together with the expression for steady-state share in the Euler equation of employed (26), we obtain the aggregate Euler equation used in text. The parameters \( \delta \) (capturing discounting) and \( \sigma^{DE} \) (capturing the lower elasticity of aggregate demand to interest rates) can be calibrated by choosing \( u \) and \( b_c \).

The microfounded welfare function is now different, because the income and consumption of unemployed are exogenous. In particular, it can be easily shown that a second-order approximation of a social welfare function around an efficient steady state (with a subsidy that induces marginal cost pricing in steady state) delivers a loss function of the form

\[
W_t = -\frac{U_C C}{2} \frac{\theta}{\psi} E_t \left\{ \sum_{i=t}^{\infty} \left\{ \gamma + \varphi \frac{\psi}{1 - ub_c} \frac{\psi}{1 - ub_c} c^2_{t+i} + \pi^2_{t+i} \right\} \right\}.
\]

However, because of the time-invariant nature of the solution the welfare objective for our problem is still unaffected, \( W = \frac{1}{1 - \beta} \left[ c^2_L + \omega(q) c^2_F \right] \), where the optimal weight \( \omega(q) \) is the same as in text. For the case of optimal discounting, the first-order condition is still of the form (20), with the difference that the composite parameter \( \nu^{DE} \) defined in text now replaces \( \nu \). The third panel of Figure 6 plots the optimal transition probability as a function of the share of unemployed.