Quantitative Easing without Rational Expectations

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Abstract

We study the effects of risky assets purchases financed by issuance of riskless debt by the government (quantitative easing) in a model with nominal frictions but without rational expectations. We use bounded rationality in the form of level-k thinking and the associated reflective equilibrium that converges to the rational expectations equilibrium in the limit. This equilibrium notion rationalizes the idea that it is difficult to change expectations about economic outcomes even if it is easy to shift expectations about the policy. Quantitative easing policy increases the price of risky assets and stimulates output in the reflective equilibrium, while it is neutral in the rational expectations equilibrium.

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1 Introduction

Quantitative easing (QE)—central bank purchases of private risky and long-term public assets financed by issuing reserves—has become an important tool of monetary policy after the standard monetary policy tool, short-term nominal interest rate, reached its lower bound in several developed countries. The Federal Reserve has increased its balance sheet from less than $900 billion in 2008 to $4.4 trillion in February 2017. Outstanding liabilities of the ECB were below 1 trillion Euros before the onset of the global financial crisis and they reached 3.7 trillion Euros in February 2017.1

There are at least two channels through which QE is believed to affect the economy: so-called “signaling” and “portfolio balance” channels (Bernanke, 2012; Draghi, 2015; Yellen, 2016). According to the signaling channel view, QE helps build credibility of the central bank’s announcement about future interest rates (forward guidance) by taking concrete actions instead of relying purely on communication. The portfolio-balance channel, instead, posits that changes in the supplies of different assets available to private sector affect their prices perhaps due to imperfect assets substitutability related to differences in risk. For example, central bank purchases of private risky assets are argued to stimulate the economy by increasing the price of risky assets.

The standard macro models, however, imply that QE is either unnecessary or has only a minimal effect on the economy. First, if QE works mainly as a support for forward guidance, then a credible implementation of forward guidance makes QE redundant. Second, the portfolio balance channel logic neglects that the risks associated with the purchased assets do not disappear from the economy. On the contrary, if agents form their expectations rationally, they must correctly understand that the risky assets on the central bank’s balance sheet will eventually lead to losses or gains that will be transferred to the fiscal authority and ultimately to private agents through taxes. To hedge against these tax risks, the agents will lower their demand for these assets and, thus, undo the effect of initial purchases. This intuition represents a version of the Modigliani-Miller irrelevance result applied to central banks (Wallace, 1981).2

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1The numbers are taken from the Federal Reserve statistical release (February 2, 2017) and the consolidated financial statement of the Eurosystem (February 3, 2017).

2Segmented assets markets may lead to a break-down of the Modigliani-Miller result. For example, if the assets purchased by the central bank are only traded by certain specialists but taxes, associated with losses or gains on these assets, are spread across a broader group of people, then non-specialists will not be able to reduce their holdings of these assets. As a result, the price of the purchased assets goes up. However, at the same time, market segmentation hypothesis implies that QE will have only a limited effect on the overall economy, as the higher price of assets that the central bank purchases will not fully be spread to the rest of the economy. See, for example, Woodford (2012).

An increase in outstanding liabilities of the central bank associated with implementation of QE, a classic monetarist expansion of money supply, are unlikely to have effects on the economy once the short-term
There is growing evidence that QE policies affect assets markets and the real economy. Studies that looked at high frequency financial data found that large-scale MBS purchases by the Fed have affected mortgage market yields and have spread to other assets markets. Moreover, Gagnon et al. (2011) argues that a substantial part of the measured effect is not related to the signaling channel of QE. Some recent studies found evidence of the effects of Fed’s MBS purchases on mortgage lending. Fieldhouse et al. (2017) present evidence that purchases of MBS by the government sponsored enterprises in the US affected not only mortgage rates and lending but also residential investments.

In this paper, we theoretically study the effects government assets purchases on financial markets and the real economy in the context of a model with standard New Keynesian ingredients but without rational expectations. More specifically, we assume that agents form their expectations based on a deductive procedure, which we call “level-k thinking.” This form of bounded rationality was introduced in a modern macro setting in Woodford (2013) and used to resolve the “forward-guidance puzzle” in García-Schmidt and Woodford (2015) and Farhi and Werning (2016).

In the model we propose, households observe government assets purchases. However, since endogenous equilibrium variables are not announced by the government, agents have to form expectations about these variables. Specifically, these variables include future taxes (or subsidies) through which the government will disburse losses (or gains) on its portfolio. A specific situation we have in mind is given by a central bank conducting assets purchases and the fiscal authority taxing agents due to losses (or gains) on the central bank portfolio. In the proposed model, agents make an effort to understand the effects of the asset purchases on future taxes, but they do not achieve fully rational expectations. This deviation from rationality can be motivated by the relative novelty of QE policies, which implies that historical evidence about the effects of such policies is not available and, thus, equilibrium implications are particularly hard to determine.

nominal interest rate is at its lower bound.

3See, for example, Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2011), Hancock and Passmore (2011). At the same time, Stroebel and Taylor (2012) find no effects of MBS purchases in the first round of quantitative easing by the Fed.

4To control for the changes in the expected future short-term rates, the authors use an estimated arbitrage-free dynamic term-structure model.

5See, for example, Di Maggio et al. (2016), Chakraborty et al. (2016).

6In the recent influential paper, Gabaix (2016) assumes that current consumption is less sensitive to changes in future income which helps resolve the “forward guidance puzzle.” With “level-k thinking,” current aggregate consumption is also less sensitive to future aggregate income but this arises “endogenously” due to non-rational expectations about future income. Angeletos and Lian (2016) show how alternative deviation from rational expectations, lack of common knowledge, helps to solve the forward guidance puzzle.

7Similarly, this deviation from rationality may be reasonable when analyzing the consequences of forward guidance due to its novelty, as argued by García-Schmidt and Woodford (2015) and Farhi and Werning (2016).
cognitive process that we use in this paper does not necessarily require the knowledge of the past QE effects and it is totally forward-looking in nature.

Level-\( k \) thinking and the equilibrium notion associated with it—which we refer to as “reflective equilibrium”—work as follows. Agents are perfectly aware of current implementation of QE policy (number of assets purchased by the government) and their own income and assets. They have to form expectations about the effects of QE on future taxes and hence on their future income. They form expectations according to the following iterative procedure. “Level-1 thinking” assumes that agents keep expectations identical to those before the change in policy. As a result, “level-1 thinkers” do not internalize the Modigliani-Miller result for central bank.\(^8\) Given these expectations, agents choose consumption and portfolio holdings. In equilibrium, agents demand goods and assets conditional on their expectations and markets clear. Future market clearing will reveal future realized government taxes. However, by assumption, “level-1 thinkers” cannot incorporate the consequences of future market-clearing into their expectations.

“Level-2 thinking” assumes that an agent understands equilibrium outcomes of the QE policy conditional on believing that economy is populated by level-1 thinkers only. As a result, this agent will correctly expect future taxes to depend on QE, however, his expectations can still deviate from rational expectations. Conditional on these updated expectations, agents make consumption and portfolio decisions and markets clear.

These deduction rounds can be immediately generalized to “level-\( k \) thinking” and carried over to infinity. Following García-Schmidt and Woodford (2015), we assume that the economy is populated by agents with all levels of thinking, with the mass of agents of each level of thinking given by an exogenous distribution. This results in a notion of reflective equilibrium which we precisely define in Section 2.

In section 3, we solve the model with CARA utility, which delivers a closed form solution and illustrates the main results of the model in the clearest way. We show that private assets purchases increase the assets price and output in the reflective equilibrium while it has no effect in the rational expectations equilibrium. What is more, as the average level of sophistication in the economy increases, equilibrium outcomes approach to the corresponding rational expectations outcomes. In addition, output converges faster than asset prices to the rational expectations equilibrium. Finally, we show that agents with higher levels of thinking do not necessarily enjoy higher welfare in equilibrium.

In Section 4, we add a second friction, limited participation in assets markets. When assets markets are sufficiently segmented, limited participation is known to make QE policies effective also under rational expectations (Curdia and Woodford, 2011). We show

\(^8\)In other words, these agents do not hold “Ricardian expectations.” See Woodford (2013) for the discussion of non-Ricardian expectations.
that limited participation not only adds another channel through which QE can work, but also interacts with $k$-level thinking. This finding echoes the result of Farhi and Werning (2016) who show that “level-$k$ thinking” interacts with incomplete markets to dampen forward guidance even more. In our case, however, the interaction effect acts to weaken the joint effects of limited participation and $k$-level thinking.

The rest of the paper is organized as follows. Section 5 extends the model to infinite horizon and studies the implication of the duration of QE policy on the economy. Section 6 concludes.

2 A Finite-Period Model

The economy lasts for two periods indexed by $t = 0, 1$. There is a continuum of households of measure one. There are perishable consumption good and two types of assets: a risky and a safe one. Each household has a unit endowment of a risky asset. A unit of risky asset pays $D(s)$ units of consumption good in period 1 in aggregate state $s \in S$, a safe asset pays a unit of consumption in period 1. Consumption goods in period 0 are produced by labor owned by households. Prices are absolutely sticky in period 0, and fully flexible in period 1. As a result, output is determined by “aggregate demand” in period 0 and “aggregate supply” in period 1. Sticky prices in period 0 together with monetary authority ability to affect the safe interest rate will determine the real interest rate on safe asset in the economy.

Households. A household has expectations of future (lump-sum) taxes given by $T_1^e : S \to \mathbb{R}$, which can potentially differ from actual taxes $T_1^e : S \to \mathbb{R}$, and solves the following problem taking prices as given:

$$\max_{C_0, B, X, \{\tilde{C}_1(s)\}_{s \in S}} \mathbb{E}_s \left[ u(C_0) + \beta u\left(\tilde{C}_1(s)\right) \right],$$

s.t.: $C_0 + \frac{B}{R} + qX \leq q + Y_0,$ \hspace{1cm} (1)

$$\tilde{C}_1(s) + T_1^e(s) \leq D(s)X + B,$$ \hspace{1cm} (2)

where $R$ is the real interest rate set by the central bank and $q$ is the price of the risky asset. Households income in period 0 consists of the value of risky assets endowment and income from production of output $Y_0$. This income is spend on consumption $C_0$, purchases of safe bonds $B$, and risky assets $X$. In period 1, the household receives return on its safe and risky assets holdings and spends it on taxes and consumption $\tilde{C}_1(s)$ in state $s$. Note that we use a tilde to denote the household’s choice of period-1 consumption $\tilde{C}_1(s)$. 


This notation is necessary in this context where the desired choice of consumption, which we also call “planned consumption”, can potentially differ from actual equilibrium consumption $C_1(s)$, which we also refer to as “realized consumption”. On the other hand, period-0 choice variables do not feature a tilde because they will always be identical to their equilibrium counterparts in the “temporary” equilibrium defined below. The solution to this problem satisfies (1), (2), and the standard Euler equations

$$u'(C_0) = \beta RE \left[ u' \left( \tilde{C}_1(s) \right) \right],$$

$$u'(C_0) = \frac{\beta}{q} E \left[ D(s) u' \left( \tilde{C}_1(s) \right) \right].$$

The solution to these equations will give following policy functions

$$C_0 = C_0^* (Y_0, q, T_1^*), \quad (3)$$

$$\tilde{C}_1(s) = \tilde{C}_1^* (s; Y_0, q, T_1^*), \quad (4)$$

$$B = B^* (Y_0, q, T_1^*), \quad (5)$$

$$X = X^* (Y_0, q, T_1^*). \quad (6)$$

Policy functions are defined for any value of their arguments. The latter will have to satisfy different conditions which depend on the definition of equilibrium.

**Government.** The government consists of a monetary and a fiscal authorities. The monetary authority sets interest rate $R$, which we assume to be fixed. This assumption allows us to study the effects of assets purchases for a given stance of conventional monetary policy. The monetary authority can purchase $X^G \geq 0$ units of private risky assets at market price $q$ by issuing reserves $B$. The fiscal authority levies lump-sum taxes on households. The consolidated government budget constraints in both periods are

$$qX^G = \frac{B}{R}, \quad (7)$$

$$B = T_1(s) + D(s)X^G, \forall s \quad (8)$$

### 2.1 Temporary Equilibrium

We begin with a definition of temporary equilibrium (TE). This definition takes as given the households’ expectations about future taxes and requires only that households optimize given their beliefs and that markets clear. Crucially, this notion of equilibrium differs from the standard notion of rational expectations equilibrium (REE) as expectations need
not be consistent with the equilibrium outcomes.

**Definition 1.** Given government policy \( \{ X^G, R \} \) and households’ expectations of future taxes \( T^e_1 \), a temporary equilibrium is a collection \( \{ C_0, B, X, \tilde{C}_1(s), C_1(s), Y_0, q, T_1(s) \} \), such that:

1. Households optimize: \( \{ C_0, B, X, \tilde{C}_1(s) \} \) are given by (3)-(6);
2. Government budget constraints hold: \( \{ B, T_1(s) \} \) satisfy (7)-(8);
3. Markets clear:
   - goods market: \( C_0 = Y_0, C_1(s) = D(s), \forall s; \)
   - risky asset market: \( X + X^G = 1. \)

The key assumption in the definition of TE is that expectations of future taxes are taken as given. They can thus potentially be inconsistent with agents’ optimizing behavior and market clearing conditions, which may result in the difference between planned consumption \( \tilde{C}_1(s) \) and realized consumption \( C_1(s) \). This inconsistency contrasts a TE from a standard REE. Specifically, a rational expectations equilibrium is a temporary equilibrium with the further requirement that beliefs \( T^e_1 \) are correct:

\[
T_1(s) = T^e_1(s), \forall s.
\]

### 2.2 Level-k Thinking and Reflective Equilibrium

We follow García-Schmidt and Woodford (2015) and use the notion of reflective equilibrium (RE). We first introduce the concept of level-k thinking using the example of QE policy and then we give a formal definition of RE. Assume that the economy is in a REE and that the government decides to start private assets purchases. Before the policy is implemented, since we assumed that there is no government spending, in a REE taxes must be zero \( T_1(s) = T^e_1(s) = 0, \forall s. \) Suppose now the government buys a quantity \( X^G > 0 \) of risky assets. Suppose that, in addition, after the policy is implemented, households hold the same expectations about future taxes as in the REE, that is, \( T^e_1(s) = 0 \forall s. \) We call “level-1 thinkers” the households who do not change expectations after a policy change. Given expectations of the level-1 thinkers, which we denote by \( \hat{T}^e_1 \), we can compute a TE.

Suppose now that there is a more sophisticated household in the economy. This household thinks that all other households are level-1 thinkers and he is the only one who can solve for the TE of the economy and, in particular, compute future realized taxes. Let
\( \hat{T}_1(s) \) be the value of these taxes in state \( s \) and call this more sophisticated household a “level-2 thinker”. Since a level-2 thinker believes that all other households are level-1 thinkers and can compute the value of \( \hat{T}_1(s) \), he will solve his consumption problem with expectations \( \hat{T}_{1}^{e,2}(s) \) given by \( \hat{T}_1(s) \).

We can similarly continue this process and introduce level-\( k \) thinkers. These agents can solve for a TE under the assumption that all other households in the economy are level-\((-k-1)\) thinkers and, thus, level-\( k \) thinkers will hold expectations \( \hat{T}_{1}^{e,k}(s) = \hat{T}_1^{k-1}(s), \forall s \). Crucially, level-\( k \) thinkers not only know the structure of the economy, but can also reflect on how other households update their beliefs up to the order \( k-1 \).

In order to define a RE we need to know the mass of level-\( k \) thinkers in the economy for any order \( k \). We assume that the economy is populated by thinkers of all levels. More specifically, we assume that the mass of level-\( k \) thinkers is given by \( f(k) \), where \( f(k) > 0 \), for all \( k \), and \( \sum_{k \geq 1} f(k) = 1 \). We can now give a formal definition of RE.

**Definition 2.** Given government policy \( \{X^G,R\} \), a reflective equilibrium is a collection \( \{C^k_0,B^k,X^k,\hat{C}^k_1(s),C^k_1(s),Y_0,q,B,T_1(s),\hat{T}_1^{e,k}(s),\mathcal{E}^k\} \) such that, for \( k \geq 1 \), \( \mathcal{E}^k = \{\hat{C}^k_0,\hat{B}^k,\hat{X}^k,\hat{C}^k_1(s),\hat{C}^k_1(s),\hat{Y}_0,\hat{q}^k,\hat{T}_1^{e,k}(s)\} \) is a TE given beliefs \( \hat{T}_1^{e,k} \), and

1. Households optimize:
   - \( \{C^k_0,B^k,X^k,\hat{C}^k_1(s)\} \) are given by (3)-(6), for \( k \geq 1 \);
   - \( C^k_1(s) = D(s)(1-X^G) + B^k - T_1(s) \), for \( k \geq 1 \);

2. Government budget constraint holds: \( \{B,T_1(s)\} \) satisfy (7)-(8);

3. Markets clear:
   - Goods market: \( \sum_{k=1}^{\infty} f(k)C^k_0 = Y_0, \sum_{k=1}^{\infty} f(k)C^k_1(s) = D(s), \forall s \);
   - Risky asset market: \( \sum_{k=1}^{\infty} f(k)X^k + X^G = 1 \);
   - Bond market: \( B = \sum_{k=1}^{\infty} f(k)B^k \);

4. Beliefs satisfy the recursion:
   \[
   \hat{T}_1^{e,1}(s) = 0, \hat{T}_1^{e,k}(s) = \hat{T}_1^{k-1}(s), \forall s, k \geq 2.
   \]

The definition of RE captures the idea that the economy is populated by different groups of agents, identified by \( k \). Agents in a group share the same beliefs about the degree of sophistication of all other agents in the economy and hold the same expectations about
future taxes. In turn, these expectations are formed by solving for the TE of the economy. More specifically, each agent in group $k$ will solve for the TE of the economy under the assumption that all other agents belong to group $k-1$, $E^{k-1}$. He will then use this equilibrium to form his expectations about future taxes, $\tilde{T}^e_{1,k}(s) = \tilde{T}^{k-1}_{1}(s)$.

3 A Finite-Period Model with CARA Utility

In this section, we make extra assumptions on the utility function and the distribution of risk and solve for the RE of the model. Specifically, we assumed that

$$u(C) = -\frac{1}{\alpha} e^{-\alpha C}, \alpha > 0,$$

and that dividends are normally distributed,

$$D(s) \sim N(\mu_D, \sigma_D^2).$$

The combination of CARA utility and Gaussian distribution will deliver closed-form expressions for the main variables of interest. We also conjecture that in the RE households’ expectations about future taxes will be a linear function of the underlying risk in the economy:

$$\tilde{T}^e_{1,k}(s) = \mu_{T,k} + \frac{\sigma_{T,k}}{\sigma_D} (D(s) - \mu_D),$$

for some values $\mu_{T,k}$ and $\sigma_{T,k}$, which will vary across thinkers of different levels. The advantage of (9) is that the beliefs and, hence, the behavior of each household are fully characterized by only two values: the mean and the standard deviation of expected future taxes. The following lemma gives a solution to the consumption/saving problem of a level-$k$ thinker.

Lemma 1. The households’ policy functions (3)-(6) are given by

$$X^*(Y_0, q, \tilde{T}^e_{1,k}) = \frac{\mu_D - qR}{\alpha \sigma_D^2} + \frac{\sigma_{T,k}}{\sigma_D},$$

$$B^*(Y_0, q, \tilde{T}^e_{1,k}) = \frac{R}{1 + R} \left[ Y_0 - \frac{\rho - r}{\alpha} + \left( \mu_{T,k} - \frac{\sigma_{T,k}}{\sigma_D} \mu_D \right) - \frac{\mu_D^2 - q^2R^2}{2 \alpha \sigma_D^2} + \left( 1 - \frac{\mu_D - qR}{\alpha \sigma_D^2} - \frac{\sigma_{T,k}}{\sigma_D} \right) q \right].$$

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9 This conjecture is easily verified by starting from $\tilde{T}^e_{1,1}(s) = 0$, $\forall s$, which implies $\mu_{T,1} = \sigma_{T,1} = 0$, and then proceeding recursively for $k \geq 2$. 

9
\[
C_0^* \left( Y_0, q, \tilde{T}_1 \right) = \frac{1}{1 + R} \left[ R Y_0 + \frac{\rho - r}{\alpha} + q R \left( 1 - \frac{\sigma_{T,k}}{\alpha} \right) \right.
\]
\[
- \left( \mu_{T,k} - \frac{\sigma_{T,k}}{\alpha} \mu_D \right) + \frac{(\mu_D - q R)^2}{2 \alpha \sigma_D^2} \right] ,
\]
\[
\tilde{C}_{1}^* \left( s; Y_0, q, \tilde{T}_1 \right) = D(s) \frac{\mu_D - q R}{\alpha \sigma_D^2} - \frac{R}{1 + R} \left[ - Y_0 + \frac{\rho - r}{\alpha} + \frac{1}{R} \left( \mu_{T,k} - \frac{\sigma_{T,k}}{\alpha} \mu_D \right) \right.
\]
\[
+ \frac{\mu_D^2 - q^2 R^2}{2 \alpha \sigma_D^2} - \left( 1 - \frac{\mu_D - q R}{\alpha \sigma_D^2} - \frac{\sigma_{T,k}}{\alpha} \right) q \left] ,
\]
\[
where \ r \equiv \log R .
\]

### 3.1 Rational Expectations Equilibrium

We begin with the characterization of the REE of this economy. In a REE every household holds the same correct expectations, which satisfy \( \tilde{T}_1(s) = T_1(s), \forall s, k \). In the appendix we show that these expectations take the form in (9) with \( \mu_{T,k} = -\alpha \sigma_D^2 X^G \) and \( \sigma_{T,k} = -X^G \sigma_D \), for all \( k \). The following lemma characterizes the REE of the economy.

**Lemma 2.** In the REE, the price of the risky asset and equilibrium output at time 0 are, respectively, given by

\[
q^{REE} = \frac{\mu_D - \alpha \sigma_D^2}{R}
\]

and

\[
Y_0^{REE} = \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2}.
\]

The key property of the REE is that both the price of the risky asset and the equilibrium output are independent of the quantity of risky assets purchased by the central bank. This result essentially implies that, when agents hold correct expectations about equilibrium variables, then a policy of QE is completely ineffective. This result is an application of Modigliani-Miller equivalence to unconventional monetary policy Wallace (1981). The intuition is also immediate. Rational-expectations households understand that the return on the assets purchased by the central bank will affect taxes. Risky assets owned by the central bank are, therefore, still on the households’ balance sheets, except that now take the form of taxes. Clearly, equilibrium prices and output cannot be affected by QE.

This stark conclusion also suggests that the only way to make QE effective is to break
the Modigliani-Miller equivalence result. This is what happens in the TE we consider below.

3.2 Temporary Equilibrium

In a TE all agents hold the same beliefs about future taxes and, thus, behave in the same way. In particular, when beliefs take the simple form in (9), Lemma 1 implies that the market-clearing condition for the risky asset takes the following simple form:

\[ 1 - X^G = \frac{\mu_D - qR}{\alpha \sigma_D^2} + \frac{\sigma_{T,k}}{\sigma_D} \]  

(10)

The right-hand side of (10) is the households’ demand for the risky asset. When taxes are perceived to be risky (\( \sigma_{T,k} \neq 0 \)), agents trade the risky asset to vary their exposure to the aggregate risk in the economy. Suppose, for example, that \( \sigma_{T,k} < 0 \). Then taxes contribute to make future consumption more sensitive to aggregate risk (remember taxes enter the budget constraint with a negative sign). Households will then respond by lowering their demand for the risky asset. Since all households behave in the same way and supply of risky assets is exogenously fixed, in equilibrium the price will have to adjust. We can invert equation (10) to solve for the equilibrium price:

\[ q^{TE} = q^{REE} + \alpha \sigma_D \frac{\sigma_D X^G + \sigma_{T,k}}{R}. \]

The equilibrium price in a TE differs from its REE counterpart and, importantly, it now varies with central bank purchases. The reason is simple. In a TE agents’ expectations are not required to be consistent with the model. Thus, following a policy of QE, agents may not fully internalize that taxes will be a function of the realized return on the central bank holdings of the risky asset. This is enough to break the Ricardian equivalence result.

To gain more intuition about this result, note that, since the risk-free rate is assumed to be fixed, the price \( q \) is immediately related to the economy’s risk-premium:

\[ rp \equiv \frac{\mu_D}{q} - R. \]

In particular, purchases of risky assets lower the risk-premium in the economy. In addition, when households believe that, after the central bank intervention, there is less aggregate risk in the economy, they will have less desire for precautionary saving and will thus consume more. Under our assumption that output at time 0 is driven by demand, central bank purchases can also affect output.
Lemma 3. Output in period 0 is given by

\[ Y_0 = \mu_D - \mu_{T,k} + \frac{\rho - r}{\alpha} + \frac{\alpha \sigma_D^2}{2} \left[ \left( X^G \right)^2 - \left( \frac{\sigma_{T,k} - \sigma_D}{\sigma_D} \right)^2 \right]. \] (11)

In a TE expected taxes are allowed to differ from realized taxes. Agents, however, do not use the model to update their beliefs, which are completely free. A RE, instead, puts more structure on such beliefs by letting some agents solve the model under different assumptions about the degree of sophistication of the other agents in the economy. In particular, level-\(k\) thinkers will expect future taxes to be those generated by the model under the assumption that the economy is populated only by level-(\(k-1\)) thinkers. The next lemma, which we prove in the appendix, combines Lemma 1, market-clearing conditions, and the government budget constraint (8) to back out realized future taxes.

Lemma 4. Assume agents expect taxes to be given by (9), for some \(\mu_{T,k}, \sigma_{T,k}\), then realized taxes are

\[ T_1(s) = Rq^{TE}X^G - D(s)X^G. \] (12)

3.3 Reflective Equilibrium

We use Lemma 4 to derive expectations of any level-\(k\) thinker. Start with a level-2 agent. By assumption this agent believes that all other agents are level-1 thinkers, who expect taxes to be 0 or, equivalently, \(\mu_{T,k} = \sigma_{T,k} = 0\). These expectations, together with Lemma 4 and the government budget constraint, deliver realized taxes when there are only level-1 thinkers in the economy. In turn, these realized taxes will coincide with the beliefs held by the level-2 thinker. Formally, we have

\[ \mu_{T,2} = -\alpha \sigma_D^2 (1 - X^G) X^G, \]
\[ \sigma_{T,2} = -\sigma_D X^G. \]

We can then use \(\mu_{T,2}\) and \(\sigma_{T,2}\) to compute realized taxes for an economy populated only by level-2 thinkers, which will give us expectations of level-3 thinkers. By iterating over \(k\) we can derive the expectations for any level of sophistication. What is more, under the assumptions of this section, the iterative process just described is rather simple. First, after the first round of reflection, agents already fully understand the standard deviation of taxes. Second, while level-2 thinkers are still wrong about the mean of taxes, all thinkers with a higher degree of sophistication will also hold correct expectations about the mean of taxes. In other words, level-3 thinkers and above will believe that the economy is
populated by agents with rational expectations. Formally, we have

\[ \mu_{T,k} = -\alpha \sigma^2_D X^G, \]
\[ \sigma_{T,k} = -\sigma_D X^G, \]

for all \( k \geq 3 \).

Equilibrium variables are then determined by finding the fixed-point of the following market-clearing conditions:

\[ 1 - X^G = \sum_{k=1}^{\infty} f(k) \left( \frac{\mu_D - q R}{\alpha \sigma^2_D} + \frac{\sigma_{T,k}}{\sigma_D} \right), \]
\[ Y_0 = \sum_{k=1}^{\infty} f(k) C_0^{*}(Y_0, q, \{\mu_{T,k}, \sigma_{T,k}\}), \]

where, with a slight abuse of notation, we used \( \{\mu_{T,k}, \sigma_{T,k}\} \) instead of \( \hat{T}^e_{1,k} \) to highlight that agents’ beliefs are fully characterized by the mean and the standard deviation. These conditions depend on the mass of agents with different levels of sophistication, which we need to assume. The next lemma provides a solution to the fixed-point above for a specific distribution.

**Proposition 1.** Assume the distribution of levels of thinking is given by the discrete exponential distribution \( f(k) = (1 - \lambda) \lambda^{k-1}, k \geq 1 \), then

\[ q = q^{REE} + \frac{\alpha \sigma^2_D (1 - \lambda)}{R} X^G, \]
\[ Y_0 = Y_0^{REE} + \frac{\alpha \sigma^2_D}{2} (X^G)^2 (1 - \lambda)^2. \]

Aside from the specific expressions, the main conclusion of Proposition 1 is that both the price of the risky asset and output are increasing functions of central bank purchases. In other words, unlike in the REE of Lemma 2, when some agents in the economy have limited ability to determine the effect of policies on future variables, central bank purchases can be expansionary. More specifically, by purchasing risky assets, the central bank hides some aggregate risk from the economy. A fraction of less sophisticated agents will react by saving less and consuming more. Figure 1 summarizes these results: As the central bank buys more assets, the price of the risky asset (the risk premium) increases (decreases) and output in period 0 increases.
Figure 1: Left and right panels plot, respectively, the price $q$ and output $Y_0$ as a function of the Central Bank’s demand of the risky asset for different values of $\lambda$. The REE values correspond to $\lambda = 1$.

Figure 1 plots the equilibrium price and output for different values of $\lambda$. Remember that the fraction of level-1 thinkers in the population is given by $1 - \lambda$, thus, when $\lambda$ increases, there are more sophisticated households in the economy. The effects of asset purchases are more muted as $\lambda$ increases towards 1, the value that corresponds to the REE.

### 3.4 Cross-Sectional Implications

After analyzing the behavior of aggregate variables, we discuss the cross-sectional implications of QE policy for households of different levels of thinking. Since households of different levels of thinking hold different beliefs about future taxes, they will not make identical choices, thus, the model has non-trivial predictions for the cross-section. An interesting question is whether higher order thinkers fare better than lower level ones? To address this question we need to compute households’ welfare. However, since in our model households make mistakes in computing future variables, there is no obvious measure for welfare. We compute ex-ante welfare conditional on different expectations that correspond to different levels of sophistication.

We start by deriving a useful property of CARA utility. Let $V$ be a household’s lifetime utility evaluated at the optimum. For brevity, we suppress the dependence of $V$ on expectations of future taxes, market prices, aggregate income, and government policy. The optimality condition with respect to safe bonds allows to conveniently rewrite the $V$ as
follows:

\[ V \equiv -\frac{1}{\alpha} e^{-\alpha C^*_0(Y_0, q, \hat{T}_1)} - \beta \frac{1}{\alpha} E \left[ e^{-\alpha C^*_1(s; Y_0, q, \hat{T}_e^k)} \right] = -\frac{11 + R}{R} e^{-\alpha C^*_0(Y_0, q, \hat{T}_1)} . \]

Thus, welfare of agents with different beliefs can be summarized by their individual choices of time-0 consumption. Moreover, cross-sectional welfare differences coincide with the behavior of cross-sectional period-0 consumption. For example, welfare of level-2 thinkers decreases as a function of \( X^G \) when \( \lambda \) is low enough, however, welfare of the rest of the population is always increasing in the quantity of purchased assets.

**Lemma 5.** Equilibrium welfare is a one-to-one and increasing function of time-0 consumption which, for households with different levels of thinking, is given by

\[
C^*_0(Y_0, q, \hat{T}_1^1) = Y_0^{REE} + \frac{\alpha \sigma^2_D}{2} (1 - \lambda)^2 \left( X^G \right)^2 , \\
C^*_0(Y_0, q, \hat{T}_1^2) = Y_0^{REE} + \frac{\alpha \sigma^2_D}{2} \left( X^G \right)^2 \left[ \frac{(1 - \lambda)^2}{2} - \frac{\lambda}{1 + R} \right] , \\
C^*_0(Y_0, q, \hat{T}_1^k) = Y_0^{REE} + \frac{\alpha \sigma^2_D}{2} \left( X^G \right)^2 \left[ \frac{(1 - \lambda)^2}{2} + \frac{1 - \lambda}{1 + R} \right] , \quad k \geq 3 .
\]

This lemma shows that consumption and, hence, welfare of level-1, level-3, and higher levels thinkers must necessarily increase in response to government purchases of risky assets. On the other hand, consumption of level-2 thinkers is ambiguous and it depends on the size of \( \lambda \). Remember, however, that by Lemma 1 overall consumption, output and, hence, welfare increase.

### 4 A Model with Limited Participation

In this section, we extend our model to allow for heterogeneity in access to financial markets. This extension allows us to study the interaction of limited participation and level-\( k \) thinking.

We will assume that there are two types of agents: those who can trade only riskless bonds, households \( a \), with mass \( \omega_a \), and those who can trade both safe and risky assets, households \( b \), with mass \( \omega_b = 1 - \omega_a \). Households \( a \) income in period 0 consists only of labor income, household \( b \) income is a sum of labor income, and the value of a risky assets endowment. Labor income is shared equally across the households. Endowment of risky assets is inversely proportional to the fraction of households \( b \) to keep the aggregate
endowment equal to one. Households $a$ solve the following optimization problem

$$\max_{C_0^a, B^a, \{\tilde{C}_1^a(s)\}} \mathbb{E}_s \left[ u(C_0^a) + \beta u\left(\tilde{C}_1^a(s)\right)\right],$$

s.t.: $C_0^a + \frac{B^a}{R} \leq Y_0,$

$$\tilde{C}_1^a(s) + T_1^{a,e}(s) \leq B^a,$$  

where $T_1^{a,e}(s)$ is household $a$’s expectation of taxes in period 1 and state $s$. The solution to this problem in the case of CARA utility function is summarized in the next lemma.

**Lemma 6.** Households $a$’s policy functions are given by

$$C_0^a = \frac{R}{1 + R} \left[ \frac{\rho - r}{\alpha R} + Y_0 - \frac{1}{R} \left( \mu_{T,a} + \frac{\alpha}{2} \sigma_{T,a}^2 \right) \right],$$

$$B^a = \frac{R}{1 + R} \left( Y_0 + \mu_{T,a} + \frac{\alpha}{2} \sigma_{T,a}^2 - \frac{\rho - r}{\alpha} \right),$$

$$C_1^a(s) = B^a - T_1^{a,e}(s).$$

Households $b$ solve the following problem

$$\max_{C_0^b, B^b, X^b, \{\tilde{C}_1^b(s)\}} \mathbb{E}_s \left[ u(C_0^b) + \beta u\left(\tilde{C}_1^b(s)\right)\right],$$

s.t.: $C_0^b + \frac{B^b}{R} + qX^b \leq \frac{q}{\omega_b} + Y_0,$

$$\tilde{C}_1^b(s) + T_1^{b,e}(s) \leq D(s)X^b + B^b,$$  

where $T_1^{b,e}(s)$ is household $b$’s expectation of taxes in period 1 and state $s$. The solution to this problem is similar to the one presented in Lemma 1 except that the endowment of risky assets is now $1/\omega_b$ instead of 1.

The government faces the following period budget constraints

$$qX^G = \frac{B}{R'},$$

$$B = \omega_a T_1^a(s) + \omega_b T_1^b(s) + D(s)X^G, \forall s,$$

where $B$ is total outstanding public debt. Finally, we assume that the government taxes both types of households equally:

$$T_1^a(s) = T_1^b(s) = T_1(s), \forall s.$$
4.1 Rational Expectations Equilibrium

We start with the characterization of the REE of this economy. In a REE every household holds the same correct expectations: \( T_{1}^{a,c}(s) = T_{1}^{b,c}(s) = T_{1}(s), \forall s. \)

**Lemma 7.** In the REE, the price of the risky asset and equilibrium output at time 0 are respectively given by

\[
q^{REE} = R^{-1} \left( \mu_{D} - \alpha \sigma_{D}^{2} \left[ \frac{1}{\omega_{b}} - \frac{\omega_{a}}{\omega_{b}} X^{G} \right] \right)
\]

and

\[
Y_{0}^{REE} = \frac{\rho - r}{\alpha} + \mu_{D} - \frac{\alpha \sigma_{D}^{2}}{2} - \frac{\alpha \sigma_{D}^{2}}{2} \cdot \frac{\omega_{a}}{\omega_{b}} + \frac{\alpha \sigma_{D}^{2}}{2} \cdot \frac{\omega_{a}}{\omega_{b}} \left[ 1 - \left( 1 - X^{G} \right)^{2} \right].
\]

These expressions can be interpreted as follows. First, consider the case where the government purchases no assets. The price of the risky asset depends inversely on the number of households of type \( b \). This is because each household \( b \) holds \( 1/\omega_{b} \) of risky assets implying that the variance of his portfolio is \( \sigma_{D}^{2}/\omega_{b}^{2} \), which reduces the demand for risky assets proportionally. Because there are \( \omega_{b} \) households of this type, the price of risky asset is proportional to \( \omega_{b} \cdot \sigma_{D}^{2}/\omega_{b}^{2} \) in equilibrium. Without government purchases, output consists of two parts: \( (\rho - r)/\alpha + \mu_{D} - \alpha \sigma_{D}^{2}/2 \) – the output when all agents are allowed to trade all assets – and \( -\alpha \sigma_{D}^{2}/2 \cdot \omega_{a}/\omega_{b} \) – a term that arises when only \( \omega_{b} \) agents can trade risky assets.

Second, when the government buys risky assets it levies taxes uniformly on all of the households. Thus, not all risk, which the government took away from households \( b \) through asset purchases, will be given back to the same households in the form of taxes. Specifically, each household \( b \) sells \( X^{C}/\omega_{b} \) risky assets, but his taxes will depend on only \( X^{C} \) units of risky assets, hence, households \( b \)’s effective risk will decline by a factor \( X^{C} \omega_{a}/\omega_{b} \). Asset purchases, therefore, have two countervailing effects on output: On the one hand, since taxes will be risky in the future, households \( a \)’s consumption demand drops; on the other hand, households \( b \) demand increases because part of the risk is taken away from their portfolio. The expression in Lemma 7 shows that the net effect is an increase in consumption demand and output.

4.2 Temporary Equilibrium

In a TE all agents hold the same beliefs about future taxes irrespective of their access to financial markets. The market-clearing condition for the risky asset takes the following
simple form:
\[ 1 - X^G = \omega_b \left( \frac{\mu_D - qR}{\alpha \sigma_D^2} + \frac{\sigma_{T,k}}{\sigma_D} \right). \]  
(13)

This can be solved to obtain the price \( q \)

\[ q_{TE} = \frac{\mu_D - \alpha \sigma_D^2 \left( \frac{1 - X^G}{\omega_b} - \frac{\sigma_{T,k}}{\sigma_D} \right)}{R}. \]

Realized taxes levied on the households are

\[ T_1(s) = R q_{TE} X^G - D(s) X^G. \]

The last two equations will be used by level-\( k \) thinkers to update their beliefs about taxes.

### 4.3 Reflective Equilibrium

We now derive expectations of any level-\( k \) thinker. Start with a level-2 agent. By assumption this agent believes that all other agents are level-1 thinkers, who expect taxes to be 0 or, equivalently, \( \mu_{T,k} = \sigma_{T,k} = 0 \). These expectations deliver realized taxes when there are only level-1 thinkers in the economy. In turn, these realized taxes will coincide with the beliefs held by the level-2 thinker. Formally, we have

\[ \mu_{T,2} = -\alpha \sigma_D^2 \frac{1 - X^G}{\omega_b} X^G, \]
\[ \sigma_{T,2} = -\sigma_D X^G. \]

We can then use \( \mu_{T,2} \) and \( \sigma_{T,2} \) to compute realized taxes for an economy populated only by level-2 thinkers, which will give us expectations of level-3 thinkers. By iterating over \( k \) we can derive the expectations for any level of sophistication. What is more, under the assumptions of this section, the iterative process just described is rather simple. First, after the first round of reflection, agents already fully understand the standard deviation of taxes. Second, while level-2 thinkers are still wrong about the mean of taxes, all thinkers with a higher degree of sophistication will also hold correct expectations about the mean of taxes. In other words, level-3 thinkers and above will believe that the economy is
populated by agents with rational expectations. Formally, we have

\[ \mu_{T,k} = -\alpha \sigma_D^2 \left( \frac{1 - X^G}{\omega_b} + X^G \right) X^G, \]

\[ \sigma_{T,k} = -\sigma_D X^G, \]

for all \( k \geq 3 \).

Equilibrium variables are then determined by finding the fixed-point of the risky-asset and goods-market clearing conditions in period 0.

**Lemma 8.** Assume the distribution of levels of thinking is given by the discrete exponential distribution \( f(k) = (1 - \lambda) \lambda^{k-1}, k \geq 1 \), then

\[ q = \frac{\mu_D - \alpha \sigma_D^2}{R} \]

\[ - \frac{\alpha \sigma_D^2 \left( \frac{1}{\omega_b} - 1 \right)}{R} \]

\[ + \frac{\alpha \sigma_D^2 \left( \frac{1}{\omega_b} - 1 \right) X^G}{R} \]

\[ + \frac{\alpha \sigma_D^2}{R} X^G(1 - \lambda), \]

\[ Y_0 = \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2} \]

\[ - \frac{\alpha \sigma_D^2}{2} \cdot \frac{\omega_a}{\omega_b} \]

\[ + \frac{\alpha \sigma_D^2}{2} \cdot \frac{\omega_a}{\omega_b} X^G \left( 2 - X^G \right) \]

\[ + \frac{\alpha \sigma_D^2}{2} \left( X^G \right)^2 (1 - \lambda)^2 \]

\[ + \frac{\alpha \sigma_D^2}{2} X^G (1 - \lambda) \omega_a \left[ \left( \frac{2}{\omega_b} - \lambda \right) X^G - \frac{2}{\omega_b} \right]. \]

The last term in the formula for output presents the interaction between limited participation and “level-k thinking.” It is easy to see that this term is negative, implying that in the presence of both friction the effect on output is less than a sum of effects of each friction separately. It is also easy to see that the overall QE effect on output is still positive, i.e., \( dY_0^{RE} / dX^G > 0 \).
5 An Infinite-horizon Model

This section extends the model to infinite horizon and study how the duration of asset purchases affect the effectiveness of QE.

We assume that there is a unit supply of infinitely lived Lucas trees that pay off \( \delta Y_t \) as dividends each period, where \( \delta \in (0, 1) \) and \( Y_t \) is an aggregate output. The remaining part of aggregate income is distributed as labor income. We assume that goods nominal prices are completely sticky. This assumption saves us from dealing with the Phillips curve and allows for a tractable non-linear analysis of the New-Keynesian model with aggregate risk. Because prices are sticky, monetary policy has a control over real interest rate on safe debt.

To be typed up.

6 Conclusion

We study the effects of government risky assets purchases financed by issuing safe debt in a model without rational expectations. The same forces that reduce the “forward guidance puzzle” in García-Schmidt and Woodford (2015) and Farhi and Werning (2016) make quantitative easing policies affect market and real economy outcomes.
References


A Appendix

A.1 Proof of Lemma 1

Let \( \rho \equiv - \log \beta, r \equiv \log R \) and use (9) to rewrite households’ objective function as

\[
-\frac{1}{\alpha} e^{-\rho C_0} - \frac{1}{\alpha} e^{-\rho - \alpha (\mathbb{E}C_1(s) - \frac{\alpha}{2} \text{Var}(\bar{C}_1(s)))},
\]

Households’ optimality conditions are a solution to

\[
C_0 = (1 - X) q + Y_0 - \frac{B}{R},
\]

\[
\bar{C}_1(s) = D(s) X + B - T_1^r(s),
\]

\[
e^{-\rho C_0} = e^{-\rho + r - \alpha (\mathbb{E}[\bar{C}_1(s)] - \frac{\alpha}{2} \text{Var}(\bar{C}_1(s)))},
\]

\[
q = e^{-r \frac{d}{dX}} \left( \mathbb{E} \left[ \bar{C}_1(s) \right] - \frac{\alpha}{2} \text{Var} \left( \bar{C}_1(s) \right) \right).
\]

We have

\[
\mathbb{E} \left[ \bar{C}_1(s) \right] = \mathbb{E} [D(s) X + B - T_1^r(s)]
\]

\[
= \mu_D X + B - \mu_{T,k}
\]

and

\[
\text{Var} \left( \bar{C}_1(s) \right) = \left( X - \frac{\sigma_{T,k}}{\sigma_D} \right)^2 \sigma_D^2.
\]

Rearranging the first Euler equation

\[
(1 - X) q + Y_0 - \frac{B}{R} = \frac{\rho - r}{\alpha} + \mathbb{E} \left[ \bar{C}_1(s) \right] - \frac{\alpha}{2} \text{Var} \left( \bar{C}_1(s) \right),
\]

\[
(1 - X) q + Y_0 - \frac{B}{R} = \frac{\rho - r}{\alpha} + \mu_D X + B - \mu_{T,k} - \frac{\alpha}{2} \left( X - \frac{\sigma_{T,k}}{\sigma_D} \right)^2 \sigma_D^2,
\]

which gives

\[
B = \frac{R}{1 + R} \left[ \frac{\rho - r}{\alpha} - \mu_D X + (1 - X) q + Y_0 + \mu_{T,k} + \frac{\alpha}{2} \left( X - \frac{\sigma_{T,k}}{\sigma_D} \right)^2 \sigma_D^2 \right].
\]  

(A.1)

Plugging back into the budget constraint at time 0,

\[
C_0 = (1 - X) q + Y_0 - \frac{B}{R}
\]

\[
= (1 - X) q + Y_0 - \frac{1}{1 + R} \left[ \frac{\rho - r}{\alpha} - \mu_D X + (1 - X) q + Y_0 + \mu_{T,k} + \frac{\alpha}{2} \left( X - \frac{\sigma_{T,k}}{\sigma_D} \right)^2 \sigma_D^2 \right]
\]

\[
= \frac{R}{1 + R} \left[ (1 - X) q + Y_0 \right] + \frac{1}{1 + R} \left[ \frac{\rho - r}{\alpha} + \mu_D X - \mu_{T,k} - \frac{\alpha}{2} \left( X - \frac{\sigma_{T,k}}{\sigma_D} \right)^2 \sigma_D^2 \right].
\]
From the second Euler equation,

\[ q = e^{-r} \frac{d}{dX} \left( E \left[ \bar{C}_1(s) \right] - \frac{\alpha}{2} Var \left( \bar{C}_1(s) \right) \right) = e^{-r} \left( \mu_D - \alpha \left( X - \frac{\sigma_{T,k}}{\sigma_D} \right) \sigma_D^2 \right), \]

which gives

\[ X = \frac{\mu_D - qR}{\alpha \sigma_D^2} + \frac{\sigma_{T,k}}{\sigma_D}. \]  

(A.2)

Combining (A.2) with (A.1) gives

\[ B = \frac{R}{1 + R} \left[ Y_0 - \frac{\rho - r}{\alpha} - \mu_D \frac{\mu_D - qR}{\alpha \sigma_D^2} - \mu_D \frac{\sigma_{T,k}}{\sigma_D} + \mu_{T,k} + \left( 1 - \frac{\mu_D - qR}{\alpha \sigma_D^2} - \frac{\sigma_{T,k}}{\sigma_D} \right) q + \frac{\alpha}{2} \left( \frac{\mu_D - qR}{\alpha \sigma_D^2} \right)^2 \sigma_D^2 \right] \]

\[ = \frac{R}{1 + R} \left[ Y_0 - \frac{\rho - r}{\alpha} + \left( \mu_{T,k} - \mu_D \frac{\sigma_{T,k}}{\sigma_D} \right) - \frac{1}{2 \alpha \sigma_D^2} \left( \mu_D^2 - q^2 R^2 \right) + \left( 1 - \frac{\mu_D - qR}{\alpha \sigma_D^2} - \frac{\sigma_{T,k}}{\sigma_D} \right) q \right]. \]  

(A.3)

Using the budget constraint at time 0,

\[ C_0 = (1 - X) q + Y_0 - \frac{B}{R} \]

\[ = \frac{R}{1 + R} \left[ Y_0 + \left( 1 - \mu_D \frac{\sigma_{T,k}}{\sigma_D} \right) q \right] + \frac{1}{1 + R} \left[ \frac{\rho - r}{\alpha} - \left( \mu_{T,k} - \mu_D \frac{\sigma_{T,k}}{\sigma_D} \right) + \frac{1}{2 \alpha \sigma_D^2} \left( \mu_D^2 - q^2 R^2 \right) \right]. \]  

(A.4)

Finally, from the budget constraint at time 1,

\[ \bar{C}_1(s) = D(s) X + B - T_1^p(s) \]

\[ = D(s) \frac{\mu_D - qR}{\alpha \sigma_D^2} + \frac{R}{1 + R} \left[ Y_0 - \frac{\rho - r}{\alpha} - \frac{1}{R} \left( \mu_{T,k} - \mu_D \frac{\sigma_{T,k}}{\sigma_D} \right) \right] \]

\[ - \frac{1}{2 \alpha \sigma_D^2} \left( \mu_D^2 - q^2 R^2 \right) + \left( 1 - \frac{\mu_D - qR}{\alpha \sigma_D^2} - \frac{\sigma_{T,k}}{\sigma_D} \right) q \].  

(A.5)

### A.2 Proof of Lemma 2

Combining (A.2), the market-clearing condition for the risky asset, and \( \sigma_{T,k} = -XG \sigma_D \), gives

\[ q_{REE} = \frac{\mu_D - \alpha \sigma_D^2}{R}. \]

Similarly, we can use the equilibrium condition \( C_0 = Y_0 \) with (A.4) and \( \mu_{T,k} = -\alpha \sigma_D^2 XG \) to obtain
or, combining with (A.2) and the market-clearing condition for the risky asset,

\[ Y_0 = \frac{R}{1+R} \left[ Y_0 + \left(1 - \frac{\sigma_{T,k}}{\sigma_D} \right) q^{REE} \right] + \frac{1}{1+R} \left[ \frac{\rho - r}{\alpha} - \left( \mu_{T,k} - \mu_D + \frac{\sigma_{T,k}}{\sigma_D} \right) \right] + \frac{1}{2\alpha \sigma_D^2} \left( \mu_D - q^{REE} \right)^2, \]

or

\[ Y_0 = R \left(1 + X^G \right) \frac{\mu_D - \alpha \sigma_D^2}{R} + \frac{\rho - r}{\alpha} - \left( -\alpha \sigma_D^2 X^G + \mu_D X^G \right) + \frac{1}{2\alpha \sigma_D^2} \left( \mu_D - \mu_D + \alpha \sigma_D^2 \right)^2, \]

or

\[ Y_0^{REE} = \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2}. \]

### A.3 Proof of Lemma 3

We can use the equilibrium condition \( C_0 = Y_0 \) with (A.4) to obtain

\[ Y_0 = R \left(1 - \frac{\sigma_{T,k}}{\sigma_D} \right) q + \frac{\rho - r}{\alpha} - \left( \mu_{T,k} - \mu_D + \frac{\sigma_{T,k}}{\sigma_D} \right) + \frac{1}{2\alpha \sigma_D^2} \left( \mu_D - qR \right)^2, \]

or, combining with (A.2) and the market-clearing condition for the risky asset,

\[ Y_0 = \left(1 - \frac{\sigma_{T,k}}{\sigma_D} \right) \left( \mu_D - \alpha \left(1 - X^G - \frac{\sigma_{T,k}}{\sigma_D} \right) \sigma_D^2 \right) + \frac{\rho - r}{\alpha} - \left( \mu_{T,k} - \mu_D + \frac{\sigma_{T,k}}{\sigma_D} \right) + \frac{\alpha \sigma_D^2}{2} \left(1 - X^G - \frac{\sigma_{T,k}}{\sigma_D} \right)^2, \]

or, rearranging,

\[ Y_0 = \mu_D - \mu_{T,k} + \frac{\rho - r}{\alpha} + \left(1 - \frac{\sigma_{T,k}}{\sigma_D} \right) \left( -\alpha \left(1 - X^G - \frac{\sigma_{T,k}}{\sigma_D} \right) \sigma_D^2 \right) + \frac{\alpha \sigma_D^2}{2} \left(1 - X^G - \frac{\sigma_{T,k}}{\sigma_D} \right)^2 \]

\[ = \mu_D - \mu_{T,k} + \frac{\rho - r}{\alpha} + \frac{\alpha \sigma_D^2}{2} \left( 2 \left(1 - \frac{\sigma_{T,k}}{\sigma_D} \right) + \left(1 - X^G - \frac{\sigma_{T,k}}{\sigma_D} \right) \left(1 - X^G - \frac{\sigma_{T,k}}{\sigma_D} \right) \right) \]

\[ = \mu_D - \mu_{T,k} + \frac{\rho - r}{\alpha} + \frac{\alpha \sigma_D^2}{2} \left( \left( X^G \right)^2 - \left( \frac{\sigma_{T,k}}{\sigma_D} - 1 \right)^2 \right). \]

### A.4 Proof of Lemma 4

\( T_1(s) = B - D(s)X^G \)

\[ = \frac{R}{1+R} \left[ -\frac{\rho - r}{\alpha} \left( \mu_{T_1} - \frac{\sigma_{T_1}}{\sigma_D} \mu_D \right) - \frac{\mu_D^2 - q^2R^2}{2\alpha \sigma_D^2} + \left(1 - \frac{\mu_D - qR}{\alpha \sigma_D^2} - \frac{\sigma_{T_1}}{\sigma_D} \right) q + Y_0 \right] - D(s)X^G \]

\[ = \frac{R}{1+R} \left[ -\frac{\rho - r}{\alpha} \left( \mu_{T_1} - \frac{\sigma_{T_1}}{\sigma_D} \mu_D \right) \right] - D(s)X^G \]

\[ + \frac{R}{1+R} \left[ \frac{\mu_D^2 - q^2R^2}{2\alpha \sigma_D^2} + \left(1 - \frac{\mu_D - qR}{\alpha \sigma_D^2} - \frac{\sigma_{T_1}}{\sigma_D} \right) q + Y_0 \right]. \]
We simplify the last line as follows

\[
- \frac{\mu_D^2 - q^2 R^2}{2\alpha \sigma_D^2} + \left(1 - \frac{\mu_D - q}{\alpha \sigma_D^2} - \frac{\sigma_T^2}{\sigma_D^2}\right) q + Y_0
\]

\[
= -\frac{\alpha \sigma_D^2}{2} \left(1 - X^G - \frac{\sigma_T^2}{\sigma_D^2}\right) \left(1 - X^G - \frac{\sigma_T^2}{\sigma_D^2}\right) + X^G R^{-1} \left[\mu_D - \alpha \sigma_D^2 \left(1 - X^G - \frac{\sigma_T^2}{\sigma_D^2}\right)\right]
\]

\[
+ \mu_D - \mu_T^1 + \frac{\rho - r}{\alpha} + \alpha \sigma_D \left[\sigma_T^2 - \sigma_D^2\right] \left(1 - X^G - \frac{\sigma_T^2}{\sigma_D^2}\right) + \frac{\alpha \sigma_D^2}{2} \left(1 - X^G - \frac{\sigma_T^2}{\sigma_D^2}\right)^2
\]

As a result, taxes in period 1 are

\[
T_1(s) = \frac{R}{1 + R} \left[-\frac{\rho - r}{\alpha} + \left(\mu_T^1 - \frac{\sigma_T^2}{\sigma_D^2} \mu_D\right)\right] - D(s) X^G
\]

\[
+ \frac{R}{1 + R} \left[\left(1 - X^G - \frac{\sigma_T^2}{\sigma_D^2}\right) \left(-\mu_D - X^G R^{-1} \alpha \sigma_D^2 + \alpha \sigma_D \left[\sigma_T^2 - \sigma_D^2\right]\right)\right]
\]

\[
+ \frac{R}{1 + R} \left[\left(1 + X^G R^{-1}\right) \mu_D - \mu_T^1 + \frac{\rho - r}{\alpha} + \alpha \sigma_D^2 \left(1 - X^G - \frac{\sigma_T^2}{\sigma_D^2}\right)^2\right]
\]

\[
= \frac{R}{1 + R} \left[\left(1 - X^G - \frac{\sigma_T^2}{\sigma_D^2}\right) \left(-\mu_D - X^G R^{-1} \alpha \sigma_D^2 + \alpha \sigma_D \left[\sigma_T^2 - \sigma_D^2\right]\right)\right]
\]

\[
+ \frac{R}{1 + R} \left[\left(1 - \frac{\sigma_T^2}{\sigma_D^2} + X^G R^{-1}\right) \mu_D + \alpha \sigma_D^2 \left(1 - X^G - \frac{\sigma_T^2}{\sigma_D^2}\right)^2\right] - D(s) X^G.
\]

\[
T_1(s) = \frac{R}{1 + R} \left[-\left(1 - X^G\right) \left(\mu_D + \alpha \sigma_D^2 \left(\frac{X^G}{R} + 1\right)\right) + \left(1 + \frac{X^G}{R}\right) \mu_D + \alpha \sigma_D^2 \left(1 - X^G\right)^2\right] - D(s) X^G
\]

\[
= \frac{R}{1 + R} \left\{\frac{\mu_D X^G}{1 + \frac{R}{R}} + \left[\left(1 - X^G\right)^2 - \left(1 - X^G\right) \left(\frac{X^G}{R} + 1\right)\right] \alpha \sigma_D^2\right\} - D(s) X^G
\]

\[
= RX^G \frac{\mu_D}{R} - \left(1 - X^G\right) \alpha \sigma_D^2 - D(s) X^G.
\]
Note that

\[ q = \frac{\mu_D - \alpha \sigma_D^2}{R} \left( 1 - X^G f(1) \right) \]
\[ = \frac{\mu_D - \alpha \sigma_D^2 + \alpha \sigma_D^2 X^G (1 -\lambda)}{R} \]
\[ = \frac{\mu_D - \alpha \sigma_D^2 (1 - X^G (1 -\lambda))}{R} \]
\[ = \frac{\mu_D - \alpha \sigma_D^2 + \alpha \sigma_D^2 X^G (1 -\lambda)}{R} \]
\[ = q^\text{REE} + \alpha \sigma_D^2 X^G(1 -\lambda) \]

Note that

\[ \mu_D - qR = \alpha \sigma_D^2 \left( 1 - X^G (1 -\lambda) \right) . \]

Output is

\[
Y_0 = \sum_{k=1}^{\infty} f(k) C_0^{(k)} = \sum_{k=1}^{\infty} f(k) \frac{1}{1+R} \left[ qR \left( 1 - \frac{\sigma_{\pi_1}^{(k)}}{\sigma_D} \right) + RY_0 + \frac{\rho - r}{\alpha} - \left( \frac{\sigma_{\pi_1}^{(k)}}{\sigma_D} \mu_D \right) + \left( \frac{\mu_D - qR}{2\alpha \sigma_D^2} \right)^2 \right]
\]
\[
Y_0 = \frac{\rho - r}{\alpha} + \frac{(\mu_D - qR)^2}{2\alpha \sigma_D^2} + \sum_{k=1}^{\infty} f(k) \left[ qR \left( 1 - \frac{\sigma_{\pi_1}^{(k)}}{\sigma_D} \right) - \left( \frac{\sigma_{\pi_1}^{(k)}}{\sigma_D} \mu_D \right) \right]
\]
\[
= \frac{\rho - r}{\alpha} + \frac{(\mu_D - qR)^2}{2\alpha \sigma_D^2} + qR \left[ f(1) + \sum_{k=2}^{\infty} f(k) \left( 1 + X^G \right) \right]
\]
\[
= \frac{\rho - r}{\alpha} + \frac{(\mu_D - qR)^2}{2\alpha \sigma_D^2} + qR \left[ (1 - \lambda) + \left( 1 + X^G \right) \lambda \right]
\]
\[
= \frac{\rho - r}{\alpha} + \frac{(\mu_D - qR)^2}{2\alpha \sigma_D^2} + qR \left[ (1 - \lambda) + \left( 1 + X^G \right) \lambda \right]
\]
\[
= \frac{\rho - r}{\alpha} + \frac{(\mu_D - qR)^2}{2\alpha \sigma_D^2} + qR \left[ (1 - \lambda) + \left( 1 + X^G \right) \lambda \right]
\]
\[
= \frac{\rho - r}{\alpha} + \frac{(\mu_D - qR)^2}{2\alpha \sigma_D^2} + qR \left[ (1 - \lambda) + \left( 1 + X^G \right) \lambda \right]
\]
\begin{align*}
&= \frac{\rho - r}{\alpha} + \frac{(\mu_D - qR)^2}{2\alpha \sigma_D^2} + qR \left[ 1 - \lambda + \left( 1 + X^G \right) \lambda \right] - R X^G \left( \frac{\mu_D - \alpha \sigma_D^2}{R} + (1 - \lambda) \lambda \frac{\alpha \sigma_D^2 X^G}{R} \right) \\
&= \frac{\rho - r}{\alpha} + \frac{(\mu_D - qR)^2}{2\alpha \sigma_D^2} + qR \left( 1 + \lambda X^G \right) - R X^G \lambda \left( \frac{\mu_D - \alpha \sigma_D^2}{R} (1 - (1 - \lambda)X^G) \right) \\
&= \frac{\rho - r}{\alpha} + \frac{(\mu_D - qR)^2}{2\alpha \sigma_D^2} + qR \left( 1 + \lambda X^G \right) - R X^G \lambda q \\
&= \frac{\rho - r}{\alpha} + \frac{(\mu_D - qR)^2}{2\alpha \sigma_D^2} + qR \\
&= \frac{\rho - r}{\alpha} + \mu_D + \frac{\alpha \sigma_D^2}{2} (1 - X^G(1 - \lambda))^2 - \alpha \sigma_D^2 + \alpha \sigma_D^2 X^G(1 - \lambda) \\
&= \frac{\rho - r}{\alpha} + \mu_D + \alpha \sigma_D^2 \left[ \frac{1 - 2X^G (1 - \lambda) + (X^G)^2 (1 - \lambda)^2}{2} - 1 + X^G(1 - \lambda) \right] \\
&= \frac{\rho - r}{\alpha} + \mu_D + \alpha \sigma_D^2 \left[ \frac{-1 + (X^G)^2 (1 - \lambda)^2}{2} \right] \\
&= \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2} + \frac{\alpha \sigma_D^2}{2} \left( X^G \right)^2 (1 - \lambda)^2. \\
\end{align*}

### A.6 Proof of Lemma 5

Equilibrium price and output

\[ q = \frac{\mu_D - \alpha \sigma_D^2}{R} + \frac{\alpha \sigma_D^2 (1 - \lambda)}{R} X^G, \]

\[ Y_0 = \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2} + \frac{\alpha \sigma_D^2}{2} \left( X^G \right)^2 (1 - \lambda)^2. \]

This implies

\[ \mu_D - qR = \alpha \sigma_D^2 \left[ 1 - (1 - \lambda)X^G \right]. \]

Consumption of level-k thinkers

\[ C_0^* \left( Y_0, q, \{ T_{i,k} (s) \} \right) = \frac{1}{1 + R} \left[ R Y_0 + \frac{\rho - r}{\alpha} + qR \left( 1 - \frac{\sigma_{T,k}}{\sigma_D} \right) - \left( \mu_{T,k} - \frac{\sigma_{T,k}}{\sigma_D} \mu_D \right) + \left( \frac{\mu_D - qR}{2 \alpha \sigma_D^2} \right) \right]. \]

Substitute in future taxes expectations
\[ C_0 \left( Y_0, q, \{ T_1^{r,1}(s) \} \right) = \frac{1}{1+R} \left\{ RY_0 + \frac{\rho - r}{\alpha} + qR + \frac{(\mu_D - qR)^2}{2\alpha e_D} \right\} \]

\[ = \frac{1}{1+R} \left\{ R\frac{\rho - r}{\alpha} + \frac{\rho - r}{\alpha} + R\mu_D + \mu_D - R\frac{a\sigma_D^2}{2} - a\sigma_D^2 \right. \]

\[ + R \frac{a\sigma_D^2}{2} \left( X^G \right)^2 (1-\lambda)^2 + a\sigma_D^2 (1-\lambda)X^G + \frac{a\sigma_D^2}{2} (1 - (1 - \lambda)X^G)^2 \right\} \]

\[ = \frac{1}{1+R} \left\{ R\frac{\rho - r}{\alpha} + \frac{\rho - r}{\alpha} + R\mu_D + \mu_D - R\frac{a\sigma_D^2}{2} - a\sigma_D^2 + R \frac{a\sigma_D^2}{2} \left( X^G \right)^2 (1-\lambda)^2 \right. \]

\[ + a\sigma_D^2 (1-\lambda)X^G + \frac{a\sigma_D^2}{2} (1 - (1 - \lambda)X^G)^2 \right\} \]

\[ = \frac{1}{1+R} \left\{ (1+R) \left( \frac{\rho - r}{\alpha} + \mu_D - \frac{a\sigma_D^2}{2} \right) + \frac{a\sigma_D^2}{2} \left[ 1 + R \right] (1 - \lambda)^2 \left( X^G \right)^2 \right\} \]

\[ = \frac{\rho - r}{\alpha} + \mu_D - \frac{a\sigma_D^2}{2} + \frac{a\sigma_D^2}{2} (1-\lambda)^2 \left( X^G \right)^2 , \]

\[ C_0 \left( Y_0, q, \{ T_1^{r,2}(s) \} \right) = \frac{1}{1+R} \left\{ RY_0 + \frac{\rho - r}{\alpha} + qR + \left( qR - \mu_D + a\sigma_D^2 (1-X^G) \right) X^G + \frac{(\mu_D - qR)^2}{2a e_D} \right\} \]

\[ = \frac{1}{1+R} \left\{ RY_0 + \frac{\rho - r}{\alpha} + qR + \frac{(\mu_D - qR)^2}{2a e_D} - \lambda a\sigma_D^2 \left( X^G \right)^2 \right\} \]

\[ = C_0 \left( Y_0, q, \{ T_1^{r,1}(s) \} \right) \frac{\lambda a\sigma_D^2 (X^G)^2}{1+R} \]

\[ = \frac{\rho - r}{\alpha} + \mu_D - \frac{a\sigma_D^2}{2} + \frac{a\sigma_D^2}{2} \left[ \lambda^2 - 2\lambda \frac{1+1}{1+R} + 1 \right] \left( X^G \right)^2 , \]

\[ C_0 \left( Y_0, q, \{ T_1^{r,k}(s) \} \right) = \frac{1}{1+R} \left\{ RY_0 + \frac{\rho - r}{\alpha} + qR + \left( qR - \mu_D + a\sigma_D^2 \right) X^G + \frac{(\mu_D - qR)^2}{2a e_D} \right\} \]

\[ = \frac{1}{1+R} \left\{ RY_0 + \frac{\rho - r}{\alpha} + qR + \frac{(\mu_D - qR)^2}{2a e_D} + (1-\lambda)\lambda a\sigma_D^2 \left( X^G \right)^2 \right\} \]

\[ = \frac{\rho - r}{\alpha} + \mu_D - \frac{a\sigma_D^2}{2} + \frac{a\sigma_D^2}{2} (1-\lambda)^2 \left( X^G \right)^2 + \frac{(1-\lambda)\lambda a\sigma_D^2 \left( X^G \right)^2}{1+R} . \]

### A.7 Proof of Lemma 6

\[ C_0^a = Y_0 - \frac{B^a}{R}, \]

\[ C_1^a(s) = B^a - T_{1,s}^a(s), \]

\[ e^{-aC_0} = \mathbb{E} \left[ e^{-\rho + r - aC_1(s)} \right] . \]

The Euler equation can be rewritten as follows
\[
C_0^a = \rho - r = a + B^a - \mu T^{a,e} - \frac{\alpha}{2} \sigma^2 T^{a,e}.
\]

Combine it with the budget constraint in period 0
\[
B^a = \frac{R}{1 + R} \left( -\rho - \frac{r}{\alpha} + Y_0 + \mu T^{a,e} + \frac{\alpha}{2} \sigma^2 T^{a,e} \right).
\]

Consumption in period 0 can be then written as
\[
C_0^a = \rho - r = a + B^a - \mu T^{a,e} - \frac{\alpha}{2} \sigma^2 T^{a,e} = \frac{1}{1 + R} \left( -\rho - \frac{r}{\alpha} + R \left( Y_0 + \mu T^{a,e} + \frac{\alpha}{2} \sigma^2 T^{a,e} \right) \right).
\]

A.8 Proof of Lemma 7

The government budget constraint implies
\[
T_1(s) = qRX^G - D(s)X^G.
\]

As a result, in REE agents have the following belief
\[
\frac{\sigma_T}{\sigma_D} = -X^G,
\]
\[
\mu_T = (qR - \mu_D) X^G.
\]

Risky asset market clearing is
\[
1 - X^G = \omega_b \left( \frac{\mu_D - qR}{\omega_b} + \frac{\sigma_T}{\sigma_D} \right).
\]

It can be solved for the equilibrium price
\[
q = \frac{\mu_D - \alpha \sigma^2_D \left( \frac{1-X^G}{\omega_b} \frac{\sigma_T}{\sigma_D} \right)}{R} \left( \frac{X^G}{\omega_b} + \right)
\]
\[
= \frac{\mu_D - \alpha \sigma^2_D \left( 1 - X^G \right)}{\omega_b} \left( 1 - \omega_b \right)
\]
\[
= \frac{\mu_D - \alpha \sigma^2_D}{\omega_b} + \frac{\alpha \sigma^2_D X^G \omega_b}{\omega_b}
\]
\[
= \mu_D - \alpha \sigma^2_D \frac{1 - X^G}{\omega_b} \left( 1 - \omega_b \right).
\]

Observe that
\[
qR - \mu_D = -\alpha \sigma^2_D \frac{1 - X^G}{\omega_b} \left( 1 - \omega_b \right).
\]
Goods market clearing in period 0 is

\[ Y_0 = \omega_a C_a^0 + \omega_b C_b^0 \]

\[ = \omega_a \frac{R}{1+R} \left[ \frac{\rho - r}{\alpha R} + Y_0 - \frac{1}{R} \left( \mu_{1\gamma} + \alpha \frac{\sigma_{1\gamma}^2}{2} \right) \right] + \omega_b \frac{R}{1+R} \left[ \frac{\rho - r}{R\alpha} + Y_0 + \frac{\alpha}{\omega_b} \left( \frac{1}{\omega_b} - \frac{\sigma_{1\gamma}}{\sigma_D^2} \right) - \frac{1}{R} \left( \mu_{1\gamma} - \frac{\sigma_{1\gamma}}{\sigma_D^2} \mu_D \right) + \frac{1}{R} \cdot \frac{(\mu_D - q)^2}{2\sigma_D^2} \right] \]

\[ = \omega_a \frac{R}{1+R} \left[ \frac{\rho - r}{R\alpha} + Y_0 - \frac{1}{R} \left( (qR - \mu_D) \alpha + \frac{\alpha}{2} \sigma_D^2 \left( X^G \right)^2 \right) \right] + \omega_b \frac{R}{1+R} \left[ Y_0 + \frac{\rho - r}{R\alpha} + \frac{q}{\omega_b} + \frac{1}{R} \cdot \frac{(\mu_D - q)^2}{2\sigma_D^2} \right], \]

\[ \frac{1}{1+R} Y_0 = \omega_a \frac{R}{1+R} \left[ \frac{\rho - r}{\alpha R} - \frac{1}{R} \left( (qR - \mu_D) \alpha + \frac{\alpha}{2} \sigma_D^2 \left( X^G \right)^2 \right) \right] + \omega_b \frac{R}{1+R} \left[ \frac{\rho - r}{R\alpha} + \frac{q}{\omega_b} + \frac{1}{R} \cdot \frac{(\mu_D - q)^2}{2\sigma_D^2} \right], \]

\[ Y_0 = \frac{\rho - r}{\alpha} - \omega_a \left[ (qR - \mu_D) \alpha + \frac{\alpha}{2} \sigma_D^2 \left( X^G \right)^2 \right] + \omega_b \left[ \frac{q}{\omega_b} + \frac{(\mu_D - q)^2}{2\sigma_D^2} \right] \]

\[ = \frac{\rho - r}{\alpha} - \omega_a \left[ -\alpha \sigma_D^2 \frac{1 - X^G (1 - \omega_b)}{\omega_b} X^G + \frac{\alpha}{2} \sigma_D^2 \left( X^G \right)^2 \right] + \omega_b \left[ \frac{\mu_D}{\omega_b} - \alpha \sigma_D^2 \frac{1 - X^G (1 - \omega_b)}{\omega_b} \right] \left[ 1 + \alpha \sigma_D^2 \left( \frac{1 - X^G (1 - \omega_b)}{\omega_b} \right)^2 \right] \]

\[ = \frac{\rho - r}{\alpha} + \omega_a \alpha \sigma_D^2 \frac{1 - X^G (1 - \omega_b)}{\omega_b} X^G - \omega_a \alpha \sigma_D^2 \left( X^G \right)^2 \]

\[ + \frac{\mu_D}{\omega_b} - \alpha \sigma_D^2 \frac{1 - X^G (1 - \omega_b)}{\omega_b} + \omega_b \alpha \sigma_D^2 \left( \frac{1 - X^G (1 - \omega_b)}{\omega_b} \right)^2 \]

\[ = \frac{\rho - r}{\alpha} + \mu_D - \left( 1 - \omega_a X^G \right) \alpha \sigma_D^2 \frac{1 - X^G (1 - \omega_b)}{\omega_b} - \omega_a \alpha \sigma_D^2 \left( X^G \right)^2 \]

\[ + \omega_b \alpha \sigma_D^2 \left( \frac{1 - X^G (1 - \omega_b)}{\omega_b} \right)^2 \]

\[ = \frac{\rho - r}{\alpha} + \mu_D - \alpha \sigma_D^2 \left( \frac{1 - \omega_a X^G}{\omega_b} \right)^2 - \omega_a \alpha \sigma_D^2 \left( X^G \right)^2 \]

\[ + \frac{\alpha \sigma_D^2 \left( 1 - \omega_a X^G \right)^2}{2 \omega_b} = \]
Goods market clearing condition in case of no restriction on trading by households $a$

\[
\frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2 \omega_b} \left[ 2 \left( 1 - \omega_a X^G \right)^2 + \omega_a \omega_b \left( X^G \right)^2 - \left( 1 - \omega_a X^G \right)^2 \right]
\]

\[
= \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2 \omega_b} \left[ \left( 1 - \omega_a X^G \right)^2 + \omega_a \left( X^G \right)^2 \right]
\]

\[
= \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2 \omega_b} \left[ 1 - 2 \omega_a X^G + \left( \omega_a X^G \right)^2 + \omega_a \left( 1 - \omega_a \right) \left( X^G \right)^2 \right]
\]

\[
= \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2 \omega_b} \left[ 1 - 2 \omega_a X^G + \omega_a \left( X^G \right)^2 \right]
\]

\[
= \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2 \omega_b} \left[ 1 - \omega_a + \omega_a - 2 \omega_a X^G + \omega_a \left( X^G \right)^2 \right]
\]

\[
= \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2 \omega_b} \left[ 1 - \omega_a + \omega_a \left( 1 - X^G \right)^2 \right]
\]

\[
= \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2 \omega_b} + \omega_a \left( 1 - X^G \right)^2
\]

\[
= \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2 \omega_b} + \frac{\omega_a}{\omega_b} \left[ 1 - \left( 1 - X^G \right)^2 \right].
\]

**No trade restrictions.** Risky assets market clearing condition

\[
1 - X^G = \left( \frac{\mu_D - q R}{\alpha \sigma_D^2} + \frac{T}{\sigma_D} \right),
\]

\[
1 - X^G = \frac{\mu_D - q R}{\alpha \sigma_D^2} - X^G,
\]

\[
q = \frac{\mu_D - \alpha \sigma_D^2}{R}.
\]

Goods market clearing condition in case of no restriction on trading by households $a$

\[
Y_0 = \omega_a C_0^a + \omega_b C_0^b
\]

\[
= \omega_a \frac{R}{1 + R} \left[ R Y_0 + R \frac{\rho - r}{\alpha} + q R \left( 0 - \frac{T_k}{\sigma_D} \right) - \left( \mu_{T,k} - \frac{T_k}{\sigma_D} \mu_D \right) + \frac{(\mu_D - q R)^2}{2 \alpha \sigma_D^2} \right]
\]

\[
+ \omega_b \frac{1}{1 + R} \left[ R Y_0 + R \frac{\rho - r}{\alpha} + q R \left( \frac{1}{\omega_b} - \frac{T_k}{\sigma_D} \right) - \left( \mu_{T,k} - \frac{T_k}{\sigma_D} \mu_D \right) + \frac{(\mu_D - q R)^2}{2 \alpha \sigma_D^2} \right]
\]

\[
= \omega_a \frac{R}{1 + R} \left[ Y_0 + R \frac{\rho - r}{\alpha} + \frac{1}{R} \cdot \frac{(\mu_D - q R)^2}{2 \alpha \sigma_D^2} \right] + \omega_b \frac{R}{1 + R} \left[ Y_0 + R \frac{\rho - r}{\alpha} + q + \frac{1}{R} \cdot \frac{(\mu_D - q R)^2}{2 \alpha \sigma_D^2} \right]
\]

\[
= \frac{R}{1 + R} \left[ Y_0 + \frac{\rho - r}{\alpha} + \frac{1}{R} \cdot \frac{(\mu_D - q R)^2}{2 \alpha \sigma_D^2} \right] + q \frac{R}{1 + R}.
\]
\[ Y_0 = \frac{\rho - r}{\alpha} + \frac{(\mu_D - qR)^2}{2\alpha\sigma_D^2} + qR \]

\[ = \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha\sigma_D^2}{2}. \]

## A.9 Proof of Lemma 8

Let’s first determine the price

\[ 1 - X^G = \omega_b \sum_{k=1}^{\infty} f(k) \left( \frac{\mu_D - qR}{\alpha\sigma_D^2} + \frac{\sigma_{T,k}}{\sigma_D} \right) \]

\[ \frac{1 - X^G}{\omega_b} = f(1) \frac{\mu_D - qR}{\alpha\sigma_D^2} + \sum_{k=2}^{\infty} f(k) \left( \frac{\mu_D - qR}{\alpha\sigma_D^2} - X^G \right), \]

\[ \frac{1 - X^G}{\omega_b} = f(1) \frac{\mu_D - qR}{\alpha\sigma_D^2} + (1 - f(1)) \left( \frac{\mu_D - qR}{\alpha\sigma_D^2} - X^G \right), \]

\[ q = \frac{\mu_D - \alpha\sigma_D^2 \left[ \frac{1 - X^G}{\omega_b} + X^G (1 - f(1)) \right]}{R} \]

\[ = \frac{\mu_D - \alpha\sigma_D^2 \left( \frac{1 - X^G}{\omega_b} + X^G \right)}{R} + \frac{\alpha\sigma_D^2 X^G f(1)}{R}. \]

Output solves goods clearing condition in period 0

\[ Y_0 = \sum_{k=1}^{\infty} f(k) \left( \omega_a \tilde{c}_{a,k} + \omega_b \tilde{c}_{b,k} \right) \]

\[ = \sum_{k=1}^{\infty} f(k) \left( \omega_a \frac{R}{1 + R} \left[ \frac{\rho - r}{\alpha R} + \frac{\alpha - 1}{\alpha R} \left( \mu_{T,k} + \frac{\alpha}{2} \sigma_{T,k}^2 \right) \right] \right. \]

\[ + \omega_b \frac{R}{1 + R} \left[ \frac{\rho - r}{\alpha R} + Y_0 + q \left( \frac{1}{\omega_b} - \frac{\sigma_{T,k}}{\sigma_D} \right) - \frac{1}{\alpha R} \left( \mu_{T,k} - \frac{\sigma_{T,k}}{\sigma_D} \mu_D \right) + \frac{1}{R} \frac{(\mu_D - qR)^2}{2\alpha\sigma_D^2} \right] \].

Rearranging we obtain

\[ Y_0 = \sum_{k=1}^{\infty} f(k) \left( \omega_a \left[ \frac{\rho - r}{\alpha} - \left( \mu_{T,k} + \frac{\alpha}{2} \sigma_{T,k}^2 \right) \right] \right. \]

\[ + \omega_b \left[ \frac{\rho - r}{\alpha} + Rq \left( \frac{1}{\omega_b} - \frac{\sigma_{T,k}}{\sigma_D} \right) - \left( \mu_{T,k} - \frac{\sigma_{T,k}}{\sigma_D} \mu_D \right) + \frac{(\mu_D - qR)^2}{2\alpha\sigma_D^2} \right) \right) \]

\[ = \frac{\rho - r}{\alpha} + \omega_b \frac{(\mu_D - qR)^2}{2\alpha\sigma_D^2} + Rq - \sum_{k=1}^{\infty} f(k) \left( \omega_a \left[ \mu_{T,k} + \frac{\alpha}{2} \sigma_{T,k}^2 \right] + \omega_b \left[ \frac{Rq}{\sigma_D} \sigma_{T,k}^2 + \mu_{T,k} - \frac{\sigma_{T,k}}{\sigma_D} \mu_D \right] \right) \]

\[ = \frac{\rho - r}{\alpha} + \omega_b \frac{(\mu_D - qR)^2}{2\alpha\sigma_D^2} + Rq - \Theta, \]
where

\[
\Theta = \sum_{k=1}^{\infty} f(k) \left( \omega_a \left[ \mu_{T,k} + \frac{\alpha}{2} \sigma^2_{T,k} \right] + \omega_b \left[ \frac{Rq}{\sigma_D} \sigma_{T,k} + \mu_{T,k} - \frac{\sigma_{T,k}}{\sigma_D} \mu_D \right] \right)
\]

\[
= f(2) \left( \omega_a \left[ \mu_{T,k} + \frac{\alpha}{2} \sigma^2_{T,k} \right] + \omega_b \left[ \frac{Rq}{\sigma_D} \sigma_{T,k} + \mu_{T,k} \right] \right)
\]

\[
+ \sum_{k=3}^{\infty} f(k) \left( \omega_a \left[ \mu_{T,k} + \frac{\alpha}{2} \sigma^2_{T,k} \right] + \omega_b \left[ \frac{Rq}{\sigma_D} \sigma_{T,k} + \mu_{T,k} \right] \right)
\]

\[
= f(2) \mu_{T,2} + f(2) \left( \omega_a \frac{\alpha}{2} \sigma^2_{T,2} - \omega_b \sigma_{T,2} \omega (Rq - \mu_D) \right)
\]

Plugging the mean and variance of expectations gives

\[
\Theta = f(2) \mu_{T,2} + f(2) \omega_a \frac{\alpha}{2} \sigma^2_{T,2} + \sum_{k=3}^{\infty} f(k) \mu_{T,k} + \sum_{k=3}^{\infty} f(k) \omega_a \frac{\alpha}{2} \sigma^2_{T,k} - \omega_b \sigma_{T,k} (Rq - \mu_D) (1 - f(1))
\]

\[
= - \omega_a^2 \frac{1 - X^G}{\omega_b} X^G (1 - \lambda) + \lambda \omega_a \frac{\alpha}{2} \left( \frac{1}{\sigma_D} X^G \right)^2 - \omega_a^2 \left( \frac{1 - X^G}{\omega_b} + X^G \right) X^G [1 - (1 - \lambda) - \lambda(1 - \lambda)]
\]

\[
- \omega_b \sigma_{T,k} (Rq - \mu_D) \lambda
\]

\[
= - \omega_a^2 \lambda X^G \left[ - \frac{1 - X^G}{\omega_b} (1 - \lambda) + \omega_a \frac{\alpha}{2} X^G - \omega_b \frac{1 - X^G}{\omega_b} + \lambda X^G \right] - \omega_b \sigma_{T,k} (Rq - \mu_D) \lambda
\]

\[
= - \omega_a^2 \lambda X^G \left[ - \frac{1}{\omega_b} + X^G \left( \frac{1}{\omega_b} + \omega_a \frac{\alpha}{2} \right) - \lambda \right] - \omega_b \sigma_{T,k} (Rq - \mu_D) \lambda
\]

\[
= - \omega_a^2 \lambda X^G \left[ - \frac{1}{\omega_b} + X^G \left( \frac{1}{\omega_b} + \omega_a \frac{\alpha}{2} \right) - \lambda \right] + \omega_b X^G \omega_a \frac{\alpha}{2} \left( \frac{1 - X^G}{\omega_b} + X^G \lambda \right)
\]

Then

\[
Y_0 = \frac{\rho - r}{\alpha} + \omega_b \left( \frac{\mu_D - qR}{2 \alpha \sigma_D^2} \right) + Rq - \Theta
\]

\[
= \frac{\rho - r}{\alpha} + \omega_b \frac{\alpha^2 \left[ \frac{1 - X^G}{\omega_b} + X^G (1 - f(1)) \right]^2}{2} + \mu_D - \omega_a^2 \left( \frac{1 - X^G}{\omega_b} + X^G (1 - f(1)) \right) - \Theta
\]

\[
= \frac{\rho - r}{\alpha} + \mu_D + \omega_b \frac{\alpha^2 \left[ \frac{1 - X^G}{\omega_b} + X^G \lambda \right]^2}{2} - \omega_a^2 \left( \frac{1 - X^G}{\omega_b} + X^G \lambda \right) - \Theta =
\]
\[
\frac{\rho - r}{\alpha} + \mu_D + \alpha \sigma_D^2 \left\{ \frac{1 - X^G}{\omega_b} + X^G \lambda \right\}^2 - \left( \frac{1 - X^G}{\omega_b} + X^G \lambda \right) \\
- \lambda X^G \left[ 1 - \frac{1}{\omega_b} + X^G \left( \frac{1}{\omega_b} - \frac{\omega_a}{2} - \lambda \left( 1 - \omega_b \right) \right) \right] \right\} \\
= \frac{\rho - r}{\alpha} + \mu_D + \frac{\alpha \sigma_D^2}{2} \left\{ \frac{1}{\omega_b} + \omega_b \left( X^G \right)^2 \left( \lambda - \frac{1}{\omega_b} \right)^2 + 2X^G \left( \lambda - \frac{1}{\omega_b} \right) - \frac{2}{\omega_b} \\
- 2X^G \left( \lambda - \frac{1}{\omega_b} \right) - 2 \left[ \lambda X^G \left( 1 - \frac{1}{\omega_b} \right) + \lambda \left( X^G \right)^2 \left( \frac{1}{\omega_b} - \frac{\omega_a}{2} - \lambda \left( 1 - \omega_b \right) \right) \right] \right\} \\
= \frac{\rho - r}{\alpha} + \mu_D + \frac{\alpha \sigma_D^2}{2} \left\{ - \frac{1}{\omega_b} + \omega_b \left( X^G \right)^2 \left( \lambda - \frac{1}{\omega_b} \right)^2 + 2X^G \left( \lambda - \frac{1}{\omega_b} \right) \\
- 2X^G \left( \lambda - \frac{1}{\omega_b} \right) - 2 \lambda X^G \left( 1 - \frac{1}{\omega_b} \right) - 2 \lambda \left( X^G \right)^2 \left( \frac{1}{\omega_b} + \frac{2}{\omega_b} - \lambda \omega_a \right) \right\} \\
= \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2 \omega_b} + \frac{\alpha \sigma_D^2}{2} X^G \left\{ X^G \left( \omega_b \lambda^2 + \frac{1}{\omega_b} - 2 \lambda - \frac{2 \lambda}{\omega_b} \omega_a - \lambda \omega_a \omega_b + 2 \lambda^2 \omega_a \right) - 2 \lambda \left( 1 - \frac{1}{\omega_b} \right) \right\} \\
= \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2 \omega_b} + \frac{\alpha \sigma_D^2}{2} X^G \left\{ \left( \lambda - \frac{1}{\omega_b} \right) - \lambda \omega_a \right\} \\
+ \frac{\alpha \sigma_D^2}{2} X^G \left\{ X^G \left( \frac{2}{\omega_b} - \lambda \right) \right\} + a \frac{\alpha \sigma_D^2}{2} X^G \left( 1 - \lambda \right)^2 \\
= \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2} \frac{\omega_a}{\omega_b} + \frac{\alpha \sigma_D^2}{2} \frac{\omega_a}{\omega_b} X^G \left( 2 - X^G \right) + \frac{\alpha \sigma_D^2}{2} X^G \left( 1 - \lambda \right)^2 \\
+ \frac{\alpha \sigma_D^2}{2} X^G \left( 1 - \lambda \right) \omega_a \left\{ \left( \frac{2}{\omega_b} - \lambda \right) X^G - \frac{2}{\omega_b} \right\} \\
= \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2} \frac{\omega_a}{\omega_b} + \frac{\alpha \sigma_D^2}{2} \frac{\omega_a}{\omega_b} X^G \left( 2 - X^G \right) + \frac{\alpha \sigma_D^2}{2} \left( X^G \right)^2 \left( 1 - \lambda \right)^2 \\
- \frac{\alpha \sigma_D^2}{2} \frac{\omega_a}{\omega_b} 2X^G \left( 1 - X^G \right) \cdot \left( 1 - \lambda \right) - \frac{\alpha \sigma_D^2}{2} \left( X^G \right)^2 \left( 1 - \lambda \right) \lambda \omega_a \\
\]

Note that

\[
Y_0 \left( \lambda \right) = \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2 \omega_b} + \frac{\alpha \sigma_D^2}{2} X^G \left( 2 - X^G \right) \frac{\omega_a}{\omega_b}, \\
Y_0 \left( \omega_b \right) = \frac{\rho - r}{\alpha} + \mu_D - \frac{\alpha \sigma_D^2}{2} + \frac{\alpha \sigma_D^2}{2} \left( X^G \right)^2 \left( 1 - \lambda \right)^2.
\]