Complex Asset Markets*

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Abstract

We develop a dynamic equilibrium model of complex asset markets with endogenous entry and exit in which the investment technology of investors with more expertise is subject to less asset-specific risk. The joint equilibrium distribution of financial expertise and wealth then determines risk bearing capacity. Higher expert demand lowers equilibrium required returns, reducing overall participation. In a dynamic industry equilibrium, investor participation in more complex asset markets with more asset-specific risk is lower, despite higher market-level Sharpe ratios, as long as asset complexity and expertise are complementary. We analyze how asset complexity affects the stationary wealth distribution of complex asset investors. Because of selection, increased asset complexity reduces expertise heterogeneity and wealth concentration, even though the wealth distribution for more expert investors has fatter tails.

Key Words: limits of arbitrage, segmented markets, slow moving capital, risky arbitrage, hedge funds, industry equilibrium, firm size distribution, financial expertise, intellectual capital, intermediary asset pricing.

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1 Introduction

Complex investment strategies, such as those employed by hedge funds and other sophisticated investors, appear to generate persistent alphas, high Sharpe ratios[1], and to feature limited participation, despite free entry. Fixed income arbitrage is a good example of a complex strategy, because its implementation requires intellectual capital (see Duarte, Longstaff, and Yu, 2006). More complex strategies within fixed income arbitrage, such as mortgage-backed security (MBS) strategies, have superior performance.[2] We develop an industry equilibrium model of the complex asset management industry that is consistent with these facts. The model generates additional testable predictions about the industrial organization of complex asset markets.

We define a complex asset as one that imposes a significant amount of non-diversifiable but idiosyncratic risk on risk-averse investors. More complex assets impose more asset-specific risk on investors holding the asset. Merton (1987) was first to point out that idiosyncratic risk will be priced when there are costs associated with learning about or hedging a specific asset. There is a growing empirical literature that documents the importance of idiosyncratic risk in complex asset strategies.[3] Complex assets impose idiosyncratic risk on investors because the acquisition and management of complex assets require a joint investment in the asset itself and in an investment technology which comprises the investor's personnel, data, hedging and risk management technologies, back office operations and trade clearing processes, relationships with dealers, and relationships with clients. We derive a specific micro-foundation for the idiosyncratic risk in complex assets based on the returns earned from long position in a fundamental asset with some mispricing or "alpha" with respect to standard models, and a short position in an imperfect tracking portfolio.

Our model economy is populated by a continuum of risk-averse agents who choose to be either non-experts who can invest only in the risk free asset, or experts who can invest in both the risk free and risky assets. On average, all expert investors in the market earn the common

[1]See Sharpe (1966).

[2]See Duarte, Longstaff, and Yu (2006) for evidence. Gabaix, Krishnamurthy, and Vigneron (2007) provide evidence that MBS returns are driven in large part by limits to arbitrage.

[3] Several papers provide evidence for the importance of idiosyncratic risk in hedge fund returns (see, e.g., Titman and Tiu, 2011; Lee and Kim, 2014). Pontiff (2006) investigates the role of idiosyncratic risk faced by arbitrageurs in a review of the literature and argues that "The literature demonstrates that idiosyncratic risk is the single largest cost faced by arbitrageurs". Greenwood (2011) states that "Arbitrageurs are specialized and must be compensated for idiosyncratic risk," and lists this first as the key friction investors in complex strategies face. To paraphrase Emanuel Derman, if you are using a model, you are short volatility, since you will lose money when your model is wrong, (see Derman, 2016).

equilibrium return that clears the market, but their returns are subject to asset-specific (or strategy-specific) shocks. Expertise shrinks the asset-specific volatility of the complex asset return, and, as a result, more expert investors earn a higher Sharpe ratio. Thus, expertise may be interpreted as the ability to implement complex strategies better either by developing a superior model or information technology, hiring better employees, or by gathering superior information.

In our model, risk is asset-specific and idiosyncratic risk is priced.[4] Funds cannot be reallocated across individual risk-averse investors. Since the risk in our economy is idiosyncratic, pooling this risk would eliminate the risk premium that experts require to hold it. For incentive reasons, asset managers cannot hedge their own exposure to their particular portfolio.[5] This motivates why we endow expert investors in our model with CRRA preferences, but we do not model the principal-agent relation between the outside investors and asset managers.

We characterize the equilibrium mapping from the endogenous joint distribution of expertise and financial wealth to complex asset prices. Importantly, in our dynamic equilibrium model, the stationary distribution for wealth conditional on expertise is endogenous. Market clearing returns must compensate participating investors for the asset-specific risk they face, but less expert investors may not be adequately compensated, because equilibrium wealth-weighted demand from higher expertise investors depresses required returns. In other words, expert demand and risk bearing capacity act as a barrier to entry below a threshold level of expertise.

More complex assets impose more idiosyncratic risk on investors and earn higher equilibrium returns, but the compensation per unit of risk may either increase or decrease as complexity increases. In our model, market-level Sharpe ratios, which aggregate the individual Sharpe ratios of market participants, increase with asset complexity only if expertise and complexity are complementary, meaning that a marginal increase in idiosyncratic risk increases the rents from expertise. To clear the market at the higher risk level, excess returns must increase. However, if expertise and complexity are complementary, as complexity and therefore risk increase, participation declines as inexpert investors are driven out. The selection effect–the exclusion of lower expertise investors–attenuates the negative effect of increased risk on the market-level Sharpe ratio.

[4] This is, of course, a useful assumption technically. We could, equivalently, assume that all investors have the same exposure to systematic risk, applying the results in Krueger and Lustig (2009).

[5] In fact, Panageas and Westerfield (2009) and Drechsler (2014) provide important results for the portfolio choice of hedge fund managers who earn fees based on assets under management and portfolio performance. In particular, they show that these managers behave like constant relative risk aversion investors. These results extend the analysis of the impact of high-water marks in Goetzmann, Ingersoll, and Ross (2003).

Our model is an example of an "industry equilibrium" model in the spirit of Hopenhayn (1992a,b).[6] This literature focuses in large part on explaining firm growth, and moments describing the firm size distribution. Recent progress in the firm dynamics literature using continuous time techniques to solve for policy functions and stationary distributions include Miao (2005); Luttmer (2007); Gourio and Roys (2014); Moll (2014); Achdou, Han, Lasry, Lions, and Moll (2014). We draw on results in these papers as well as discrete time models of firm dynamics, as in recent work by Clementi and Palazzo (2016), which emphasizes the role of selection in explaining the observed relationships between firm age, size, and productivity. We are the first to use a model in this class to study the size or wealth distribution of financial intermediaries.

In our model, expertise varies in the cross-section but is fixed for each agent over time. This allows us to solve our model analytically, including the joint stationary wealth and expertise distribution, in closed form, up to the equilibrium fixed point for expected returns.[7] The joint distribution is determined by the deep parameters which describe preferences, endowments, and technologies. We analyze the stationary joint distribution of wealth and expertise to examine the relation between size and skill in complex asset markets. The stationary wealth distribution of participants is Pareto conditional on each expertise level. Because investors with higher expertise choose a higher exposure to the risky asset, both the drift and the volatility of their wealth will be greater, leading to a fatter tailed distribution at higher expertise levels. However, our model also predicts, under natural conditions, that more complex asset markets will have less concentrated wealth distributions. This is because, in equilibrium, selection reduces expertise heterogeneity, and more so for more complex assets when expertise and complexity are complementary. We provide evidence for this ancillary prediction using data from Hedge Fund Research on size distributions across different strategies.

Recently, dynamic heterogeneous agent models have also been used to study wealth distributions in the consumer sector. We build on work by Benhabib, Bisin, and Zhu (2011, 2015, 2016),[8] which study the wealth distribution of consumers subject to idiosyncratic shocks to capital and/or labor income risk. These papers contain fundamental results describing necessary conditions for fat-tailed or Pareto wealth distributions, as well as interesting positive results about the importance of capital income shocks and taxation schemes in shaping ob-

[6]Such models are typically used to study the role of firm dynamics, entry, and exit in determining equilibrium prices in an environment which builds on the heterogeneous agent framework developed in Bewley (1986).

[7]We use a numerical algorithm to solve the market clearing fixed point problem. However, the solution is straightforward given our analytical solution for policy functions and distribution over individual states.

[8]See also Kaplan, Moll, and Violante (2016) for a model with a portfolio choice over liquid and illiquid assets.

served wealth inequality.[9] A key distinction of our work is that we clear the market for the risky asset, and solve the resulting fixed point problem determining equilibrium excess returns and market volatility, consistent with our interpretation of the complex risky asset market we study. Another distinction is that we study a participation decision, which plays a key role in our model's equilibrium price and allocation outcomes. Our paper also introduces heterogeneity in technologies, which leads to endogenous variation in the drift and volatility of wealth across agents.[10] Finally, another distinction is that we draw on the techniques used in Gabaix (1999), who studies city size distributions in a model in which relative sizes follow a reflecting, or regulated, geometric Brownian motion. Because the stationary distribution in our model depends on the equilibrium fixed point excess return on the risky asset, it is more convenient for us to employ a reflecting barrier for relative wealth levels, rather than Poisson elimination as in the Benhabib, Bisin, Zhu (and many other) papers.

In terms of its asset pricing implications, our paper contributes to a large and growing literature on segmented markets and asset pricing. Relative to the existing literature, we provide a model with endogenous entry, a continuous distribution of heterogeneous expertise, and a rich distribution of expert wealth that is determined in stationary equilibrium. Thus, we have segmented markets, but allow for a participation choice. Our market has limited risk bearing capacity, determined in part by expert wealth, but in addition to the amount of wealth, the efficiency of the wealth distribution also matters for asset pricing.

We group this existing literature into three main categories, namely investor heterogeneity, financial constraints and limits to arbitrage, and segmented market models with alternative micro-foundations to agency theory. There is closely related work on heterogeneity in trading technologies and risk aversion (see, e.g., Dumas, 1989; Basak and Cuoco, 1998; Kogan and Uppal, 2001; Chien, Cole, and Lustig, 2011, 2012).

Our study shares the goal of understanding the returns to complex assets and strategies, and the features of segmented markets and trading frictions, with the literature on limits of arbitrage. Gromb and Vayanos (2010a) provide a recent survey of the theoretical literature on limits to arbitrage, starting with the early work by Brennan and Schwartz (1990) and Shleifer and Vishny (1997).[11] Shleifer and Vishny (1997) emphasize that arbitrage is conducted by a

[9]See Gabaix (2009) for a review of the empirical evidence and theoretical foundations for power law distributions in economics and finance.

[10]Gabaix, Lasry, Lions, and Moll (2016) also emphasize the importance of heterogeneity in income growth rates in generating realistic inequality. They do not feature a portfolio choice, or endogenous variation in growth rates, and thus our study complements theirs.

[11]See also Aiyagari and Gertler (1999); Froot and O'Connell (1999); Basak and Croitoru (2000); Xiong (2001);

fraction of investors with specialized knowledge, but similar to He and Krishnamurthy (2012), they focus on the effects of the agency frictions between arbitrageurs and their capital providers. Although we do not explicitly model risks to the liability side of investors' balance sheets, one can interpret the shocks agents in our model face to include idiosyncratic redemptions. The asset pricing impact of financially constrained intermediaries has been studied in the literature on intermediary asset pricing following He and Krishnamurthy (2012, 2013) (see also, for example, Adrian and Boyarchenko, 2013). For empirical applications, see for example, Adrian, Etula, and Muir (2014), and Muir (2014).

Finally, several papers develop alternative micro-foundations to agency theory for segmented markets. Allen and Gale (2005) provides an overview of their theory of asset pricing based on "cash-in-the-market". Plantin (2009) develops a model of learning by holding. Duffie and Strulovici (2012) develop a theory of capital mobility and asset pricing using search foundations. Glode, Green, and Lowery (2012) study asset price dynamics in a model of financial expertise as an arms race in the presence of adverse selection. Kurlat (2016) studies an economy with adverse selection in which buyers vary in their ability to evaluate the quality of assets on the market, and, like us, emphasizes the distribution of expertise on the equilibrium price of the asset. Grleanu, Panageas, and Yu (2015) derive market segmentation endogenously from differences in participation costs. Edmond and Weill (2012), Haddad (2014), and DiTella (2016) study the effects of idiosyncratic risk from concentrated holdings on asset prices. Kacperczyk, Nosal, and Stevens (2014) construct a model of consumer wealth inequality from differences in investor sophistication.

The paper proceeds as follows. Section 2 contains the construction and analysis of our dynamic model, and finally Section 6 concludes. Most proofs appear in the Appendix.

2 Model

According to its general definition, α cannot be generated by bearing systematic risk. However, capturing α is risky. Complex assets expose their owners to idiosyncratic risk through several channels. First, any investment in a complex asset requires a joint investment in the front and back office infrastructure necessary to implement the strategy. Second, their constituents tend

Gromb and Vayanos (2002); Yuan (2005); Gabaix, Krishnamurthy, and Vigneron (2007); Mitchell, Pedersen, and Pulvino (2007); Acharya, Shin, and Yorulmazer (2009); Kondor (2009); Duffie (2010); Gromb and Vayanos (2010b); Hombert and Thesmar (2014); Mitchell and Pulvino (2012); Pasquariello (2014); Kondor and Vayanos (2014).

to be significantly heterogeneous, so that no two investors hold exactly the same asset. Third, the risk management of complex assets typically requires a hedging strategy that will be subject to the individual technological constraints of the investor. Hedging portfolios, to cite just one example, tend to vary substantially across different investors in the same asset class.[12] Fourth, firms which manage complex assets may be exposed to key person risk due to the importance of specialized traders, risk managers, and marketers.[13]

2.1 Preferences, Endowments, & Technologies

We study a model with a continuum of investors of measure one, with CRRA utility functions over consumption:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

Investment Technology Investors are endowed with a level of expertise which varies in the cross section, but is fixed for each agent over time. Each individual investor is born with a fixed expertise level, x, drawn from a distribution with pdf $\lambda(x)$, and a continuous cdf $\Lambda(x)$. Investors can choose to be experts, and have access to the complex risky asset, or non-experts, who can only invest in the risk free asset.

Each investor's complex risky asset delivers a stochastic return which follows a geometric Brownian motion:

$$dR_i(t,s) = [r_f + \alpha(s)]\,ds + \sigma(\sigma_\nu, x)\,dB_i(t,s) \tag{1}$$

where $\alpha(s)$ is the common excess return on the risky asset and $B_i(t,s)$ is a standard Brownian motion which is investor-specific and i.i.d. in the cross section. Hereafter, we suppress the dependence of the Brownian shock on investor i in our notation for parsimony. The effective volatility of the risky technology, $\sigma(\sigma_\nu, x)$ is a function of the volatility of the complex asset before expertise is applied, and the specific investor's level of expertise, x. We use the notation σ_ν and the language "total volatility" to describe the overall level of volatility of a complex

[12]For example, in MBS, there is no agreed upon method to hedge mortgage duration risk, though most all active investors do so. Some hedge according to empirical durations, using various estimation periods and rebalancing periods. Others hedge according to the sensitivity of MBS prices yield curve shifts using their own (widely varying) proprietary model of MBS prepayments and prices.

[13]Finally, complex assets may introduce or amplify idiosyncratic risk on the liability side of the balance sheet, through the fact that they are difficult for outside investors to understand, but tend to be funded with external finance. Broadly interpreted, these risks may come either from the asset side, or from the liability side, since funding stability likely varies with expertise. However, we abstract from the micro-foundations of risks from the liability side of funds' balance sheets, and model risk on the asset side.

asset class, before expertise is applied. "Effective volatility" is, then, a function of the asset's "total" volatility, and the investor's level of expertise. For now, we focus on describing the equilibrium for a single asset, and we suppress the positive dependence of $\sigma(x)$ on the total volatility of the asset class σ_ν. Below, we describe comparative statics across assets with varying complexity, with more complex assets characterized by a higher σ_ν, or "total volatility". We refer to $\sigma(x)$ as "effective volatility", meaning the remaining volatility the investor faces after expertise has been applied. In particular, we assume that the volatility of the risky technology $\sigma(x)$ decreases in the investor's level of expertise x, i.e. $\frac{\partial \sigma(x)}{\partial x} < 0$. For convenience, we assume that the support of expertise is bounded above by $\bar{x}$, although most of our results only require that $\lim_{x \to \infty} \sigma(x) = \underline{\sigma} > 0$. The implied lower bound on volatility, $\underline{\sigma}$, represents complex asset risk that cannot be eliminated even by the agents with the greatest expertise, and it guarantees that the growth rate of wealth is finite.

A simple motivation for the return process in Equation (1) is that in order to invest in the risky asset and to earn the common market clearing return, an investor must also jointly invest in a technology with a zero mean return and an idiosyncratic shock. This technology represents each investor's specific hedging and financing technologies, as well as the unique features of their particular asset. We also provide a specific formal micro-foundation for Equation (1) in Lemma 2.1. Investors take a long position in an underlying asset with some alpha or mis-pricing relative to its systematic risk exposure, and a short position in an imperfectly correlated, investor-specific, tracking portfolio. The long-short position is designed to "harvest the alpha" in the underlying asset while hedging out unnecessary aggregate risk exposures to the agent's best ability. It can also be interpreted as a relative value strategy aimed at over and under weighting different parts of the cross section in order to capture misvaluation while attempting to hedge out market level risk, as is common practice in quantitative equity and fixed income strategies. Investors with more expertise have superior tracking portfolios which hedge out more of the risk in the underlying asset. As a result, their total net position is less risky and they earn the common market clearing alpha while bearing less risk. An additional contribution of our paper is to provide a precise explanation for the idiosyncratic risk that the prior literature has argued is important for understanding complex asset returns.[14]

[14]We thank Peter Kondor for providing a closely related static micro-foundation in his discussion of our paper. See Pontiff (1996), Pontiff (2006), and Greenwood (2011) for evidence of idiosyncratic risk as a limit to arbitrage. Consistent with our model and micro-foundation of complex asset returns, Pontiff (1996) finds that closed end mutual funds with portfolios that are more difficult to replicate have values that deviate more from fundamentals.

Lemma 2.1 *Each investor's asset-specific return process, given by (1), can be derived as the net return on a strategy in which the investor takes a long position in an underlying fundamental asset, and a short position in a tracking portfolio. The returns to the fundamental asset are given by:*

$$\frac{dF(t,s)}{F(t,s)} = [r_f + \alpha(s) + a(s)]\,dt + \sigma^F dB^F(t,s). \tag{2}$$

The returns to each investor's best per-unit tracking portfolio is given by:

$$\frac{dT_i(t,s)}{T_i(t,s)} = a(s)dt + \rho_i(x)\sigma^F dB^F(t,s) - \sigma^T(x)dB_i^T(t,s), \tag{3}$$

where $dB^F(t,s)$ and $dB_i^T(t,s)$ are independent Brownian motions, and $\rho_i(x)$ represents the dependence of the investor's tracking portfolio returns on the fundamental risk in the underlying asset. The correlation between the investor's tracking portfolio return and underlying asset return is given by

$$corr\left(\frac{dT_i}{T_i}, \frac{dF}{F}\right) = \frac{\rho_i(x)\sigma^F}{\sqrt{\left(\left(1-\rho_i(x)\right)\sigma^F\right)^2 + \left(\sigma^T(x)\right)^2}},$$

which is increasing in $\rho_i(x)$. We assume that $\frac{\partial|\rho_i(x)-1|}{\partial x} < 0$, so that higher expertise investors have access to tracking portfolios which hedge out more fundamental risk. The net asset returns are given by:

$$dR_i(t,s) = \frac{dF(t,s)}{F(t,s)} - \frac{dT_i(t,s)}{T_i(t,s)} = [r_f + \alpha(s)]\,dt + \sigma(x)dB_i(t,s), \tag{4}$$

where

$$\sigma(x)dB_i(t,s) \equiv \left(1-\rho_i(x)\right)\sigma^F dB^F(t,s) + \sigma^T(x)dB_i^T(t,s). \tag{5}$$

The return process in Equations (4)-(5) generates a single process for expertise level wealth dynamics in which aggregate risk washes out, and all investors with expertise level x face the same amount of effective risk, under the sufficient condition in Assumption 1:

Assumption 1 *At each level of expertise, half of investors over-hedge (type o with $\rho_o(x) > 1$), and half under-hedge (type u with $\rho_u(x) < 1$). That is,*

$$\frac{\rho_o(x) + \rho_u(x)}{2} = 1.$$

We omit the proof, which is given in the calculations provided. Lemma A.1, which we write formally and prove in the Appendix, states that, given the micro-foundation for the return process in Equation (1) provided in Lemma 2.1, aggregate shocks do not affect equilibrium policy functions or prices, and therefore the return dynamics Equation (1) are taken as given in the remainder of the main text.

Note that, by definition, if the tracking portfolio returns are not perfectly correlated with the underlying asset returns (in which case there would exist a risk-less arbitrage opportunity), then the tracking portfolio introduces risk which is independent from fundamental risk. We assume this independent risk is uncorrelated across investors. Because each investor has their own model and strategy implementation, tracking portfolios introduce investor-specific shocks. Formally, we use the fact that any Brownian shock which is partially correlated with the underlying fundamental Brownian shock $dB^F(t,s)$ can be decomposed into a linear combination of a correlated shock and an independent shock. We denote this independent, investor-specific shock $dB_i^*(t,s)$. The amount of idiosyncratic risk the tracking portfolio introduces is smaller the closer to one is $\rho_i(x)$, which is intuitive. We assume that agents with higher expertise have better tracking portfolios which eliminate more risk from the underlying fundamental shock. Across asset classes, more complex assets are characterized by more imperfect models and tracking portfolios, and hence highly complex assets impose more total risk σ_ν on investors. This higher total risk can result from overall lower quality tracking portfolios (lower $\rho_i(x)$). Higher total risk can also result from the greater model and execution risk of highly complex assets, which can be considered part of $\sigma^T(x)$.

To be an expert, an investor must pay the entry cost F_{nx} to set up their specific technology for investing in the complex risky asset. Experts must also pay a maintenance cost, F_{xx} to maintain continued access to the risky technology. We specify that both the entry and maintenance costs are proportional to wealth:

$$\begin{aligned} F_{nx} &= f_{nx} w, \\ F_{xx} &= f_{xx} w, \end{aligned}$$

which yields value functions which are homogeneous in wealth.

Optimization We first describe the Bellman equations for non-experts and experts respectively, and characterize their value functions, as well as the associated optimal policy functions.

With the value functions of experts and non-experts in hand, we then characterize the entry decision.

We begin with non-experts, who can only invest in the risk free asset. Let $w(t,s)$ denote the wealth of investors at time s with initial wealth W_t at time t. The riskless asset delivers a fixed return of r_f. All investors choose consumption, and an optimal stopping, or entry time according to the Bellman Equation:

$$\begin{aligned} V^n(w(t,s),x) &= \max_{c^n(t,s),\tau} \mathbb{E}\left[\int_t^\tau e^{-\rho(s-t)} u(c^n(t,s))\,ds + e^{-\rho(\tau-t)} V^x(w(t,s)-F_{nx},x)\right] \quad (6) \\ \text{s.t. } dw(t,s) &= (r_f w(t,s) - c^n(t,s))\,ds \quad (7) \end{aligned}$$

where ρ is their subjective discount factor, $c^n(t,s)$ is consumption at time s, F_{nx} is the entry cost, and τ is the optimal entry date.

Under the assumptions of linear entry and maintenance costs, and expertise which is fixed over time, the optimal entry date in a stationary equilibrium will be either immediately, or never. Thus, assuming an initial stationary equilibrium, investors who choose an infinite stopping time are then non-experts, and investors who choose a stopping time $\tau = t$ are experts.[15]

Experts allocate their wealth between current consumption, a risky asset, and a riskless asset. They also choose an optimal stopping time T to exit the market.

$$\begin{aligned} V^x(w(t,s),x) &= \max_{c^x(x,t,s),T,\theta(x,t,s)} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)} u(c^x(x,t,s))\,ds + e^{-\rho(T-t)} V^n(w(t,s),x)\right] \quad (8) \\ \text{s.t. } dw(t,s) &= [w(t,s)(r_f + \theta(x,t,s)\alpha(t,s)) - c^x(x,t,s) - F_{xx}]\,ds \quad (9) \\ &\quad + w(t,s)\theta(x,t,s)\sigma(x)\,dB(t,s), \end{aligned}$$

where $\alpha(s)$ is the equilibrium excess return on the risky asset, $\theta(x,t,s)$ is the portfolio allocation to the risky asset by investors with expertise level x at time s, $c^x(x,t,s)$ is consumption, F_{xx} is the maintenance cost. We include exit for completeness. However, exit will not occur in this homogeneous model with fixed expertise.

The following proposition states the analytical solutions for the value and policy functions in our model. We prove this Proposition by guess and verify in the Appendix.

[15]Outside of a stationary equilibrium, because α is not constant, both entry and exit are possible.

Proposition 2.1 ***Value and Policy Functions:*** *The value functions are given by*

$$V^x\left(w\left(t,s\right),x\right) \;=\; y^x\left(x,t,s\right)\frac{w\left(t,s\right)^{1-\gamma}}{1-\gamma} \tag{10}$$

$$V^n\left(w\left(t,s\right),x\right) \;=\; y^n\left(x,t,s\right)\frac{w\left(t,s\right)^{1-\gamma}}{1-\gamma} \tag{11}$$

where $y^x\left(x\right)$ *and* $y^n\left(x\right)$ *are given by:*

$$y^x\left(x\right) \;=\; \left[\frac{\left(\gamma-1\right)\left(r_f-f_{xx}\right)+\rho}{\gamma}+\frac{\left(\gamma-1\right)\alpha^2}{2\gamma^2\sigma^2\left(x\right)}\right]^{-\gamma} \;\; and \tag{12}$$

$$y^n\left(x\right) \;=\; \left[\frac{\left(\gamma-1\right)r_f+\rho}{\gamma}\right]^{-\gamma}. \tag{13}$$

The optimal policy functions $c^x\left(x,t,s\right), c^n\left(t,s\right),$ *and* $\theta\left(x\right)$ *are given by:*

$$c^x\left(x,t,s\right) \;=\; \left[y^x\left(x\right)\right]^{-\frac{1}{\gamma}}w\left(t,s\right), \tag{14}$$

$$c^n\left(t,s\right) \;=\; \left[y^n\left(x\right)\right]^{-\frac{1}{\gamma}}w\left(t,s\right) \;\; and \tag{15}$$

$$\theta\left(x,t,s\right) \;=\; \frac{\alpha(t,s)}{\gamma\sigma^2\left(x\right)}. \tag{16}$$

Furthermore, the wealth of experts evolves according to the law of motion:

$$\frac{dw\left(t,s\right)}{w\left(t,s\right)}=\left(\frac{r_f-f_{xx}-\rho}{\gamma}+\frac{\left(\gamma+1\right)\alpha^2\left(t,s\right)}{2\gamma^2\sigma^2\left(x\right)}\right)dt+\frac{\alpha\left(t,s\right)}{\gamma\sigma\left(x\right)}dB\left(t,s\right) \tag{17}$$

Finally, investors choose to be experts if the excess return earned per unit of wealth exceeds the maintenance cost per unit of wealth:

$$\frac{\alpha^2\left(t,s\right)}{2\sigma^2\left(x\right)\gamma}\geq f_{xx}. \tag{18}$$

We define $\underline{x}$ as the lowest level of expertise amongst participating investors, for which Equation (18) holds with equality. Note that the law of motion for wealth is a sort of weighted average of the return to the risky and riskless assets, as determined by portfolio choice, net of consumption. The drift and volatility of investors' wealth are increasing in the allocation to the risky asset. This mechanism has important implications for the wealth distribution in the stationary equilibrium of our model.

2.2 The Distribution(s) of Expert Wealth

The total amount of wealth allocated to the complex risky asset, as well as the distribution of expert wealth across expertise levels, are key aggregate state variables for the the first and second moments of the equilibrium returns to the complex risky asset. Once the participation decision has been made, given that we do not clear the market for the riskless asset, the wealth of non-experts is irrelevant for the returns to the complex risky asset. We solve for the cross sectional distribution of expert wealth in a stationary equilibrium of our model. Given that expertise is fixed over time for each investor, constructing the wealth distribution at each expertise level is sufficient to obtain the cross-sectional joint distribution of wealth and expertise.

First, we note that in order to construct a stationary equilibrium given that experts' wealth on average grows over time, it is convenient to study the ratio $z\left(t,s\right)$ of individual wealth to the mean wealth of agents with highest expertise, $\mathbb{E}\left[w|_{\bar{x}}\left(t,s\right)\right]$.

$$z\left(t,s\right)\equiv\frac{w\left(t,s\right)}{\mathbb{E}\left[w|_{\bar{x}}\left(t,s\right)\right]}.$$

We can replace wealth w with wealth share z in the value and policy functions without consequence, due to the homogeneity described in Proposition 2.1. Next, note that the law of motion for the growth rate of mean wealth of agents with a given level of expertise x is given by

$$\frac{d\mathbb{E}[w|_{x}\left(t,s\right)]}{\mathbb{E}[w|_{x}\left(t,s\right)]}\equiv\left[g\left(x\right)\right]dt.$$

where $g\left(x\right)$ is determined in equilibrium. Define the average growth rate amongst agents with the "highest" level of expertise as $g(\bar{x})\equiv\sup_{x}g(x)$. Then, the ratio $z\left(t,s\right)$ follows a geometric Brownian motion given by

$$\frac{dz\left(t,s\right)}{z\left(t,s\right)}=\left(\frac{r_{f}-f_{xx}-\rho}{\gamma}+\frac{\left(\gamma+1\right)\alpha^{2}\left(t,s\right)}{2\gamma^{2}\sigma^{2}\left(x\right)}-g\left(\bar{x}\right)\right)dt+\frac{\alpha\left(t,s\right)}{\gamma\sigma\left(x\right)}dB\left(t,s\right), \qquad (19)$$

where $\frac{r_{f}-f_{xx}-\rho}{\gamma}+\frac{(\gamma+1)\alpha^{2}(t)}{2\gamma^{2}\sigma^{2}(x)}-g\left(\bar{x}\right)<0$ represents the negative drift, or growth rate.

Let the cross-sectional p.d.f. of expert investors' wealth and expertise at time t be denoted by $\phi^{x}\left(z,x,t\right)$. Without additional assumptions, the relative wealth of lower expertise agents will shrink to zero. Two methods are commonly used to generate a stationary distribution. The first, for example used in Benhabib, Bisin, and Zhu (2016), is to employ a life cycle model, or Poisson elimination of agents. The second, employed by Gabaix (1999), is to introduce a reflective or regulating barrier at a minimum relative size.[16] We impose a reflective, or regulating, barrier at a minimum wealth share, $z_{\min}$, which is relative to the average wealth of agents with the highest level of expertise. This reflective barrier implies that the growth rate of any individual agent not at the barrier, even those with the highest level of expertise, will grow more slowly than the mean wealth of the highest expertise agents. We adopt the assumption of a minimum wealth share because it leads to a more elegant expression for the wealth distribution. In particular, wealth shares conditional on expertise are characterized by a simple Pareto distribution. At the same time, this assumption conveniently avoids the need to solve the fixed point problem determining the initial conditions for wealth needed to generate a Pareto distribution with Poisson death.[17] Since the reflective boundary mainly affects low wealth investors, decisions near the boundary matter little for equilibrium pricing, so elegance and parsimony comes at a low cost. Importantly, using reflecting dynamics, we are able to construct stationary distributions that can be easily studied analytically.

We adopt an interpretation of possible punishment vs. rescue at $z_{\min}$ which ensures that policies are not distorted there. Then, since both time and state variables are continuous in our model, if policies are not distorted at $z_{\min}$, then they will not be distorted elsewhere. The strategy we employ is to ensure that the value at $z_{\min}$ from adopting the optimal policy functions under Brownian wealth share dynamics is equal to the value of adopting those policies given that with some probability the investor will be punished by being forced to exit, and with some probability the investor will be rewarded by being able to remain in the market and maintain a wealth share of $z_{\min}$. In the case of exit, we assume the investor is replaced by a new entrant with wealth share $z_{\min}$ and the same level of expertise x as the exiting agent.

Proposition 2.2 *The value and policy functions under reflecting, or regulated, Brownian dynamics for wealth shares, given by:*

$$dz_t/z_t = \mu_z dt + \sigma_z dB_t \text{ for } z_t > z_{\min}$$

[16]See Harrison (2013), who also notes that the "reflected geometric Brownian motion process" might more precisely be called a regulated Brownian motion. Gabaix (1999) constructs a model of the city size distribution, and thus his share variable represents relative population shares. See also the Appendix of that paper for a related method of constructing a stationary distribution using a Kesten (1973) process, which introduces a random shock with a positive mean to normalized city size.

[17]Adopting the assumption of Poisson death with a fixed initial wealth, for example, would instead lead to a double Pareto distribution, with a cutoff at the initial value of wealth. For example, see Benhabib, Bisin, and Zhu (2016) for the wealth distribution under the alternative assumption of Poisson elimination in a closely related model. Formulas for wealth distributions with Poisson death are available upon request.

and

$$dz_t/z_t = \max\left(\mu_z dt + \sigma_z dB_t, 0\right) for z_t \leq z_{\min},$$

with μ_z and σ_z defined in Equation (19), are given by the solution to the alternative model in:

$$\begin{aligned} V^r\left(z\left(t,s\right),x\right) &= \max_{c^r(x,t,s),T,\theta^r(x,t,s)} \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u\left(c^r\left(x,t,s\right)\right) ds \right. \\ & \left. + e^{-\rho(s'-t)}\left[\left(1-q\right)V^r\left(z_{\min},x\right) + qV^n\left(z\left(t,s\right),x\right)\right]\right] \\ s.t.\ dz\left(t,s\right) &= \left[z\left(t,s\right)\left(r_f + \theta^r\left(x,t,s\right)\alpha\left(t,s\right)\right) - c^r\left(x,t,s\right) - F_{xx}\right] ds \\ & + z\left(t,s\right)\theta^r\left(x,t,s\right)\sigma\left(x\right) dB\left(t,s\right), \end{aligned}$$

where $s' < T$ is the first time that an agent's wealth falls below $z_{\min}$. The value and policy functions for this alternative model are equivalent to those under the true Brownian dynamics in the model of Equations (8) - (9), for the appropriately defined probability, q.

The proof, along with the definition of the punishment probabilities are given in the Appendix.[18]

With this result in hand, we derive the stationary distributions for wealth shares. The Kolmogorov forward equations describing the evolution of the wealth distributions, conditional on expertise level x and taking $\alpha(t)$ as given, can be stated as follows:[19]

$$\begin{aligned} \partial_t \phi^x\left(z,x,t\right) = &-\partial_z\left[\left(\left(r_f + \theta\left(x,t\right)\alpha\left(t,s\right)\right) - \left[y^x\left(x\right)\right]^{-\frac{1}{\gamma}} - f_{xx} - g\left(\bar{x}\right)\right) z\phi^x\left(z,x,t\right)\right] \\ &+ \frac{1}{2}\partial_{zz}\left(\left[z\theta\left(x,t\right)\sigma(x)\right]^2 \phi^x\left(z,x,t\right)\right) \\ &= -\partial_z\left[\left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\left(\gamma+1\right)\alpha^2\left(t,s\right)}{2\gamma^2\sigma^2\left(x\right)} - g\left(\bar{x}\right)\right) z\phi^x\left(z,x,t\right)\right] \\ &+ \frac{1}{2}\partial_{zz}\left[\left(z\frac{\alpha\left(t,s\right)}{\gamma\sigma\left(x\right)}\right)^2 \phi^x\left(z,x,t\right)\right]. \end{aligned} \tag{20}$$

We then study the stationary distribution of wealth shares, in which $\partial_t \phi^x\left(z,x,t\right) = 0\ \forall x$. We take as given, for now, that $\alpha(t,s)$ will be constant, as in the stationary equilibrium we define in the following section. This will be true given a stationary distribution over investors' individual state variables. A stationary distribution of wealth shares $\phi^x\left(z,x\right)$ satisfies the

[18]Our proof should enable more researchers to use reflecting dynamics to ensure stationarity in models with endogenous policies, especially if analytically tractable Pareto distributions are desired. In the model of Gabaix (1999), which introduces the method of using reflecting dynamics to generate a stationary Pareto distribution, cities do not choose size. This is in contrast to models such as ours with endogenous state variables.

[19]See Dixit and Pindyck (1994) for a heuristic derivation, or Karlin and Taylor (1981) for more detail.

following equation:

$$0 = -\partial_z \left[\left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\,\alpha^2}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right) z\phi^x(z,x) \right] + \frac{1}{2}\partial_{zz}\left[\left(z\frac{\alpha}{\gamma\sigma(x)} \right)^2 \phi^x(z,x) \right]. \tag{21}$$

We use guess and verify to show that the stationary distribution of wealth shares at each level of expertise is given by a Pareto distribution with an expertise-specific tail parameter. This tail parameter, which we denote by β, is determined by the drift and volatility of the expertise-specific law of motion for wealth shares. Intuitively, the larger the drift and volatility of the expertise-specific wealth process, the fatter the tail of the wealth distribution at that level of expertise will be.

Proposition 2.3 *The stationary distribution of wealth shares $\phi^x(z,x)$ has the following form:*

$$\phi(z,x) \propto C(x) z^{-\beta(x)-1},$$

where

$$\begin{aligned}
\beta(x) &= C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma \geq 1, \\
C_1 &= 2\gamma\left(f_{xx} + \rho - r_f + \gamma g(\bar{x})\right), \\
C(x) &= \frac{1}{\int z^{-\beta(x)} dz} = \frac{C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma}{z_{\min}^{-C_1 \frac{\sigma^2(x)}{\alpha^2} + \gamma}}.
\end{aligned}$$

See the Appendix for the Proof. Note that, in this proof, we also show that, in the stationary distribution, $\beta > 1$, which ensures a finite integral, and confirms that the distribution satisfies stationarity. The following Corollary solves for the tail parameter of the highest expertise agents, as well as the average growth rate amongst these agents. Given this growth rate, we can write the tail parameter for any level of expertise as a function of the coefficient of relative risk aversion, the minimum wealth share, and the ratio the effective variance relative to the effective variance for the highest expertise agents. This is convenient, because it clearly shows the dependence of the thickness of the right tail of the relative wealth distribution on expertise.

Corollary 2.1 *For the highest expertise agents, we have*

$$\beta\left(\bar{x}\right)=\frac{1}{1-z_{\min}/\bar{z}}=C_1\frac{\sigma^2\left(\bar{x}\right)}{\alpha^2}-\gamma$$

where $\bar{z}$ is mean of normalized wealth of experts with highest expertise,

$$\bar{z}=\int_{z_{\min}}^{\infty}z\phi(z,\bar{x})dz=z_{\min}\left[1+\frac{1}{\beta\left(\bar{x}\right)-1}\right]$$

and

$$g\left(\bar{x}\right)=\frac{r_f-f_{xx}-\rho}{\gamma}+\frac{\alpha^2}{2\gamma\sigma^2\left(\bar{x}\right)}+\frac{\alpha^2}{2\gamma^2\sigma^2\left(\bar{x}\right)}\frac{1}{1-z_{\min}/\bar{z}}$$

For all other expertise levels, we have

$$\beta\left(x\right)=\left(\gamma+\frac{z_{\min}/\bar{z}}{1-z_{\min}/\bar{z}}\right)\frac{\sigma^2\left(x\right)}{\sigma^2\left(\bar{x}\right)}-\gamma.$$

The parameter $\beta(x)$ controls the rate of decay in the tail of each expertise-specific wealth distribution. We now have two expressions for this parameter:

$$\beta(x)=2\gamma\left(f_{xx}+\rho-r_f+\gamma g\left(\bar{x}\right)\right)\frac{\sigma^2\left(x\right)}{\alpha^2}-\gamma, \tag{22}$$

from Proposition 2.3 and

$$\beta\left(x\right)=\left(\gamma+\frac{z_{\min}/\bar{z}}{1-z_{\min}/\bar{z}}\right)\frac{\sigma^2\left(x\right)}{\sigma^2\left(\bar{x}\right)}-\gamma. \tag{23}$$

from Corollary 2.1. The smaller is β, the more slowly the wealth share distribution decays, and the fatter is the upper tail. Both equations clearly show that $\beta(x)$ an increasing function of expertise level volatility, $\sigma\left(x\right)$, and thus a decreasing function of expertise, x. That is, the wealth distribution of experts with a higher level of fixed expertise has a fatter right tail. Investors with higher expertise allocate more wealth to the risky asset, which increases the mean and volatility of their wealth growth rate. Both a higher drift, and a wider distribution of shocks, lead to a fatter upper tail for wealth. Moreover, Equation (23), in which the dependence of the tail parameter on expertise is given by $\frac{\sigma^2(x)}{\sigma^2(\bar{x})}$, shows that if the relation between expertise and effective volatility is steeper, then the difference in the size of the right tails of the wealth distribution across expertise levels increases. In equilibrium, variation in effective volatilities in complex asset markets will be driven both by the functional form for effective volatility, and by participation decisions which determine how different effective volatilities of participating agents can be given equilibrium pricing. Finally, we note that the degree of wealth inequality within each expertise level can be measured by $\frac{1}{2\beta(x)-1}$, which is the Gini coefficient for each expertise level wealth distribution, representing twice the area between the Lorenz curve and the equidistribution line. High expertise levels exhibit greater size "inequality". Note that if the relation between expertise and effective volatility is steeper, then the difference in size inequality across expertise levels increases.

It is intuitive that investing more in the risky asset leads to a fatter tailed wealth distribution. However, perhaps surprisingly, as Lemma 2.2 illustrates, not every parameter which increases the difference in the fraction of wealth allocated to the risky asset leads to an increase in the degree of fat tails of the expertise-specific wealth distributions. We show in Lemma 2.2 that, while differences in portfolio choice driven by differences in effective volatilities lead to greater differences in decay parameters, this is not true for variation in portfolio choice driven by higher excess returns or lower risk aversion. This result offers a unique prediction for our model of complexity as differences in risk vs. risk aversion. See the Appendix for the proof.

Lemma 2.2 ***Relation Between*** $\theta(x)$ ***and*** $\beta(x)$

Consider two level of expertise, $x_H > x_L$*. We have that:*

$$\theta(x_H) - \theta(x_L) = \frac{\alpha}{\gamma}\frac{\sigma^2(x_L) - \sigma^2(x_H)}{\sigma^2(x_H)\,\sigma^2(x_L)},$$

and

$$\beta(x_H) - \beta(x_L) = 2\gamma^2\left(f_{xx} + r - r_f + \gamma g(\bar{x})\right)\frac{\sigma^2(x_H)\,\sigma^2(x_L)}{\alpha^3}\left[\theta(x_L) - \theta(x_H)\right].$$

If a larger difference in portfolio choice is due to either a higher excess return or a lower risk aversion, the dispersion in β *is smaller. If it is due to an increase in the difference in effective volatilities, then the difference in* β*'s is larger.*

2.3 Aggregation and Stationary Equilibrium

The equilibrium market clearing α is determined by equating supply and demand:

$$S(t) = \int \lambda(x)\,\theta(x,t)\,W(x,t)\,dx.$$

In order to ensure that the supply of the complex risky asset does not become negligible as investor wealth grows, we assume that the supply grows proportionally to the mean wealth of the highest expertise investors. That is, we assume:

$$\frac{dS\left(t\right)}{S} = g\left(\bar{x}\right)dt.$$

For convenience, we assume that the support of expertise is bounded above by $\bar{x}$, although most of our results only require that $\sigma(x)$ satisfies $\lim_{x\to\infty}\sigma(x) = \underline{\sigma} > 0$.

We define a stationary equilibrium for detrended economies. We define detrended aggregate investment in the complex risky asset to be I, defined as:

$$I \equiv \int \lambda\left(x\right) I\left(x\right) dx, \tag{24}$$

where $I\left(x\right)$ is the detrended total expertise level investment in the complex risky asset, namely,

$$I\left(x\right) = \frac{\alpha}{\gamma\sigma^{2}\left(x\right)} Z\left(x\right), \tag{25}$$

where $Z\left(x\right)$ is the total expertise level wealth share,

$$Z\left(x\right) = z_{\min}\left(1 + \frac{1}{\beta(x) - 1}\right).$$

This is the well-known expression for the mean of a Pareto distribution.[20] The condition which determines the market clearing α in a stationary equilibrium of a detrended economy equates detrended investment to detrended asset supply.

Definition 2.1 *A stationary equilibrium consists of a market clearing α, policy functions for all investors, and a stationary distribution over investor types $i \in \{x, n\}$, expertise levels x, and wealth shares z, $\phi(i, z, x, t)$, such that given an initial wealth distribution, an expertise distribution $\lambda(x)$, and parameters describing risk aversion, investors' discount rate, the risk free rate, entry and maintenance costs, total volatility, minimum wealth shares, and detrended risky asset supply, $\{\gamma, \rho, r_f, f_{nx}, f_{xx}, \sigma_\nu, z_{\min},$ and $S\}$, the economy satisfies:*

1. *Investor optimality: Investors choose participation in the complex risky asset market according to Equation (18), as well as optimal consumption and portfolio choices*

[20] For a simple derivation, see the proof of Corollary 2.1.

$\{c^n(t), c^x(x,t), \theta(x,t)\}_{t=0}^{\infty}$ *according to Equations (14)-(16), such that their utilities are maximized.*

2. *Market clearing: In a stationary equilibrium, we have:*

$$I \equiv \int \lambda(x) I(x) dx = S, \tag{26}$$

3. *The distribution over all individual state variables is stationary, i.e.* $\partial_t \phi(i, z, x, t) = 0$.

3 Results

3.1 Analytical Asset Pricing Results

With policy functions, stationary distributions, and the equilibrium definition in hand, we turn to our asset pricing results. We define a more complex asset as one that introduces more idiosyncratic risk. Comparing across assets, we use σ_ν to denote the total volatility of the asset before expertise is applied, so that the risk in each investor's asset is $\sigma(\sigma_\nu, x)$, and is increasing in the first argument, and decreasing in the second. We provide specific examples below, but begin with any general function satisfying these two properties. Importantly, we describe conditions under which more complex assets, or assets which introduce more idiosyncratic risk, have lower participation despite higher α's and higher Sharpe ratios. A key requirement is complementarity of expertise and complexity, meaning that the higher risk of more complex assets more negatively impacts investors with lower expertise.

We begin by studying comparative statics over the equilibrium market clearing α. Although we focus on comparative statics over total volatility, we also provide results for the market clearing α for changes other parameters which might proxy for asset complexity, such as the cost of maintaining expertise, or investor risk aversion. Next, we analyze individual Sharpe ratios. We emphasize across-investor heterogeneity in changes in the risk return tradeoff as total volatility changes. Because other parameters which might also vary with complexity, such as γ or f_{xx}, do not change investor-specific volatility, the results for individual Sharpe ratios are the same as those for α. Finally, we study market level Sharpe ratios, with a focus on the effects of changes in total volatility on the intensive and extensive margins of participation by investors with heterogeneous expertise.

Investor Demand, Aggregate Demand, and Equilibrium α We first describe the comparative statics for demand conditional on investors' expertise levels in Lemma 3.1.

Lemma 3.1 *Using Equation (25) for investor demand conditional on expertise, x, we have following comparative statics, $\forall x$:*

1. $\frac{\partial I(x)}{\partial \sigma^2(x)} < 0$
2. $\frac{\partial I(x)}{\partial \sigma_\nu} < 0$
3. $\frac{\partial I(x)}{\partial \alpha} > 0$
4. $\frac{\partial I(x)}{\partial \gamma} < 0$
5. $\frac{\partial I(x)}{\partial f_{xx}} < 0$.

Demand for the risky asset at each level of expertise is increasing in the squared investor-specific Sharpe ratio, and it is increasing in α. Demand is decreasing in effective variance, total volatility, risk aversion, and the maintenance cost.

With expertise level total demands in hand, we can construct comparative statics for aggregate demand. We cannot express the equilibrium excess return in closed form. However, the following Proposition shows that the equilibrium excess return, α, and aggregate demand, I, form a bijection. This uniqueness result in turn ensures that α can easily be solved for numerically as the unique fixed point to Equation (26).

Proposition 3.1 *Aggregate market demand for the complex risky asset is an increasing function of the excess return, α, and α and I form a bijection. Mathematically,*

$$\frac{\partial I}{\partial \alpha} > 0.$$

Proposition 3.2 provides comparative statics over the aggregate demand for the complex risky asset, I. Using the result in Proposition 3.1, these comparative statics also hold for α.

Proposition 3.2 *Using the market clearing condition, we have that the following comparative statics hold for aggregate investment in the complex risky asset in partial equilibrium:*

1. $\frac{\partial I}{\partial \sigma_\nu} < 0$, *thus α is an increasing function of total risk*

2. $\frac{\partial I}{\partial \gamma} < 0$, *thus* α *is an increasing function of risk aversion*

3. $\frac{\partial I}{\partial f_{xx}} < 0$, *thus* α *is an increasing function of the maintenance cost.*

In partial equilibrium, demand for the risky asset is decreasing in total volatility, risk aversion, and the maintenance cost. As a result, in general equilibrium α is increasing in total volatility, risk aversion, and the maintenance cost. An increase in these parameters proxies for greater asset complexity, and thus that our model predicts that α will be higher in more complex asset markets.

We now turn to the effect of the efficiency of the joint distribution of wealth and expertise on equilibrium pricing. In particular, we demonstrate that the equilibrium required excess return on the complex risky asset is decreasing in the amount of wealth commanded by agents with higher levels of expertise. The proof appears in the Appendix.

Proposition 3.3 *If* $\frac{\partial \sigma(x)}{\partial x} < 0$, *and* Λ_1 *exhibits first-order stochastic dominance over* Λ_2, $I(\Lambda_1) \geq I(\Lambda_2)$. *As a result* $\alpha(\Lambda_1) < \alpha(\Lambda_2)$.

The wealth distribution at each expertise level is a Pareto distribution with an expertise-specific tail parameter. By shifting the distribution of expertise rightward, leading to a new distribution with a relatively larger fraction of higher expertise investors, relatively more wealth will reside with agents with higher expertise. Thus, with any rightward shift, the joint distribution of wealth and expertise becomes more efficient. Moreover, because the wealth distribution at higher expertise levels exhibits fatter right tails, there is an additional direct effect on overall wealth from a rightward shift in the distribution of expertise. Accordingly, Proposition 3.3 shows that if the density of experts shifts to the right, then demand for the complex risky asset will increase, and the required equilibrium excess return will decrease. The equilibrium excess return is decreasing in the amount of wealth which resides in the hands of agents with higher expertise. Note that this result can also be interpreted to state that in asset markets in which higher levels of expertise are more widespread, or less rare, equilibrium required returns will be lower. We argue that the scarcity of relevant expertise is increasing with asset complexity, again implying a higher α in more complex markets.

Investor-specific Sharpe ratios, Investor Participation, and Market-level Sharpe ratios With the analysis of equilibrium excess returns in hand, we now turn to the equilibrium risk-return tradeoff at the investor and market-level as described by the investor-specific, and market-level Sharpe ratios. We emphasize the variation across individual Sharpe ratios as a function of expertise; all investors face a common market clearing α, but their effective risk varies. For the market-level Sharpe ratio, two effects are present. First, there is the effect of any changes on parameters on the individual Sharpe ratios of participants. Second, there is a selection effect, or the effect on participation. We provide an intuitively appealing condition, complementarity between expertise and complexity, under which participation declines as the asset becomes more complex. We focus on the equally weighted market-level equilibrium Sharpe ratio in our analysis. In addition to offering cleaner comparative statics because it does not depend on investor portfolio choices and market shares, the equally weighted Sharpe ratio represents the expected Sharpe ratio that an investor who could pay a cost to draw from the expertise distribution above the entry cutoff would earn. In that sense, it is the "expected Sharpe ratio". Note that the Sharpe ratio for non-experts is not defined.

Investor-specific Sharpe ratios: We define the investor-specific Sharpe Ratio as:

$$SR(x) = \frac{\alpha}{\sigma(x)}.$$

We provide results for how investor-specific Sharpe ratios change as total volatility changes under the three possible cases for the elasticity of investor-specific risk with respect to total volatility in Proposition 3.4. The sign of this elasticity is a key determinant of our Sharpe ratio results.

Proposition 3.4 *The comparative statics for the investor-specific Sharpe ratios with respect to total volatility depend on whether expertise and complexity display complementarity. Specifically, the results depend on which of the following three possible cases for the elasticity of investor-specific risk with respect to total volatility, applies:*

1. *Case 1, Constant Elasticity: If* $\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}$ *is a constant, that is*

$$\frac{\partial \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}}{\partial x} = 0,$$

we must have that $SR(x)$ *is either an increasing or a decreasing function of total risk for all expertise levels.*

2. *Case 2, Increasing Elasticity: If* $\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}$ *is an increasing function of expertise, that is*

$$\frac{\partial\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}}{\partial x} > 0,$$

then there is a cutoff level x^**, such that for all* $x < x^*$*, we have* $\frac{\partial SR(x)}{\partial\sigma_\nu} > 0$*; and for all* $x > x^*$*, we have* $\frac{\partial SR(x)}{\partial\sigma_\nu} < 0$*. Further,* x^* *exists if for any small* $\varepsilon < 10^{-6}$

$$(0,\varepsilon) \subseteq \left\{\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} | \text{ for all } x\right\} \subseteq [0,\infty).$$

3. *Case 3 Decreasing Elasticity (complementarity): If* $\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}$ *is a decreasing function of expertise, that is*

$$\frac{\partial\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}}{\partial x} < 0,$$

then there is a cutoff level x^**, such that for all* $x < x^*$*, we have* $\frac{\partial SR(x)}{\partial\sigma_\nu} < 0$*; and for all* $x > x^*$*, we have* $\frac{\partial SR(x)}{\partial\sigma_\nu} > 0$.

Since individual Sharpe ratios are ratios of excess returns relative to effective volatilities, variation in Sharpe ratios as total volatility changes depend on the elasticity of excess returns relative to the elasticities of effective volatilities as total volatility changes. This is intuitive, as these elasticities measure the equilibrium percentage change in excess returns vs. the percentage change in effective volatilities as underlying total risk varies. The change in α is aggregate, the same for all investors. So, the changes in individual Sharpe ratios with respect to changes in total volatility depend on the expertise-specific percentage changes in effective volatility relative to the percentage change in total volatility. Proposition 3.4 demonstrates that the effect of an increase in total volatility on individual Sharpe ratios varies in the cross section, except in the special case of Case 1. If the elasticity of effective volatilities as total volatilities changes is the same for all investors (Case 1), then the percentage change in α relative to the percentage change in effective volatility is the same for all investors. On the other hand, if the elasticity of effective volatilities with respect to total volatility is increasing in expertise (Case 2), then Sharpe ratios increase below a cutoff level of expertise and decrease above as total volatility increases. Finally, if this elasticity is declining in expertise, so that higher expertise investors face smaller increases in effective volatility as total volatility increases –expertise and complexity are complementary–, (Case 3), then Sharpe ratios increase above a cutoff level of expertise and decrease below. In this case, the economic rents from expertise are higher in more complex markets, characterized by higher total volatility.

We focus our analysis on Case 3, because it leads to the empirically plausible implication that more complex assets, with higher total volatilities, have *lower* participation despite having persistently elevated excess returns. Thus, we argue that the decreasing elasticity case is the most relevant for describing a long-run, stationary equilibrium in a complex asset market. Moreover, it seems intuitive that the difference in effective volatilities between more and less complex assets would be smaller for higher expertise investors.

As a concrete example, consider the following two fixed income arbitrage strategies considered by Duarte, Longstaff, and Yu (2006). Mortgages are highly complex securities containing embedded prepayment options. MBS payoffs are affected by consumer behavior, house prices, and credit conditions, as well as interest rates. There is no agreed upon pricing model, and investors' strategy implementations vary widely as a result. Recent work by Boyarchenko, Fuster, and Lucca (2014) emphasizes the role of prepayment model risk in explaining the "smile" in option adjusted spreads, extending the early work by Gabaix, Krishnamurthy, and Vigneron (2007). By contrast, swap spread arbitrage follows a fairly straightforward long-short rule based on current LIBOR swap rates relative to Treasury yields and repo rates. The way this strategy is implemented is quite similar across investors. Accordingly, Duarte, Longstaff, and Yu (2006) show that mortgage related strategies (MBS) earn higher alphas, and Sharpe ratios, than simple swap spread arbitrage strategies. We argue, in agreement with their motivation and findings, that expertise is more valuable in MBS arbitrage. Put another way, the difference in the risk which investors face in MBS vs. Treasuries is decreasing in investor expertise. To be sure, highly sophisticated investors face more risk in MBS than in treasuries. However, the difference in effective risk across these two fixed income strategies is not as great for expert investors as it is for an inexpert investor, consistent with Case 3 in Proposition 3.4.

Investor Participation Before analyzing market-level Sharpe ratios, we first describe investor participation. There are two key inputs into the market-level risk return tradeoff. First, incumbents' individual Sharpe ratios change. Second, as equilibrium α changes, participation also changes. This selection effect plays a key role in determining comparative static results in general equilibrium. We show in the Appendix that participation increases with total volatility in Cases 1 and 2 of Proposition 3.4. If all elasticities of $\sigma(x)$ with respect to σ_ν are the same, or if they are lower for lower expertise investors, then participation will increase with total volatility. This is intuitive because α must increase with total volatility σ_ν by enough for lower expertise investors to help to clear the market. Because it is intuitive that participation should decrease with asset complexity, we thus focus on results under Case 3 of Proposition 3.4. For this case, we provide a natural condition under which participation declines as the asset becomes more complex and total volatility increases.

Proposition 3.5 *Define the entry cutoff $\underline{x}$ as in Equation (18). Participation declines*

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} > 0$$

if the following conditions hold:

1. $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} < 0$, *(complementarity of complexity and expertise, Case 3 of Proposition 3.4) and*

2. $l_{\text{sup}}^{\sigma_\nu} > \left(1 + \frac{1}{1+\mathcal{B}(\underline{x})}\right) E\left[\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} | x \geq \underline{x}\right]$,

   where $l_{\text{sup}}^{\sigma_\nu}$ is defined to be the highest elasticity of all participating investors' effective volatility with respect to total volatility, and

$$\mathcal{B}(x) = \frac{2}{\beta(x)} \frac{\beta(x) + \gamma}{\beta(x) - 1}.$$

The first condition, namely that the elasticity of effective volatility with respect to total volatility is decreasing in expertise, is necessary for participation to decline as complexity, and total volatility, increase. To see this, consider the market clearing condition. For the market to clear without the participation by, and demand from, lower expertise investors as total volatility increases, it must be that higher expertise investors' demand is less adversely affected. The second condition gives a sufficient condition which states that the elasticity of the agent with the highest sensitivity of effective volatility to total volatility, which in Case 3 of Proposition 3.4 will be the lowest expertise agent who participates, must be sufficiently different from the average across participants, times a constant. Note that the constant will be near one if β is close to one, which it will be as it is the tail parameter from a Pareto distribution. The term $\mathcal{B}(\underline{x})$ gives the elasticity of the mean wealth at expertise level x with respect to total volatility. Intuitively, what is necessary for participation to decline as total volatility increases is that there is enough variation in the effect of the change in total volatility across agents with high and low expertise so that α does not need to increase enough to satisfy the marginal investor or to entice lower expertise investors to participate.

In sum of the results so far, under the conditions for complementarity between expertise and complexity in Proposition 3.5, our model generates higher persistent α's and lower participation, despite free entry, as total volatility and asset complexity increase. Thus, we argue that more complex assets, in addition to exposing investors to more risk overall, pose a larger difference in risk across investors with different levels of expertise.

Equilibrium market-level Sharpe Ratio We define the equally weighted market equilibrium Sharpe ratio as:

$$SR^{ew} = E\left[\frac{\alpha}{\sigma(x)} | x \geq \underline{x}\right].$$

We focus on comparative statics for the equally weighted market equilibrium Sharpe ratio, however we note the following Corollary holds for the value weighted market Sharpe ratio: [21]

Corollary 3.1 *In Case 3 of Proposition 3.4, in which expertise and complexity are complementary, and in which participation declines with complexity, we have that an increase in the equally weighted Sharpe ratio as total volatility increases implies an increase in the value weighted Sharpe ratio.*

We omit the formal proof, which is straightforward. Intuitively, the value weighted Sharpe ratio weights higher expertise investors' Sharpe ratios more heavily, since they are wealthier and they allocate a greater fraction of wealth to the risky asset. Because the value weighted Sharpe ratio is a more complex equilibrium object, we focus on the equally weighted Sharpe ratio, SR^{ew} to develop our results. The following Proposition provides conditions under which SR^{ew} increases with total volatility σ_ν.

Proposition 3.6 *The equally weighted market Sharpe ratio is increasing with total risk in general equilibrium, i.e.,*

$$\frac{\partial SR^{ew}}{\partial \sigma_\nu} > 0,$$

if:

[21] See the Appendix for the definition of the value-weighted market equilibrium Sharpe ratio.

1. $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} \leq 0$, *(Cases 1 and 3 of Proposition 3.4) and*

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} \underbrace{\frac{\lambda(\underline{x})}{1-\Lambda(\underline{x})}}_{\text{fraction of threshold experts}} \overbrace{\left(1 - \frac{1}{E\left[\frac{\sigma(\underline{x})}{\sigma(x)} | x \geq \underline{x}\right]}\right)}^{\text{difference btw marginal vol and avg}} > \underbrace{-\frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \underline{x}}}_{\text{sensitivity of threshold vol w.r.t. expertise}} \frac{\partial \underline{x}}{\partial \sigma_\nu} \tag{27}$$

or

2. $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} > 0$, *(Case 2 of Proposition 3.4) and*

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} \underbrace{\frac{\lambda(\underline{x})}{1-\Lambda(\underline{x})}}_{\text{fraction of threshold experts}} \overbrace{\left(1 - \frac{1}{E\left[\frac{\sigma(\underline{x})}{\sigma(x)} | x \geq \underline{x}\right]}\right)}^{\text{difference btw marginal vol and avg}} >> \underbrace{-\frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \underline{x}}}_{\text{sensitivity of threshold vol w.r.t. expertise}} \frac{\partial \underline{x}}{\partial \sigma_\nu} \tag{28}$$

Consider Equation (27) for Case 3 of Proposition 3.4, in which $\frac{\partial \underline{x}}{\partial \sigma_\nu} > 0$. The first term on the right hand side describes how much the effective volatility of the marginal agent changes with expertise. In Case 3, this term will be negative before applying the negative sign in the equation; effective volatilities decline with expertise. Thus, the condition in Equation (27) states that if the effective volatility of the marginal agent changes substantially as $\underline{x}$ changes, then the right hand side will be larger, and it is positive. Then, on the left hand side, the term involving the conditional pdf of expertise gives the fraction of experts which reside at the threshold for entry. The next term on the left hand side measures the difference between the elasticity of the marginal agents' effective volatilities, and the average elasticity. This term is close to its maximum value of one when the second term in parentheses is near zero, which occurs when the elasticity of the marginal agent's effective volatility is very different from the average. Thus, the left hand side indicates that if there are many agents at the threshold when participation decreases, these agents must be very different from the average agent in order for the equally weighted market Sharpe ratio to increase. This is because marginal agents display the highest elasticity of effective volatility and hence have the most negatively impacted individual Sharpe ratios. For the overall Sharpe ratio to increase, the effect on their Sharpe ratio cannot be representative. Then, what is required for the condition to be satisfied is that the more agents there are at the participation threshold, the more different these marginal agents must be from the average agent. The condition is easier to satisfy if the marginal agent's effective volatility on the right hand side does not increase too much.

The above conditions give us the sufficient conditions for deep parameters and functions which yield an increasing Sharpe ratio when total volatility increases. Equivalently, we can rewrite the Equation (27) as the following:[22]

$$\underbrace{-\frac{\partial \log\left[1-\Lambda\left(\underline{x}\right)\right]}{\partial \log \sigma_{\nu}}}_{\text{elasticity of participation}} \overbrace{\left(1-\frac{SR\left(\underline{x}\right)}{SR^{ew}}\right)}^{\text{difference btw marginal SR and average}} > \underbrace{-\frac{\partial \log SR\left(\underline{x}\right)}{\partial \log \sigma_{\nu}}}_{\text{elasticity of marginal SR}} \tag{29}$$

Equation (29) has three terms. On the left hand side, the first term represents the elasticity of participation with respect to total volatility. When participation is increasing, $\frac{\partial \log[1-\Lambda(\underline{x})]}{\partial \log \sigma_{\nu}}$ is positive, and vice versa. The second term is the relative value of the marginal Sharpe ratio to the population weighted average. On the right hand side is the elasticity of the marginal Sharpe ratio with respect to total volatility. When participation increases, the new entrants increase the overall density of experts with lower Sharpe ratios, which has a negative effect on the equally weighted market-level Sharpe ratio. The general equilibrium effect from a higher α increases the average Sharpe ratio of incumbents, which has a positive effect on the equally weighted market-level sharpe ratio. The condition in Equation (29) states that, when participation increases, the change in the measure of experts has to be smaller than the change in the marginal Sharpe ratio. Conversely, when participation decreases, we have that the change in the measure of experts has to be larger than the change in the marginal Sharpe ratio. If there are many investors around the entry threshold, the value of the marginal Sharpe ratio is closer to the population weighted average. When participation declines (increases), we require that the decrease in the Sharpe ratio of these investors is not so large (small) that the market Sharpe ratio is overwhelmed by the change in participation of investors around the threshold. Intuitively, this condition is easier to satisfy if: First, participation is very sensitive to total volatility, because then many low Sharpe ratio agents will drop out of the market. Second, if the difference between the marginal and average Sharpe ratios is large, because then the lower Sharpe ratios of the marginal agents are not representative, and finally if the marginal Sharpe ratio is not too sensitive to total volatility.

In summary, we have shown that for the equally weighted market Sharpe ratio to increase with total volatility while participation declines, the model requires complementarity of com-

[22]See the proof of Proposition 3.6 in the Appendix for the derivation.

plexity and expertise, namely, that agents with less expertise are more sensitive to increases in volatility. Thus, under the conditions in Proposition 3.6, our model delivers a rational explanation for why more complex assets can generate a higher α, a higher equally-weighted equilibrium market Sharpe ratio, but have low participation, despite free entry. Intuitively, as in a standard industrial organization model, the superior volatility reduction technologies of more expert investors provide them with an excess of (risk-bearing) capacity, which serves to reduce the entry incentives of newcomers despite attractive conditions for incumbents.

4 Extension: Two Complex Assets

In this section, we describe how the intuition from comparative statics across stationary equilibria can be applied to a single economy with multiple complex assets. In particular, we consider an economy with two risky assets with different total volatilities given by σ_ν^L and σ_ν^H, where $\sigma_\nu^H > \sigma_\nu^L$. We assume that agents can only invest in one risky asset; they must specialize to take advantage of their expertise. The following Proposition gives a condition for which higher expertise agents specialize in the more complex risky asset, with total volatility σ_ν^H.[23]

Proposition 4.1 *In a two complex asset economy, higher expertise agents choose to specialize and invest in the more complex asset, with total volatility σ_ν^H if and only if*

$$\frac{\partial \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}}{\partial x} < 0.$$

The proof appears in the Appendix. The intuition behind this result is very similar to the intuition for the single asset economy results for the variation in individual Sharpe ratios as total volatility varies. The condition is the same as Condition 3 for declining elasticities, or complementarity between complexity and expertise, in Proposition 3.4. Under this condition, for higher expertise agents, the market clearing α will adequately compensate them for the higher total risk. Lower expertise agents are not adequately compensated for the higher risk of the more complex asset. Note that this necessarily implies that the α of the more complex asset will be higher, because otherwise the high expertise agents would choose the less complex asset.

[23] The analysis can be extended to multiple risky assets, and we omit the conditions under which agents with higher expertise will invest in σ_ν^L.

5 Numerical Examples and Concentration Results

This section presents numerical results and comparative statics. We provide additional intuition for our analytical results, and develop new results for wealth concentration across assets with different complexity. We focus on a functional form for effective volatility which satisfies the condition in Case 3 from Proposition 3.4, in which the elasticity of effective volatility with respect to total volatility declines with expertise.[24] Specifically, and notationally reintroducing the dependence of $\sigma(x)$ on total volatility σ_ν, we specify that:

$$\sigma^2(\sigma_\nu, x) = a + x^{-b}\sigma_\nu^2. \tag{30}$$

This function is increasing in total volatility at a rate that decreases with expertise, i.e. it satisfies:

$$\frac{\partial \sigma(\sigma_\nu, x)}{\partial \sigma_\nu} > 0,$$

and,

$$\frac{\partial \frac{\partial \sigma(\sigma_\nu, x)/\sigma(\sigma_\nu, x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} < 0.$$

Numerical Comparative Statics: Asset Pricing and Participation We begin with simple, illustrative numerical comparative statics for our main asset pricing and participation results. The model generates closed form policy functions and wealth distributions conditional on expertise levels. To provide intuition for the effects of equilibrium pricing, we provide the comparative statics in both partial equilibrium and general equilibrium. In partial equilibrium, the excess return is given exogenously, and held fixed, while aggregate demand (and hence implicitly supply) varies. In general equilibrium, the excess return is computed endogenously given a fixed supply of the risky asset. Because α and I form a bijection (Proposition 3.1 provides conditions for which they are one to one and onto), for any given supply of the complex risky asset, we can solve for the market equilibrium α in the following steps:

1. Choose an upper and a lower bound for α, namely α_1 and α_2, ($\alpha_1 > \alpha_2$).

[24] Results for the other cases are available upon request.

2. Let $\alpha = \frac{\alpha_1+\alpha_2}{2}$, and compute the total demand for the risky asset

$$\int \lambda(x)\, I(x)\, dx$$

3. If $S - \int \lambda(x)\, I(x)\, dx < -10^{-4}$, let $\alpha_1 = \alpha$ and back to step 1; if $S - \int \lambda(x)\, I(x)\, dx > 10^{-4}$, let $\alpha_2 = \alpha$ and back to step 1; otherwise, STOP.

Our baseline parameters are summarized in Table 1. The time interval is one quarter. The risk-free rate, subjective discount factor, and the maintenance cost are all set to 1%. The coefficient of relative risk-aversion is 5. The total standard deviation of the risky asset return is 40%. We set $a = 0.48\%$ and $b = 1.5$. This implies that the highest expertise investors can eliminate 83% of total risk, and face an effective standard deviation of 6.8%. And the marginal expertise investors at entry can eliminate 45% of total risk, and face an effective standard deviation of 22.1%, which is about half that of total volatility. The minimum wealth share is set to 0.01.

The log-normal distribution of expertise has a mean of 0 and volatility of 4. We choose 500 grid points for expertise levels between the lowest and highest levels of expertise. The lowest expertise level is pinned down by Equation (18) for participation. We choose a value of 10,000 for the highest expertise level. The value ensures that the policy functions for the most expert investors very closely approximate the limiting case of infinite expertise, and that there is no mass point at the upper bound. The density of expertise on the grid points is then chosen to approximate a truncated log-normal distribution.

Figure 1 studies the effects of changes in fundamental volatility, with more complex assets characterized by higher fundamental volatility. Starting in the top row, as fundamental volatility increases, demand for the risky asset in partial equilibrium decreases, implying a higher α in general equilibrium. The left hand side of the second row displays the entry cut-off, which is increasing in fundamental volatility, consistent with our result in Proposition 3.5. Accordingly, participation, graphed on the right hand side of the second row, declines. We note that participation declines by less in general equilibrium, due to the positive effect of fundamental volatility on α, but still the decline is nearly as large as in partial equilibrium given our parametric assumptions. Finally, the third row plots the equally weighted standard deviation of the risky asset returns, which are increasing in both partial and general equilibrium. In partial equilibrium, α is held constant, so participation declines substantially as total volatility increases. The selection effect then significantly attenuates the increase in the equally weighted effective volatility. By contrast, in general equilibrium the increase in α compensates for the increase in total risk somewhat, participation declines by less, and so the equally weighted effective volatility increases by more than in partial equilibrium. Finally, the bottom right panel of Figure 1 shows that despite the fact that the equally weighted standard deviation is increasing, the larger, positive effect of the increase in α in general equilibrium implies that the equally weighted Sharpe ratio increases, consistent with Proposition 3.6. Thus, the numerical example confirms the model's ability to generate persistently higher α's and larger Sharpe ratios, but lower participation despite free entry, for more complex assets characterized by higher fundamental volatility, consistent with the empirical findings in Pontiff (1996) and Duarte, Longstaff, and Yu (2006).

Numerical Comparative Statics: Wealth Distributions Finally, we present results on the size distribution of funds in our model, and in the data, across asset classes which are more and less complex. While we rely on the empirical results in Duarte, Longstaff, and Yu (2006) for average returns and Sharpe ratios of complex asset strategies, we must use primary data to study the size distribution of complex asset investors. While in the model it is straightforward to define a complex asset as one with a higher total volatility, total volatility (before expertise is applied) is unobservable in the data. Therefore, we use the implication of our model that equally weighted Sharpe ratios are higher in more complex asset classes. We use the subset of the Hedge Fund Research (HFR) data which describes Relative Value fund performance for single strategy funds, namely the asset-backed fixed income, sovereign fixed income, convertible bond arbitrage, volatility, and corporate fixed income relative value strategies.[25] This choice is motivated by two main considerations. First, we build on the results in Duarte, Longstaff, and Yu (2006), who show that excess returns and Sharpe ratios are higher for more complex fixed income arbitrage strategies. Second, relative value funds should be comprised of long-short strategies, consistent with the micro-foundation we provide for complex asset investment strategies. Moreover, long-short strategies are designed to be market neutral and thus average returns are excess returns. For each strategy, we compute the "pseudo" Sharpe ratio as the ratio of the average industry level return to the cross section average of the fund level standard deviations of returns. We use the industry level return because in our model there is one common market clearing alpha. We use the ratio of the industry level return to the average fund level standard deviation, rather than an average of fund level Sharpe ratios, as a parsimonious way of eliminating the effect of

[25]We apply one filter, requiring funds to have assets greater than ten million, which is standard.

outliers. We then rank strategies from most to least complex by these pseudo Sharpe ratios, and provide summary statistics in Table 2. The last column of Table 2 gives the Gini coefficient describing the concentration of wealth within each strategy. We compute the Gini coefficient using the wealth share by wealth deciles. These Gini coefficients appear to be higher for less complex strategies, which display lower pseudo Sharpe ratios. This might seem surprising given that we have argued that there is a natural force toward wealth concentration in more complex strategies, if expertise and complexity are complementary, due to the portfolio choice effect described in Lemma 2.2. While the portfolio choice effect is important, the effect of limited participation tends to dominate in equilibrium. Equilibrium asset prices limit the amount of heterogeneity amongst participants, driving lower expertise investors out, especially in more complex asset classes. As a result, investors who participate in the most complex asset classes, with higher participation cutoffs, are not as different from each other as those in less complex asset markets, which drives the equilibrium wealth concentration down on net.

Table 3 and Figure 6 present a comparison of empirical and model generated wealth distributions across strategies. For this comparison, we use the pseudo Sharpe ratio to rank strategies in the data. We use the average Sharpe ratio and Gini coefficient over all strategies to describe a complex asset with medium complexity. The asset pricing moments for this average, or medium, complexity asset motivate our baseline calibration in Table 1, and are given in the columns labeled "model" in Table 3. For the high complexity strategy, in the top row of Table 3, we target the average Sharpe ratio for the two highest pseudo Sharpe ratio strategies. Similarly, we target the average Sharpe rato for the lowest pseudo Sharpe ratio strategies for the least complex strategy in the bottom row. We vary total volatility, σ_ν, between 0.04 and 1.4, holding all other parameters in Equation 30 constant, to match these empirical average pseudo Sharpe ratios. The implied marginal volatility, which can be computed using the entry condition described in Equation (18), is 35.6% in the high complexity strategy, 22.08% in the medium complexity strategy, and 11.73% in the low complexity strategy.

In Figure 6, the top left panel plots the Lorenz curves, or average cumulative wealth shares by wealth decile for these three strategy groupings. As can be seen, wealth concentration increases as the average Sharpe ratio decreases. The top right panel compares the concentration of wealth in the data, and in the model for the baseline parameters, which match the overall average Sharpe ratio. As is typical, it is difficult to match both the top and bottom of the wealth distribution simultaneously, however Table 3 shows that the model generates the empirical average Gini coefficient at the baseline parameters. In addition, the model can closely match the wealth shares of the top deciles if we do not target the Gini coefficient directly.[26] Finally, the bottom panel of Figure 6 shows that in the model, as in the data, more complex strategies, with higher Sharpe ratios, display less wealth concentration. Table 3 contains the respective Gini coefficients, and shows that the lower concentration is driven by the selection effect from lower participation rates.

6 Conclusion

We study the equilibrium properties of complex asset markets. A complex asset is defined as an investment which requires a model and implementation strategy, thereby exposing investors to idiosyncratic risk. We provide a specific micro-foundation for how complex assets impose idiosyncratic risk on investors through imperfect hedging or tracking portfolios in long-short strategies designed to maximize returns while minimizing risk. In our equilibrium model, required returns increase with asset complexity, as proxied for by higher total volatility. We emphasize heterogeneity in the risk-return tradeoff faced by investors with different levels of expertise. Accordingly, we show that in our model, if expertise and complexity are complementary, improvements in market-level Sharpe ratios can be accompanied by lower market participation, consistent with empirical observations. Finally, we describe the implications of our model for the industrial organization of markets for complex risky assets. Limited participation and selection effects imply that more complex assets display less concentrated equilibrium size distributions, which we show is consistent with data on relative value hedge fund strategies.

[26] Our result mirrors that in other prominent studies of wealth distributions. See Castaneda, Gimenez, and Rull (2003) for a review of the literature, and specifically Table 1 for the errors in six prominent models for the low end of the wealth distribution.

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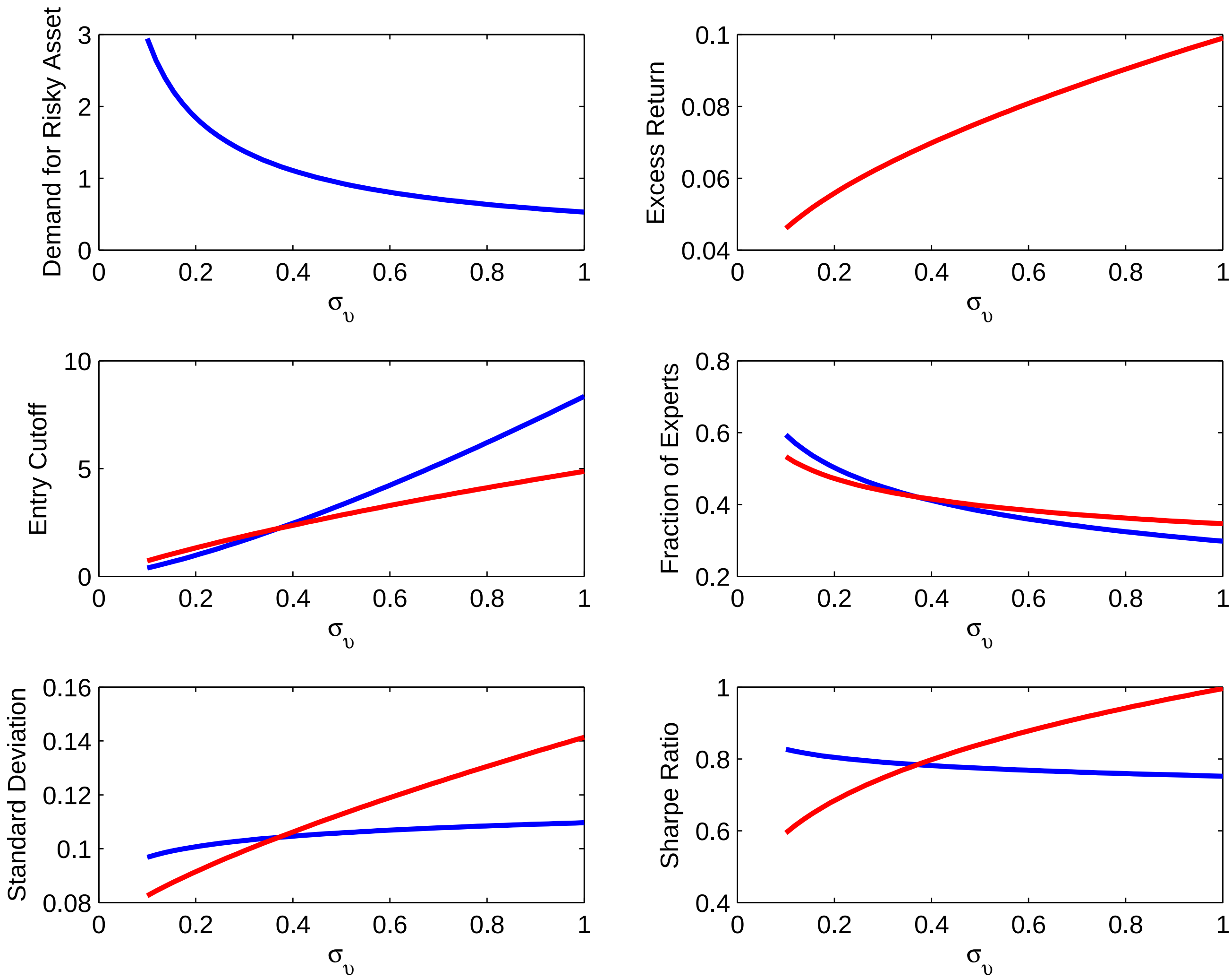

Figure 1: Model comparative statics over variation in total risk. Blue lines plot partial equilibrium comparative statics, red lines plot general equilibrium comparative statics. Economies satisfy Case 3 of Proposition 3.4, i.e. expertise and complexity are complementary.

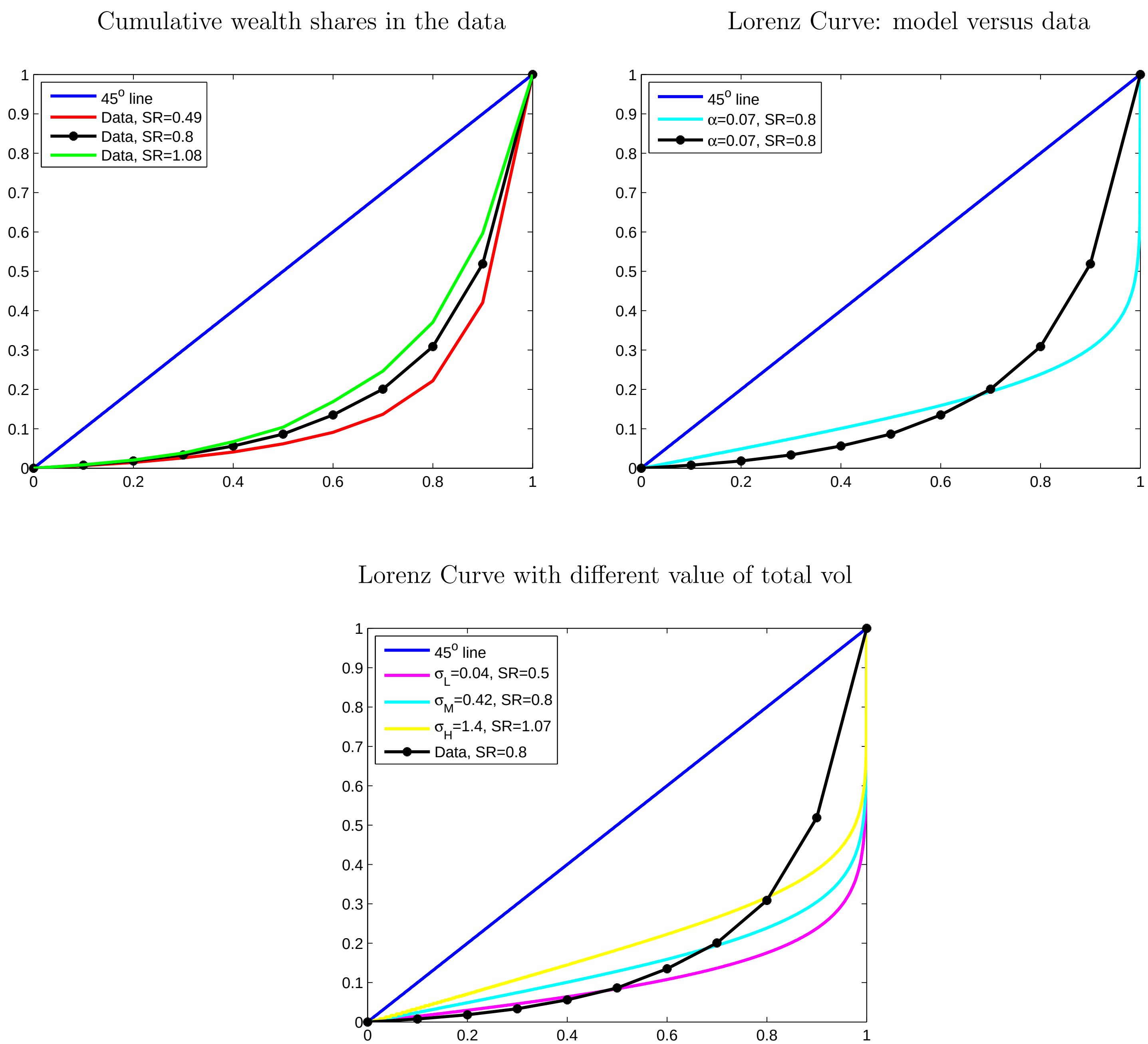

Figure 2: Concentration in complex asset markets, data and model: The top left panel shows Lorenz curves for Relative Value Fixed Income Strategies in the Hedge Fund Research Database. Strategies are grouped into complexity rankings by pseudo Sharpe ratios across funds within strategies, $\frac{\alpha}{E[\sigma_i]}$. The top right panel compares the wealth distribution of the model at baseline parameters to the average of all five relative value strategies. The bottom panel plots the average wealth distribution over all five relative value strategies against the model targeted to match the average Sharpe ratios in the top left panel. As in the data, wealth concentration decreases with complexity.

Table 1: Numerical Example: Baseline Parameter Values

Parameter	Symbol	Value	Target
Discount factor	ρ	0.01	Annual interest rate
Risk-free rate	r_f	0.01	Annual interest rate
Risk aversion	γ	5	
Entry cost	fnx	0.03	
Maintenance cost	fxx	0.01	
Supply of risky asset	S	1.20	Average excess return $\alpha = 6.98\%$
Total volatility of risky asset return	σ_ν	40%	
Mean of expertise distribution	μ_x	0	
Volatility of expertise distribution	σ_x	4	
Constant in $\sigma^2(x)$	a	0.48%	Average SR
Slope of $\sigma^2(x)$	b	1.5	Average Gini
Minimum wealth share	$z_{\min}$	0.01	

Table 2: Relative Value Hedge Fund Strategies: Summary Statistics. This table provides data on the five single strategy relative value fund classifications in the Hedge Fund Research database. The first column provides time series average returns by fund classification. The second column provides the pseudo Sharpe ratios across funds withing strategies, $\frac{\alpha}{E[\sigma_i]}$.

Strategy Name	TS AVG Industry Returns	Pseudo Sharpe Ratio	Gini
FI Asset Backed	9.06%	1.15	0.66
FI Sov	7.94%	1	0.62
FI Conv Bond Arb	7.78%	0.87	0.66
Volatility	6.32%	0.52	0.75
FI Corp	3.80%	0.45	0.79

Table 3: Model vs. Data: Asset Pricing and Wealth Distributions. We classify the five single strategy relative value fund classifications in the Hedge Fund Research database into three groups, by pseudo Sharpe ratios across funds within strategies, $\frac{\alpha}{E[\sigma_i]}$. The medium complexity strategy grouping averages over all strategies, the high complexity strategy averages over the highest two pseudo Sharpe ratio strategies, and the lowest strategy averages over the lowest two strategy groupings. α measures time series average returns by fund classification grouping.

	α		Pseudo Sharpe ratio		Gini		Participation
	Model	Data	Model	Data	Model	Data	Model
σ_ν^H	11.25	8.50	1.07	1.08	0.60	0.64	0.32
σ_ν^M	6.98	6.98	0.80	0.80	0.70	0.70	0.41
σ_ν^L	3.66	5.06	0.50	0.49	0.78	0.77	0.61

A Appendix

Proof. Proposition 2.1. We prove this Proposition by guess and verify. First, we write the HJB equations of our model

$$\begin{aligned}
\max_{c^x(x,t,s),\theta(x,t,s)} 0 &= u\left(c^x(x,t,s)\right) + V_w^x\left[w(t,s)\left(r_f + \theta(x,t,s)\alpha(t,s)\right) - c^x(x,t,s) - f_{xx}w(t,s)\right] \\
&\quad + \frac{\theta^2(x)\sigma^2(x)w(t,s)^2}{2}V_{ww}^x - \rho V^x \\
\max_{c^n(t,s)} 0 &= u\left(c^n(t,s)\right) + V_w^n\left(r_f w(t,s) - c^n(t,s)\right) - \rho V^n
\end{aligned}$$

The first order conditions for optimality are given by:

$$\begin{aligned}
u'\left(c^x(x,t,s)\right) &= V_w^x, \\
u'\left(c^n(t,s)\right) &= V_w^n, \\
V_w^x \alpha(t,s) + \theta(x,t,s)\sigma^2(x)w(t,s)V_{ww}^x &= 0.
\end{aligned}$$

Next, we guess that the value functions have the following form:

$$\begin{aligned}
V^x(w(t,s),x) &= y^x(x,t,s)\frac{w(t,s)^{1-\gamma}}{1-\gamma}, \\
V^n(w(t,s),x) &= y^n(t,s)\frac{w(t,s)^{1-\gamma}}{1-\gamma}.
\end{aligned}$$

Given these conjecture, it follows from the Benveniste-Scheinkman condition that the optimal consumption choices are given by:

$$\begin{aligned}
c^x(x,t,s) &= \left[y^x(x,t,s)\right]^{-\frac{1}{\gamma}} w(t,s), \\
c^n(t,s) &= \left[y^n(t,s)\right]^{-\frac{1}{\gamma}} w(t,s),
\end{aligned}$$

and that the optimal portfolio choice is given by

$$\theta(x,t,s) = \frac{\alpha(t,s)}{\gamma\sigma^2(x)}.$$

Plugging these choices into the HJB equations, we get

$$\begin{aligned}
0 &= \left[y^x(x,t,s)\right]^{-\frac{1-\gamma}{\gamma}} + y^x(x,t,s)\left(r_f + \frac{\alpha^2(t,s)}{\gamma\sigma^2(x)} - \left[y^x(x,t,s)\right]^{-\frac{1}{\gamma}} - f_{xx}\right)(1-\gamma) \\
&\quad - \frac{\alpha^2(t,s)}{2\gamma\sigma^2(x)}y^x(x,t,s)(1-\gamma) - \rho y^x(x,t,s) \\
&= \gamma\left[y^x(x,t,s)\right]^{-\frac{1-\gamma}{\gamma}} + y^x(x,t,s)\left(r_f + \frac{\alpha^2(t,s)}{2\gamma\sigma^2(x)} - f_{xx}\right)(1-\gamma) - \rho y^x(x,t,s), \\
0 &= \gamma\left[y^n(t,s)\right]^{-\frac{1-\gamma}{\gamma}} + y^n(t,s)(1-\gamma)r_f - \rho y^n(t,s).
\end{aligned}$$

Rearranging the equations, we solve for $y^x(x,t,s)$ and $y^n(x,t,s)$,

$$\begin{aligned}
y^x(x,t,s) &= \left[\frac{(\gamma-1)(r_f - f_{xx}) + \rho}{\gamma} + \frac{(\gamma-1)\alpha^2(t,s)}{2\gamma^2\sigma^2(x)}\right]^{-\gamma}, \\
y^n(t,s) &= \left[\frac{(\gamma-1)r_f + \rho}{\gamma}\right]^{-\gamma}.
\end{aligned}$$

Given all policy functions, we get the experts' wealth growth rates:

$$\frac{dw(t,s)}{w(t,s)} = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2(t,s)}{2\gamma^2\sigma^2(x)}\right)dt + \frac{\alpha(t,s)}{\gamma\sigma(x)}dB(t,s)$$

Finally, given homogeneity of the value functions in wealth, the participation cutoff is constructed by direct comparison between $y^x(x,t,s)$ and $y^n(t,s)$. ■

Proof of Proposition 2.2. Equivalence of value and policy functions under the reflecting barrier $\mathbf{z}_{\min}$.

Interpretation of $z_{\min}$: We assume that one of two things can happen to an investor at $z_{\min}$. With probability q, the investor is given a low value as punishment, and is eliminated from the market and replaced with a new agent with wealth share $z_{\min}$ and the same expertise as the exiting agent. Note that punishment and elimination in isolation would cause the incumbent agent to be conservative, to avoid $z_{\min}$. With probability $1-q$, the agent is rewarded by remaining in the market with wealth share equal to $z_{\min}$, which we implement with a reflected geometric Brownian Motion. Note that this reward in isolation would cause the agent to risk shift, to take advantage of limited liability at $z_{\min}$. Intuitively, we require that $E[V^x(z,x)] = qE[V^{\text{punishment}}] + (1-q)E[V^{\text{reward}}]$, conditional on the optimal policies under the true wealth share dynamics. Since the value under the true, non-reflecting, dynamics lies between the punishment value of dying and the reward value of reflection, we conjecture (and verify below) that there exists some probability, conditional on parameters, that this is the case. It seems realistic that investors face uncertainty about what will happen to them if their assets reach a lower threshold. Will they be liquidated, or rescued? Note that our proof offers a technical contribution for models with endogenous state variables following a reflecting geometric Brownian motion.

We show that the optimal policies in the model with reflecting barrier $z_{\min}$ are equivalent to those in the original model under our assumptions for probabilities and values of exit vs. reflection. Our proof assumes an optimal voluntary exit date. This is without loss of generality in a stationary equilibrium with no entry or exit.

The model with geometric Brownian motion wealth share dynamics is given by the true Model 1 as follows:

$$\begin{aligned} V^x(z(t,s),x) &= \max_{c^x(x,t,s),T,\theta^x(x,t,s)} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)}u(c^x(x,t,s))\,ds + e^{-\rho(T-t)}V^n(z(t,s))\right] \\ \text{s.t. } dz(t,s) &= [z(t,s)(r_f+\theta^x(x,t,s)\alpha(t,s)) - c^x(x,t,s) - F_{xx}]\,ds \\ &\quad + z(t,s)\theta^x(x,t,s)\sigma(x)\,dB(t,s), \end{aligned}$$

The model with reflected Brownian motion wealth share dynamics is given by the alternative Model 2 as follows:

$$\begin{aligned} V^r(z(t,s),x) &= \max_{c^r(x,t,s),T,\theta^r(x,t,s)} \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)}u(c^r(x,t,s))\,ds + e^{-\rho(s'-t)}\left[\begin{array}{c}(1-q)V^r(z_{\min},x) \\ +qV^n(z(t,s),x)\end{array}\right]\right] \\ \text{s.t. } dz(t,s) &= [z(t,s)(r_f+\theta^r(x,t,s)\alpha(t,s)) - c^r(x,t,s) - F_{xx}]\,ds \\ &\quad + z(t,s)\theta^r(x,t,s)\sigma(x)\,dB(t,s), \end{aligned}$$

where $s' < T$ is the first time that the agent's wealth falls below $z_{\min}$, the superscript n denotes the value of a non-expert which cannot re-enter, and the superscript r denotes the value of an expert under reflected Brownian dynamics, given by $dz_t/z_t = \mu_z dt + \sigma_z dB_t$ for $z_t > z_{\min}$ and $dz_t/z_t = \max(\mu_z dt + \sigma_z dB_t, 0)$ for $z_t \leq z_{\min}$.[27]

[27] See Harrison (2013), who also notes that the "reflected geometric Brownian motion process" might more precisely be called a regulated Brownian motion.

For parsimony, we only present the case in which the expert wealth share drops below $z_{\min}$ before the voluntary exit stopping time at which an expert would voluntarily choose to exit, which applies in the stationary equilibrium with exit time $T = \infty$ since there is no voluntary entry or exit in equilibrium.

To equate value and policy functions under the true and reflected wealth share dynamics then requires the appropriate specification of q, the probability of forced exit vs. being allowed to remain in the complex asset market with a wealth share of $z_{\min}$. Two alternative specifications for q both lead to equal values and policies. The first specifies that the value upon exit is the value of a non-expert which cannot re-enter, which is the same for all levels of expertise. In this case, the probability of exit must depend on expertise. The second alternative specifies that the value of exit is increasing in expertise. In this case, the probability of exit is the same for all agents. In what follows, we use the case in which the value upon exit is the value of a non-expert.[28] Define

$$q\left(z\left(t,s\right),x\right)=\frac{y^{x}\left(x\right)-\left[\frac{z_{\min}}{z(t,s)}\right]^{1-\gamma}}{y^{n}\left(x\right)-y^{x}\left(x\right)\left[\frac{z_{\min}}{z(t,s)}\right]^{1-\gamma}},\ \text{for } z\left(t,s\right)\leq z_{\min}. \tag{31}$$

Using this definition and equation (10) from Proposition 2.1, it is straightforward to show that

$$V^{x}\left(z\left(t,s\right),x\right)=\left(1-q\right)V^{x}\left(z_{\min},x\right)+qV^{n}\left(z\left(t,s\right)\right),\ \text{for } z\left(t,s\right)\leq z_{\min}. \tag{32}$$

Then, it is sufficient to show that

$$V^{r}\left(z\left(t,s\right),x\right)=V^{x}\left(z\left(t,s\right),x\right),\ \text{for all } x \text{ and } z\left(t,s\right)\geq z_{\min}.$$

Our proof strategy is to first show that the value function and optimal policy functions are identical when expert wealth equals $z_{\min}$. Next, we show that the two models are identical for $z > z_{\min}$.

First, in Model 2, $c^{r}\left(x,t,s\right)$ is the optimal consumption, therefore,

$$\begin{aligned}V^{r}\left(z_{\min},x\right) &= \mathbb{E}\left[\int_{t}^{s'}e^{-\rho(s-t)}u\left(c^{r}\left(x,t,s\right)\right)ds+e^{-\rho(s'-t)}\left[\left(1-q\right)V^{r}\left(z_{\min},x\right)+qV^{n}\left(z\left(t,s'\right)\right)\right]\right]\\ &\geq \mathbb{E}\left[\int_{t}^{s''}e^{-\rho(s-t)}u\left(c^{x}\left(x,t,s\right)\right)ds+e^{-\rho(s''-t)}\left[\left(1-q\right)V^{r}\left(z_{\min},x\right)+qV^{n}\left(z\left(t,s''\right)\right)\right]\right],\end{aligned}$$

where on the right hand side of the inequality we replace c^{r} with c^{x} and allow the time s' at which the wealth share declines to $z_{\min}$ to update to s'' accordingly. Rearranging terms, we have:

$$\begin{aligned}V^{r}\left(z_{\min},x\right) &= \frac{1}{1-E\left[\left(1-q\right)e^{-\rho(s'-t)}\right]}\mathbb{E}\left[\int_{t}^{s'}e^{-\rho(s-t)}u\left(c^{r}\left(x,t,s\right)\right)ds+e^{-\rho(s'-t)}qV^{n}\left(z\left(t,s'\right)\right)\right]\\ &\geq \frac{1}{1-E\left[\left(1-q\right)e^{-\rho(s''-t)}\right]}\mathbb{E}\left[\int_{t}^{s''}e^{-\rho(s-t)}u\left(c^{x}\left(x,t,s\right)\right)ds+e^{-\rho(s''-t)}qV^{n}\left(z\left(t,s''\right)\right)\right].\end{aligned}$$

Second, in Model 1, we can rewrite the value of being an expert as the value of the stream of con-

[28] The alternative is to specify q to be independent of the level of expertise, and given by $q\left(z\left(t,s\right),x\right)=\frac{1-\left[\frac{z_{\min}}{z(t,s)}\right]^{1-\gamma}}{y^{r}-\left[\frac{z_{\min}}{z(t,s)}\right]^{1-\gamma}}$. In this case, the value of exit must be specified to be proportional to the expertise-specific value, i.e. $V^{r}_{exit}\left(z\left(t,s\right),x\right)=y^{r}V^{x}\left(z\left(t,s\right),x\right)$, where $y^{r}>1$.

sumption before s'' plus the continuation value if wealth falls below $z_{\min}$. We have:

$$\begin{aligned} V^x(z_{\min}, x) &= \mathbb{E}\left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x,t,s))\,ds + e^{-\rho(s''-t)} V^x\left(z\left(t,s''\right), x\right)\right] \\ &= \mathbb{E}\left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x,t,s))\,ds + e^{-\rho(s''-t)}\left[(1-q)V^x(z_{\min},x) + qV^n\left(z\left(t,s''\right)\right)\right]\right] \\ &\geq \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u(c^r(x,t,s))\,ds + e^{-\rho(s'-t)}\left[(1-q)V^x(z_{\min},x) + qV^n\left(z\left(t,s'\right)\right)\right]\right], \end{aligned}$$

where the second equality uses the result in Equation (32), and the inequality results from replacing c^x with c^r and allowing the expected time s'' at which the wealth share declines to $z_{\min}$ under the policy c^r to update to s' accordingly. Rearranging terms, we have

$$\begin{aligned} V^x(z_{\min}, x) &= \frac{1}{1-E\left[(1-q)e^{-\rho(s''-t)}\right]}\mathbb{E}\left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x,t,s))\,ds + e^{-\rho(s''-t)} qV^n(z(t,s))\right] \\ &\geq \frac{1}{1-E\left[(1-q)e^{-\rho(s'-t)}\right]}\mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u(c^r(x,t,s))\,ds + e^{-\rho(s'-t)} qV^n(z(t,s))\right]. \end{aligned}$$

We have established two inequalities which hold in opposite directions. Thus, we must have equality, namely:

$$\begin{aligned} &\frac{1}{1-E\left[(1-q)e^{-\rho(s''-t)}\right]}\mathbb{E}\left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x,t,s))\,ds + e^{-\rho(s''-t)} qV^n(z(t,s))\right] \\ = &\frac{1}{1-E\left[(1-q)e^{-\rho(s'-t)}\right]}\mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u(c^r(x,t,s))\,ds + e^{-\rho(s'-t)} qV^n(z(t,s))\right], \end{aligned}$$

and

$$V^r(z_{\min}, x) = V^x(z_{\min}, x).$$

Next, we show that the value functions for Model 1 and Model 2 are identical when $z > z_{\min}$. Using analogous logic, we have:

$$\begin{aligned} V^r(z(t,s), x) &= \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u(c^r(x,t,s))\,ds + e^{-\rho(s'-t)}\left[(1-q)V^r(z_{\min},x) + qV^n\left(z\left(t,s'\right)\right)\right]\right] \\ &\geq \mathbb{E}\left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x,t,s))\,ds + e^{-\rho(s''-t)}\left[(1-q)V^x(z_{\min},x) + qV^n\left(z\left(t,s''\right)\right)\right]\right] \\ &= \mathbb{E}\left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x,t,s))\,ds + e^{-\rho(s''-t)} V^x\left(z\left(t,s''\right), x\right)\right] \\ &= V^x(z(t,s), x)\,, \text{ for all } z(t,s) \end{aligned}$$

with equality iff $c^x(x,t,s) = c^r(x,t,s)$ and $\theta^x(x,t,s) = \theta^r(x,t,s)$. Also, we have:

$$\begin{aligned} V^x(z(t,s), x) &= \left[\int_t^{s''} e^{-\rho(s-t)} u(c^x(x,t,s))\,ds + e^{-\rho(s''-t)}\left[(1-q)V^x(z_{\min},x) + qV^n\left(z\left(t,s''\right)\right)\right]\right] \\ &\geq \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u(c^r(x,t,s))\,ds + e^{-\rho(s'-t)}\left[(1-q)V^r(z_{\min},x) + qV^n\left(z\left(t,s'\right)\right)\right]\right] \\ &= V^r(z(t,s), x)\,, \text{ for all } z(t,s) \end{aligned}$$

with equality iff $c^x(x,t,s) = c^r(x,t,s)$ and $\theta^x(x,t,s) = \theta^r(x,t,s)$. Therefore, our definition of the probabilities for exit vs. remaining in equation (31) yields equivalence for all value and policy functions under the true and reflected dynamics models:

$$\begin{aligned} V^x(z(t,s),x) &= V^r(z(t,s),x), \text{ for all } x \text{ and } z(t,s) \\ c^x(x,t,s) &= c^r(x,t,s), \text{ for all } x \text{ and } z(t,s) \\ \theta^x(x,t,s) &= \theta^r(x,t,s), \text{ for all } x \text{ and } z(t,s). \end{aligned}$$

Proof. Proposition 2.3 We prove this Proposition by guess-and-verify. We guess that the stationary distribution takes the following form:

$$\phi(z,x) = C(x) z^{-\beta(x)-1},$$

Then, by plugging this guess into the Kolmogorov forward equation, we obtain the following condition:

$$\begin{aligned} 0 = &-\partial_z \left(z^{-\beta(x)} \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right) \right) \\ &+ \frac{1}{2}\partial_{zz} \left(z^{1-\beta(x)} \frac{\alpha^2}{\gamma^2\sigma^2(x)} \right) \\ &= \beta(x) \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right) \\ &- \frac{1}{2}\beta(x)(1-\beta(x)) \left[\frac{\alpha}{\gamma\sigma(x)} \right]^2 \\ &= \beta(x) \left[\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2(\gamma+\beta(x))}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right] \end{aligned}$$

Thus, by collecting terms, we obtain:

$$\begin{aligned} \beta(x) &= C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma \geq 1, \\ C_1 &= 2\gamma(f_{xx} + \rho - r_f + \gamma g(\bar{x})), \\ C(x) &= \frac{1}{\int z^{-\beta-1} dz} = \frac{C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma}{z_{\min}^{-C_1 \frac{\sigma^2(x)}{\alpha^2} + \gamma}}. \end{aligned}$$

Note there are two roots of equation

$$0 = \beta(x) \left[\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2(\gamma+\beta(x))}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right].$$

The negative drift of wealth shares ensures that there will be one root of this equation which is larger than one. We then take this root in order to ensure that the mean wealth has a finite mean. ■

Proof. Corollary 2.1. For the highest expertise agents, we have

$$\begin{aligned} \bar{z} &= \int_{z_{\min}}^{\infty} z\phi(z,\bar{x}) dz = \int_{z_{\min}}^{\infty} C(\bar{x}) z^{-\beta(\bar{x})} dz \\ &= \frac{1}{-\beta(\bar{x}) - 1} C(\bar{x}) z^{-\beta(\bar{x})-1} |_{z_{\min}}^{\infty} = z_{\min} \left[1 + \frac{1}{\beta(\bar{x}) - 1} \right]. \end{aligned}$$

This gives us another expression for $\beta(\bar{x})$,

$$\beta(\bar{x}) = \frac{1}{1 - z_{\min}/\bar{z}}.$$

Also, we know that the decay coefficient is given by:

$$\beta(\bar{x}) = 2\gamma\left(f_{xx} + \rho - r_f + \gamma g(\bar{x})\right)\frac{\sigma^2(\bar{x})}{\alpha^2} - \gamma$$

Therefore, by combining these expressions, we have

$$2\gamma\left(f_{xx} + \rho - r_f + \gamma g(\bar{x})\right)\frac{\sigma^2(\bar{x})}{\alpha^2} - \gamma = \frac{1}{1 - z_{\min}/\bar{z}},$$

By rearranging the above equation, we get the following expression:

$$g(\bar{x}) = \frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2}{2\gamma\sigma^2(\bar{x})} + \frac{\alpha^2}{2\gamma^2\sigma^2(\bar{x})}\frac{1}{1 - z_{\min}/\bar{z}}.$$

We plug $g(\bar{x})$ into $\beta(x)$, to obtain:

$$\beta(x) = \left(\gamma + \frac{z_{\min}/\bar{z}}{1 - z_{\min}/\bar{z}}\right)\frac{\sigma^2(x)}{\sigma^2(\bar{x})} - \gamma.$$

■

Proof. Lemma 2.2 Recall that the risky asset share and the decay coefficients are given by:

$$\begin{aligned}
\theta(x) &= \frac{\alpha}{\gamma\sigma^2(x)}, \\
\beta(x) &= 2\gamma\left(f_{xx} + r - r_f + \gamma g(\bar{x})\right)\frac{\sigma^2(x)}{\alpha^2} - \gamma.
\end{aligned}$$

Consider two levels of expertise, $x_H > x_L$ we have the following expression for the difference in the risky asset share:

$$\begin{aligned}
\theta(x_H) - \theta(x_L) &= \frac{\alpha}{\gamma}\left[\frac{1}{\sigma^2(x_H)} - \frac{1}{\sigma^2(x_L)}\right] \\
&= \frac{\alpha}{\gamma}\frac{\sigma^2(x_L) - \sigma^2(x_H)}{\sigma^2(x_H)\sigma^2(x_L)},
\end{aligned}$$

and the following expression for the difference in decay coefficient:

$$\begin{aligned}
\beta(x_H) - \beta(x_L) &= 2\gamma\left(f_{xx} + r - r_f + \gamma g(\bar{x})\right)\frac{1}{\alpha^2}\left[\sigma^2(x_H) - \sigma^2(x_L)\right] \\
&= 2\gamma^2\left(f_{xx} + r - r_f + \gamma g(\bar{x})\right)\frac{\sigma^2(x_H)\sigma^2(x_L)}{\alpha^3}\left[\theta(x_L) - \theta(x_H)\right].
\end{aligned}$$

If a larger dispersion of portfolio choice is due to either a higher excess return or a lower risk aversion, the dispersion in β is smaller, since:

$$\begin{aligned}
\frac{\partial\left[\beta(x_H) - \beta(x_L)\right]}{\partial\alpha} &< 0, \text{ and } \frac{\partial\left[\theta(x_L) - \theta(x_H)\right]}{\partial\alpha} > 0 \\
\frac{\partial\left[\beta(x_H) - \beta(x_L)\right]}{\partial\gamma} &> 0, \text{ and } \frac{\partial\left[\theta(x_L) - \theta(x_H)\right]}{\partial\gamma} < 0
\end{aligned}$$

Consider the case where $\sigma^2(x_H)\sigma^2(x_L)$ is a constant, then

$$\frac{\partial\left[\beta(x_H) - \beta(x_L)\right]}{\partial\left[\theta(x_L) - \theta(x_H)\right]} = 2\gamma^2\left(f_{xx} + r - r_f + \gamma g(\bar{x})\right)\frac{\sigma^2(x_H)\sigma^2(x_L)}{\alpha^3} > 0.$$

A larger dispersion in portfolio choice, resulting from a larger difference between effective volatility, implies a larger dispersion of tail distribution. The condition on the product of the effective variances is not necessary, however, as can be seen by simple algebra. ■

Lemma A.1 *The micro-foundation for the return process in Equation(1) provided in Lemma 2.1, along with Assumption 1, are sufficient to ensure that neither policy functions nor equilibrium prices are functions of the realization of the fundamental shock $dB^F(t,s)$.*

Proof. Lemma A.1 First, since the innovations in $dB^F(t,s)$ are independent over time, the realizations do not affect portfolio choices or consumption decisions. Moreover, Lemma 2.1 describes how fundamental shocks are absorbed into the asset-specific risk $\sigma(x)dB_i(t,s)$ in Equation 1, which implies that the proof for our Proposition 2.1 applies as stated. Then, for asset prices to be independent of the realization of the fundamental shock $dB^F(t,s)$, all that remains is to show that the expected growth rate of wealth, conditional on each expertise level, is the same under the return process stated in (1) and the return process constructed in Lemma 2.1.

We can rewrite the expected growth rate of wealth as

$$\begin{aligned}\frac{dw\left(t,s\right)}{w\left(t,s\right)} &= \left(\frac{r_f-f_{xx}-\rho}{\gamma}+\frac{\left(\gamma+1\right)\alpha^2\left(t,s\right)}{2\gamma^2\sigma^2\left(x\right)}\right)dt+\frac{\alpha\left(t,s\right)}{\gamma\sigma\left(x\right)}dB\left(t,s\right)\\ &= \left(\frac{r_f-f_{xx}-\rho}{\gamma}+\frac{\left(\gamma+1\right)\alpha^2\left(t,s\right)}{2\gamma^2\sigma^2\left(x\right)}\right)dt+\frac{\alpha\left(t,s\right)}{\gamma\sigma\left(x\right)}\left[\left(1-\rho_i(x)\right)\sigma^F dB^F(t,s)+\sigma^T(x)dB^T(t,s)\right].\end{aligned}$$

Second, because $\frac{\rho_o(x)+\rho_u(x)}{2}=1$, we have that

$$\sigma_o(x)=\sigma_u(x)=\sqrt{\left[\left(1-\rho_o(x)\right)\sigma^F\right]^2+\left[\sigma^T(x)\right]^2}=\sqrt{\left[\left(1-\rho_u(x)\right)\sigma^F\right]^2+\left[\sigma^T(x)\right]^2}.$$

Thus, each expert's expected growth rate of wealth, conditional on the aggregate shock is given by:

$$E\left[\frac{dw\left(t,s\right)}{w\left(t,s\right)}|x,dB^F(t,s)\right]=\frac{r_f-f_{xx}-\rho}{\gamma}+\frac{\left(\gamma+1\right)\alpha^2\left(t,s\right)}{2\gamma^2\sigma^2\left(x\right)}+\frac{\alpha\left(t,s\right)}{\gamma\sigma\left(x\right)}\left(1-\rho_i(x)\right)\sigma^F dB^F(t,s).$$

If there is a positive aggregate shock, the growth rate of the wealth of under-hedging investors, with lower ρ, is higher than the growth rate of wealth for over-hedging investors, with higher ρ, and vice versa. The difference is exactly canceled out at the expertise-level aggregate. That is:

$$E\left[\frac{dw\left(t,s\right)}{w\left(t,s\right)}|x\right]=\frac{r_f-f_{xx}-\rho}{\gamma}+\frac{\left(\gamma+1\right)\alpha^2\left(t,s\right)}{2\gamma^2\sigma^2\left(x\right)}.$$

Aggregating across expertise levels, then, the equilibrium excess return α does not depend on the realization of aggregate shock. ■

Proof. Lemma 3.1 Direct calculation. We use 1 to denote a positive sign. For ease of exposition, we study the partial derivative of the log of expertise level investment in the risky asset. The sign of the partial derivative of the log and the level will be the same because log is a positive transformation. We have:

$$\begin{aligned}\log I\left(x\right) &= \log\frac{\alpha}{\gamma\sigma^2\left(x\right)}+\log Z\left(x\right)\\ &= \log\alpha-\log\gamma-\log\sigma^2\left(x\right)+\log Z\left(x\right),\end{aligned}$$

where $Z(x)$ is the total expertise level wealth share,

$$\begin{aligned} Z(x) &= \int_{z_{\min}}^{\infty} z\phi(z,x)dz = \int_{z_{\min}}^{\infty} C(x)\, z^{-\beta(x)}dz \\ &= \frac{1}{-\beta(x)-1} C(x)\, z^{-\beta(x)-1}|_{z_{\min}}^{\infty} = z_{\min}\left[1+\frac{1}{\beta(x)-1}\right]. \end{aligned}$$

Then:

1.

$$\begin{aligned} sign\left(\frac{\partial I(x)}{\partial\sigma^2(x)}\right) &= sign\left(\frac{\partial \log I(x)}{\partial\sigma^2(x)}\right) \\ &= sign\left(-1-\frac{1}{Z(x)}\frac{z_{\min}}{(\beta(x)-1)^2}C_1\frac{1}{\alpha^2}\right) \\ &= -1 \end{aligned}$$

2.

$$\begin{aligned} & sign\left(\frac{\partial I(x)}{\partial\sigma_\nu}\right) \\ =& sign\left(\frac{\partial I(x)}{\partial\sigma^2(x)}\frac{\partial\sigma^2(x)}{\partial\sigma_\nu}\right) \\ =& sign\left(\frac{\partial I(x)}{\partial\sigma^2(x)}\right) sign\left(\frac{\partial\sigma^2(x)}{\partial\sigma_\nu}\right) \\ =& -1. \end{aligned}$$

3.

$$\begin{aligned} sign\left(\frac{\partial I(x)}{\partial\alpha}\right) &= sign\left(\frac{\partial \log I(x)}{\partial\alpha}\right) \\ &= sign\left(1+\frac{2}{Z(x)}\frac{z_{\min}}{(\beta(x)-1)^2}C_1\frac{\sigma^2(x)}{\alpha^3}\right) \\ &= 1 \end{aligned}$$

4.

$$\begin{aligned} sign\left(\frac{\partial I(x)}{\partial\gamma}\right) &= sign\left(\frac{\partial \log I(x)}{\partial\gamma}\right) \\ &= sign\left(-1-\frac{1}{Z(x)}\frac{z_{\min}}{(\beta(x)-1)^2}\left(\frac{\sigma^2(x)}{\alpha^2}\left(\frac{C_1}{\gamma}+2\gamma g(\overline{x})\right)-1\right)\right) \\ &\leq sign\left(-1-\frac{1}{Z(x)}\frac{z_{\min}}{(\beta(x)-1)^2}\left(\frac{\sigma^2(x)}{\alpha^2}\frac{C_1}{\gamma}-1\right)\right) \\ &= -1 \end{aligned}$$

5.

$$\begin{aligned} sign\left(\frac{\partial I(x)}{\partial f_{xx}}\right) &= sign\left(\frac{\partial \log I(x)}{\partial f_{xx}}\right) \\ &= sign\left(-\frac{1}{Z(x)}\frac{z_{\min}}{(\beta(x)-1)^2}\frac{\sigma^2(x)}{\alpha^2}2\gamma\right) \\ &= -1 \end{aligned}$$

■

Proof. Proposition 3.1 For each level of expertise, we have

$$sign\left(\frac{\partial I\left(x\right)}{\alpha}\right)=1,\text{ for } x\geq\underline{x}.$$

And when α is higher, more experts enter. Thus

$$\frac{\partial I}{\partial\alpha}>0.$$

■

Proof. Proposition 3.2 We use direct calculations. We use 1 to denote a positive sign.

1. $$\begin{aligned}&sign\left(\frac{\partial I\left(x\right)}{\partial\sigma_{\nu}}\right)\\=\;&sign\left(\frac{\partial I\left(x\right)}{\partial\sigma^{2}\left(x\right)}\frac{\partial\sigma^{2}\left(x\right)}{\partial\sigma_{\nu}}\right)\\=\;&sign\left(\frac{\partial I\left(x\right)}{\partial\sigma^{2}\left(x\right)}\right)sign\left(\frac{\partial\sigma^{2}\left(x\right)}{\partial\sigma_{\nu}}\right).\end{aligned}$$

   We also have

   $$sign\left(\frac{\partial I\left(x\right)}{\partial\sigma^{2}\left(x\right)}\right)=-1$$

   Thus for each level of expertise, when total risk is higher, the demand for the complex risky asset is smaller. And, from Equation (18), since $\sigma(x)$ is increasing in σ_{ν}, when σ_{ν} is higher, participation is lower.

   $$\frac{\partial I}{\partial\sigma_{\nu}}<0.$$

2. For each level of expertise:

   $$sign\left(\frac{\partial I\left(x\right)}{\partial\gamma}\right)=-1,$$

   and from Equation (18), participation is lower. Thus,

   $$\frac{\partial I}{\partial\gamma}<0.$$

3. For each level of expertise:

   $$sign\left(\frac{\partial I\left(x\right)}{\partial f_{xx}}\right)=-1,$$

   and again from Equation (18), participation is lower. Thus, we also have that

   $$\frac{\partial I}{\partial f_{xx}}<0.$$

■

Proof. Proposition 3.3 We have

$$sign\left(\frac{\partial I(x)}{\partial x}\right)=sign\left(\frac{\partial I(x)}{\partial\sigma(x)}\frac{\partial\sigma(x)}{\partial x}\right)=1$$

And, using integration by parts, we obtain:

$$\begin{aligned} I(\Lambda_1)-I(\Lambda_2) &= \int[\lambda_1(x)-\lambda_2(x)]\,I(x)\,dx \\ &= -I(\underline{x})\left[\Lambda_1(\underline{x})-\Lambda_2(\underline{x})\right]-\int\frac{\partial I(x)}{\partial x}\left[\Lambda_1(x)-\Lambda_2(x)\right]dx \\ &> 0 \end{aligned}$$

■

Proof. Proposition 3.4. Writing out the partial derivative, and using the fact that, for example, $\frac{\partial\log\alpha}{\partial\log\sigma_\nu}=\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}$, we have that:

$$\begin{aligned} \frac{\partial SR(x)}{\partial\sigma_\nu} &= \frac{\frac{\partial\alpha}{\partial\sigma_\nu}\sigma(x)-\alpha\frac{\partial\sigma(x)}{\partial\sigma_\nu}}{\sigma^2(x)} \\ &= \frac{\alpha}{\sigma(x)\sigma_\nu}\left[\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right]. \end{aligned}$$

Thus,

$$\frac{\partial SR(x)}{\partial\sigma_\nu}>0 \text{ iff } \frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}>\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}$$

This result illustrates the key role of the elasticity of excess returns vs. the elasticity of effective volatilities with respect to changes in total volatility.

If $\frac{\partial\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}}{\partial x}$ is a constant, we must have either $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}>\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}$ for all x or $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}<\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}$ for all x.

If $\frac{\partial\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}}{\partial x}<0$, and assume there is a cutoff x^* such that

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}=\frac{\partial\sigma(x^*)/\sigma(x^*)}{\partial\sigma_\nu/\sigma_\nu},$$

then for all $x<x^*$, we have $\frac{\partial SR(x)}{\partial\sigma_\nu}<0$; and for all $x>x^*$, we have $\frac{\partial SR(x)}{\partial\sigma_\nu}>0$.

If $\frac{\partial\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}}{\partial x}>0$, and assume there is a cutoff x^* such that

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}=\frac{\partial\sigma(x^*)/\sigma(x^*)}{\partial\sigma_\nu/\sigma_\nu},$$

then for all $x<x^*$, we have $\frac{\partial SR(x)}{\partial\sigma_\nu}>0$; and for all $x>x^*$, we have $\frac{\partial SR(x)}{\partial\sigma_\nu}<0$. ■

Value Weighted Equilibrium Sharpe ratio The market value weighted Sharpe ratio can be written as

$$\begin{aligned} SR^{vw} &= E\left[\frac{\theta Z(x)}{I}\frac{\alpha}{\sigma(x)}|x \geq \underline{x}\right] \\ &= \frac{\alpha}{\gamma I}E\left[\frac{Z(x)}{\sigma^3(x)}|x \geq \underline{x}\right]. \end{aligned}$$

Participation: Intermediate results and proofs We begin by describing results for bounds on the elasticity of α with respect to changes in total volatility, σ_ν. As we saw in Proposition 3.4 which describe changes in individual Sharpe ratios as total volatility changes, the relative elasticity of excess returns, $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}$, vs. the elasticity of effective volatilities $\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}$ as total volatility changes,

$$\left[\frac{\partial\log\alpha}{\partial\log\sigma_\nu} - \frac{\partial\log\sigma(x)}{\partial\log\sigma_\nu}\right] \equiv \left[\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right]$$

plays an important role in our model economy. Both portfolio and participation policies depend on excess returns relative to effective risk. Sharpe ratios at both the individual and market level measure the amount of compensation per unit of risk. Thus, how each of these policies and equilibrium outcomes change as total volatilities change is driven in large part by the relative elasticity of excess returns vs. the elasticity of effective volatilities. If α is highly elastic with respect to changes in total volatility, then many agents will choose to participate even when total volatility is high. On the other hand, if low expertise agents' effective volatilities are very elastic with respect to total volatility, and in particular are more elastic than α is, then they will drop out of the complex asset market when total volatility increases.

Although we cannot solve for α in closed form, we describe bounds on its elasticity with respect to changes in total volatility. These bounds depend on the shape of the elasticity of effective volatility with respect to total volatility. Intuitively, cases in which participation increases are cases in which the difference in the increase in effective volatility for low and high expertise agents is similar. This is due to the requirement of market clearing. As volatility increases, demand decreases. If all agents' volatilities increase by a similar amount, then for demand to meet supply α must increase enough to satisfy even lower expertise agents. On the other hand, if the change in effective volatility is much smaller for high expertise agents (Case 3 of Proposition 3.4), then it may be that these agents demand a lot of the risky asset despite the higher volatility. If high expertise agents' demand is high enough, the market can clear at a level of α that does not adequately compensate lower expertise agents.

First, we establish intuition by showing that the percentage change in α has to be large enough to at least satisfy the investors whose risk-return tradeoff deteriorates the least as total volatility increases.

Lemma A.2 *In the equilibrium, we have*

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} > l_{\inf}^{\sigma_\nu},$$

where $l_{\inf}^{\sigma_\nu}$ is the lowest elasticity of all participating investors' effective volatility with respect to total volatility

$$l_{\inf}^{\sigma_\nu} \equiv \inf\left\{\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}|x \geq \underline{x}\right\}.$$

Proof. Lemma A.2 Proof by contradiction. Suppose σ_ν is increased by 1%, but the equilibrium α is increased by less than $l_{\inf}^{\sigma_\nu}$%, that is

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} \leq l_{\inf}^{\sigma_\nu}$$

The condition implies that $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} < 0 \quad \forall x \geq \underline{x}$. We have:

1. Less participation: because $\frac{\alpha^2}{2\sigma^2(\underline{x})\gamma} = f_{xx}$ and $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} < \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}$, $\underline{x}$ increases.

2. Less investment in the complex risky asset:

$$\begin{aligned}
&\frac{\partial \log I(x)}{\partial\sigma_\nu} \\
=\ & \frac{\partial\alpha/\alpha}{\partial\sigma_\nu} - 2\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} + \frac{1}{Z(x)}\frac{\partial Z(x)}{\partial\sigma_\nu} \\
=\ & \frac{\partial\alpha/\alpha}{\partial\sigma_\nu} - 2\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} - \frac{1}{Z(x)}\frac{z_{\min}}{(\beta(x)-1)^2}\frac{\partial\beta(x)}{\partial\sigma_\nu} \\
=\ & \frac{\partial\alpha/\alpha}{\partial\sigma_\nu} - 2\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} - \frac{1}{Z(x)}\frac{z_{\min}}{(\beta(x)-1)^2}2(\beta(x)+\gamma)\left[\frac{\partial\alpha/\alpha}{\partial\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu}\right] \\
=\ & -\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} + \frac{1}{\sigma_\nu}\left[1 + \frac{z_{\min}}{Z(x)}\frac{2(\beta(x)+\gamma)}{(\beta(x)-1)^2}\right]\left[\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right] \\
=\ & \frac{1}{\sigma_\nu}\left\{-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} + \left[1 + \frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right]\left[\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right]\right\} \\
<\ & 0\ ,\ \text{for all } x \geq \underline{x},
\end{aligned}$$

where for $\frac{\partial\beta(x)}{\partial\sigma_\nu}$, we use Equation (22). We also use the fact that $Z(x) = z_{\min}\left(\frac{\beta(x)}{\beta(x)-1}\right)$. The fact that we have $\beta(x) \geq 1$ is shown in the proof to Proposition 2.3.

Define

$$\mathcal{B}(x) = \frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1},$$

which describes the elasticity of the mean wealth at expertise level x with respect to total volatility. We can rewrite the partial derivative of expert's investment with respect to total volatility as

$$\frac{\partial \log I(x)}{\partial\sigma_\nu} = \frac{1}{\sigma_\nu}\left\{-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} + [1+\mathcal{B}(x)]\left[\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right]\right\} \tag{33}$$

Therefore, in the new equilibrium, the total demand for risky asset is less than the total supply. Contradiction. It must be that

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} > l_{\inf}^{\sigma_\nu}.$$

The last term in the partial derivative for expertise level investment, Equation (33), clearly illustrates the important role of the relative elasticity of excess returns vs. the elasticity of effective volatilities. It shows why our bound for the elasticity of α is that its change will be larger than the change in the effective volatility of the agents with the lowest elasticity of effective volatility with respect to total volatility (displayed by the highest expertise agents). ■

The following lemma describes more detailed bounds on the percentage change in α for a given percentage change in total volatility for the case of decreasing elasticities of effective volatility with respect to total volatility (Case 3 of Proposition 3.4). Case 3 of Proposition 3.4 is the only case which yields a decline in participation as total volatility increases. The condition for decreasing participation will be closely related related to the bounds in Lemma A.3. In particular, we show below that participation increases if Condition 1 of Lemma A.3 holds, but decreases if Condition 2 holds. Intuitively, participation will increase if the change in α is large enough to satisfy lower expertise investors in Case 3, but will decrease otherwise. Lemma A.3 provides bounds on the percentage change in α for a given percentage change in total volatility for Case 3, depending on the elasticity of the lowest expertise agent who participates relative to the elasticity of α, and, depending a condition on the ratio of the partial equilibrium change in demand with respect to total volatility to the partial equilibrium change in demand with respect to α. Intuitively, if, in partial equilibrium, demand is more sensitive to changes in effective volatility than to changes in α, then in general equilibrium participation will decline. We provide a sufficient condition for participation to decline as total volatility increases in Proposition 3.5 in the main text.

Lemma A.3 *In general equilibrium, in Case 3 of Proposition 3.4, in which* $\frac{\partial \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}}{\partial x} \leq 0$, *we have that:*

1.

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} > l_{\sup}^{\sigma_\nu} \text{ if } l_{\sup}^{\sigma_\nu} < \frac{\int_{\underline{x}}^{\overline{x}} I(x)\left(2+\mathcal{B}(x)\right)\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}}\left[I(x)\left(1+\mathcal{B}(x)\right)\right]d\Lambda(x)},$$

and

2.

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} < l_{\sup}^{\sigma_\nu} \text{ if } l_{\sup}^{\sigma_\nu} > \frac{\int_{\underline{x}}^{\overline{x}} I(x)\left(2+\mathcal{B}(x)\right)\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}}\left[I(x)\left(1+\mathcal{B}(x)\right)\right]d\Lambda(x)},$$

where

$$\mathcal{B}(x) = \frac{2}{\beta(x)}\frac{(\beta(x)+\gamma)}{\beta(x)-1}.$$

Proof. Proof of Lemma A.3 In Case 3 of Proposition 3.4, we have $\frac{\partial \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}}{\partial x} < 0$.

Using Equation (33) for the change in expertise-level investment with total volatility and accounting for the change in participation we then have for aggregate investment it must be that:

$$\begin{aligned}
0 &= \frac{\partial I}{\partial\sigma_\nu} = \int_{\underline{x}}^{\overline{x}}\frac{\partial I(x)}{\partial\sigma_\nu}d\Lambda(x) - I(\underline{x})\lambda(\underline{x})\frac{\partial\underline{x}}{\partial\sigma_\nu} \\
&= \int_{\underline{x}}^{\overline{x}}\left\{\frac{I(x)}{\sigma_\nu}\left(1+\mathcal{B}(x)\right)\left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right)\right\}d\Lambda(x) \\
&\quad - \int_{\underline{x}}^{\overline{x}}\frac{I(x)}{\sigma_\nu}\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x) - I(\underline{x})\lambda(\underline{x})\frac{\partial\underline{x}}{\partial\sigma_\nu}.
\end{aligned} \tag{34}$$

Rearranging terms, we have an expression for the change in the participation threshold, weighted by the mass of agents at the threshold, that must be satisfied in order for the risky asset market to clear:

$$\begin{aligned}
\lambda(\underline{x})\frac{\partial\underline{x}}{\partial\sigma_\nu} &= \frac{1}{I(\underline{x})}\int_{\underline{x}}^{\overline{x}}\left\{\frac{I(x)}{\sigma_\nu}\left[1+\mathcal{B}(x)\right]\left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right)\right\}d\Lambda(x) \\
&\quad - \frac{1}{I(\underline{x})}\int_{\underline{x}}^{\overline{x}}\frac{I(x)}{\sigma_\nu}\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x).
\end{aligned}$$

Collecting terms with the elasticities with respect to α and effective volatility $\sigma(x)$, we can write this as:

$$\lambda(\underline{x})\frac{\partial\underline{x}}{\partial\sigma_\nu} = \frac{1}{I(\underline{x})}\int_{\underline{x}}^{\overline{x}}\frac{I(x)}{\sigma_\nu}\left[1+\mathcal{B}(x)\right]\left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}\right)d\Lambda(x)$$
$$-\frac{1}{I(\underline{x})}\int_{\underline{x}}^{\overline{x}}\frac{I(x)}{\sigma_\nu}\left[2+\mathcal{B}(x)\right]\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x). \qquad (35)$$

Therefore, from Equation 35 we have:

$$\frac{\partial\underline{x}}{\partial\sigma_\nu} > 0 \Leftrightarrow$$
$$l_{\text{sup}}^{\sigma_\nu} = \frac{\partial\sigma(\underline{x})/\sigma(\underline{x})}{\partial\sigma_\nu/\sigma_\nu} > \frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} > \frac{\int_{\underline{x}}^{\overline{x}}I(x)\left(2+\mathcal{B}(x)\right)\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}}\left[I(x)\left(1+\mathcal{B}(x)\right)\right]d\Lambda(x)}. \qquad (36)$$

The first inequality comes from the participation condition. We know that $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} < l_{\text{sup}}^{\sigma_\nu}$ when participation decreases and $\underline{x}$ increases. The second inequality is from solving Equation (35) for the elasticity of α with respect to total volatility in the case that the left hand side of that equation is less than zero. Using similar arguments, we also have:

$$\frac{\partial\underline{x}}{\partial\sigma_\nu} < 0 \Leftrightarrow$$
$$l_{\text{sup}}^{\sigma_\nu} = \frac{\partial\sigma(\underline{x})/\sigma(\underline{x})}{\partial\sigma_\nu/\sigma_\nu} < \frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} < \frac{\int_{\underline{x}}^{\overline{x}}I(x)\left(2+\mathcal{B}(x)\right)\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}}\left[I(x)\left(1+\mathcal{B}(x)\right)\right]d\Lambda(x)}. \qquad (37)$$

■

To gain intuition for the conditions in Equations (36) and (37), note that the right hand side is the ratio of the partial equilibrium change in demand with respect to total volatility (holding α and $\underline{x}$ constant), and the partial equilibrium change in demand with respect to α. Consider first the numerator of this ratio. The numerator shows that when σ_ν increases, investors allocate a lower fraction of wealth to the risky asset. We have that $\theta = \frac{\alpha}{\gamma\sigma^2(x)}$. Thus, if all elasticities of effective volatility with respect to total volatility were one, then if σ_ν increased by 1%, investors would allocate 2% less to the risky asset. This leads to the 2 in the numerator. The second term, involving $\beta(x)$ arises because when σ_ν increases, $\beta(x)$ increases, so there is lower total wealth in partial equilibrium. The term

$$\mathcal{B}(x) = \frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}$$

gives the elasticity of the mean wealth at expertise level x with respect to total volatility. The denominator similarly shows how changes in α affects aggregate demand in partial equilibrium, holding $\underline{x}$ and effective volatilities constant. If α increases by 1%, portfolio allocations to the risky asset increase by 1%, and we have the 1 term. Then, we again see the effect of the change in α on mean wealth levels. Now, consider why this ratio matters for the elasticity of α. In Equation (36), participation declines because the effect of the change in aggregate demand from increasing effective volatilities is greater than the effect on demand from increasing α. The reverse is true in Equation (37).

We can bound the elasticity of α with respect to total volatility more tightly in Case 3 as follows:

Lemma A.4 *When* $\frac{\partial\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}}{\partial x} \leq 0$, *in the equilibrium, we have,* *1.*

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} > l_{\text{sup}}^{\sigma_\nu} \text{ if } l_{\text{sup}}^{\sigma_\nu} < \left(1+\frac{1}{1+\mathcal{B}(\overline{x})}\right) l_{\text{inf}}^{\sigma_v}.$$

and

2.

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} < l_{\text{sup}}^{\sigma_\nu} \text{ if } l_{\text{sup}}^{\sigma_\nu} > \left(1+\frac{1}{1+\mathcal{B}(\underline{x})}\right) E\left[\frac{\partial\sigma\left(x\right)/\sigma\left(x\right)}{\partial\sigma_\nu/\sigma_\nu}|x\geq\underline{x}\right],$$

where

$$\mathcal{B}(x) = \frac{1}{\beta\left(x\right)}\frac{2\left(\beta\left(x\right)+\gamma\right)}{\beta\left(x\right)-1}.$$

We show that the percentage change in α for a given percentage change in total volatility will be greater than the highest elasticity of effective volatility with respect to total volatility (displayed by the participating investor with the lowest expertise) if that highest elasticity is less than a constant times the lowest elasticity, displayed by agents with the highest level of expertise. In other words, if the highest elasticity, displayed by the marginal investor, is close enough to the elasticity of the highest expertise agents, it must be that the change in α is enough to compensate that investor. If not, the change in α will not be great enough to satisfy market clearing.

We also show the converse: The percentage change in α for a given percentage change in total volatility will be less than the highest elasticity of effective volatility with respect to total volatility (displayed by the participating investor with the lowest expertise) if that highest elasticity is greater than a constant near one times the average elasticity over participating investors. In other words, if the marginal investor is different enough from the average, the market can clear despite the fact that their individual Sharpe ratio declines, since the other investors face smaller declines, or increasing individual Sharpe ratios.

Note that the constant will be near one if β is close to one, which it will be as it is the tail parameter from a Pareto distribution. In this case, $\beta(x)-1$ goes to zero, and the denominator of the constant, $\mathcal{B}(\underline{x})$, goes to infinity, so that the entire constant becomes (1+0). Note also that we derive a sufficient condition which is based on the wealth distribution of the highest expertise agents, as using the entire distribution, a mixture of Pareto distributions, is more complicated but would yield similar intuition.

Proof. Proof of Lemma A.4 We first derive a lower bound on the ratio of the partial equilibrium change in demand with respect to total volatility (holding α and $\underline{x}$ constant), and the partial equilibrium change in demand with respect to α. This lower bound is the condition bounding $l_{\text{sup}}^{\sigma_\nu}$ in the first statement of Lemma A.4.

$$\begin{aligned}
\frac{\int_{\underline{x}}^{\overline{x}} I(x)\left(2+\mathcal{B}(x)\right)\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda\left(x\right)}{\int_{\underline{x}}^{\overline{x}}\left[I(x)\left(1+\mathcal{B}(x)\right)\right]d\Lambda\left(x\right)} &\geq \frac{\int_{\underline{x}}^{\overline{x}} I(x)\left(2+\mathcal{B}(x)\right)d\Lambda\left(x\right)}{\int_{\underline{x}}^{\overline{x}}\left[I(x)\left(1+\mathcal{B}(x)\right)\right]d\Lambda\left(x\right)} l_{\text{inf}}^{\sigma_v} \\
&= \left(1+\frac{\int_{\underline{x}}^{\overline{x}} I(x)d\Lambda\left(x\right)}{\int_{\underline{x}}^{\overline{x}}\left[I(x)\left(1+\mathcal{B}(x)\right)\right]d\Lambda\left(x\right)}\right) l_{\text{inf}}^{\sigma_v} \\
&\geq \left(1+\frac{\int_{\underline{x}}^{\overline{x}} I(x)d\Lambda\left(x\right)}{\left(1+\mathcal{B}(\overline{x})\right)\int_{\underline{x}}^{\overline{x}} I(x)d\Lambda\left(x\right)}\right) l_{\text{inf}}^{\sigma_v} \\
&= \left(1+\frac{1}{1+\mathcal{B}\left(\overline{x}\right)}\right) l_{\text{inf}}^{\sigma_v}.
\end{aligned}$$

The first inequality follows because $l_{\inf}^{\sigma_v}$ is the lowest elasticity, and it replaces the weighted average elasticity. The next equality groups terms, and the following inequality follows because $\mathcal{B}(x)$ is replaced by $\mathcal{B}(\bar{x})$ and we have that β is decreasing in x. The last equality again groups terms. So, if

$$l_{\sup}^{\sigma_\nu} < \left(1 + \frac{1}{1+\mathcal{B}(\overline{x})}\right) l_{\inf}^{\sigma_v},$$

we have

$$l_{\sup}^{\sigma_\nu} < \left(1 + \frac{1}{\mathcal{B}\left(\overline{x}\right)}\right) l_{\inf}^{\sigma_v} \leq \frac{\int_{\underline{x}}^{\overline{x}} I\left(x\right)\left(2+\mathcal{B}(x)\right) \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} d\Lambda\left(x\right)}{\int_{\underline{x}}^{\overline{x}} \left[I\left(x\right)\left(1+\mathcal{B}(x)\right)\right] d\Lambda\left(x\right)},$$

and this ratio was the bound on the elasticity of α from Lemma A.3. Thus, we have established our tighter bound on the elasticity of α in the first case of Lemma A.4 since we have from Condition 1 in Lemma A.3,

$$l_{\sup}^{\sigma_\nu} = \frac{\partial\sigma\left(\underline{x}\right)/\sigma\left(\underline{x}\right)}{\partial\sigma_\nu/\sigma_\nu} < \frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}.$$

Now, we bound the ratio of the partial equilibrium change in demand with respect to total volatility (holding α and $\underline{x}$ constant), and the partial equilibrium change in demand with respect to α from above. This upper bound is the condition bounding $l_{\sup}^{\sigma_\nu}$ in the second statement of Lemma A.4. Consider:

$$\begin{aligned}
& \int_{\underline{x}}^{\overline{x}} I\left(x\right)\left(2+\mathcal{B}(x)\right) \frac{\partial\sigma\left(x\right)/\sigma\left(x\right)}{\partial\sigma_\nu/\sigma_\nu} d\Lambda\left(x\right) \\
\leq & \int_{\underline{x}}^{\overline{x}} I\left(x\right)\left(2+\mathcal{B}(x)\right) d\Lambda\left(x\right) \frac{\int_{\underline{x}}^{\overline{x}} \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} d\Lambda\left(x\right)}{1-\Lambda\left(\underline{x}\right)} \\
= & \int_{\underline{x}}^{\overline{x}} I\left(x\right)\left(2+\mathcal{B}(x)\right) d\Lambda\left(x\right) E\left[\frac{\partial\sigma\left(x\right)/\sigma\left(x\right)}{\partial\sigma_\nu/\sigma_\nu} | x \geq \underline{x}\right].
\end{aligned}$$

The first inequality follows because the integral $\int_{\underline{x}}^{\overline{x}} I(x)\left(2+\mathcal{B}(x)\right) \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} d\Lambda\left(x\right)$ puts more weight on the average elasticity of wealth times investment relative to total volatility when $\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}$ is small and less weight when $\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}$ is large, whereas the term after the inequality sign uses an equal weighted average of all effective volatilities. The last equality groups terms. We have now bounded above the numerator in the ratio of the partial equilibrium change in demand with respect to total volatility (holding α and $\underline{x}$ constant), and the partial equilibrium change in demand with respect to α. If we divide through by the partial equilibrium change in demand with respect to α, we have, using similar logic as before:

$$
\begin{aligned}
& \frac{\int_{\underline{x}}^{\overline{x}} I(x)\left(2+\mathcal{B}(x)\right) \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}} \left[I(x)\left(1+\mathcal{B}(x)\right)\right] d\Lambda(x)} \\
\leq\ & \frac{\int_{\underline{x}}^{\overline{x}} I(x)\left(2+\mathcal{B}(x)\right) d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}} \left[I(x)\left(1+\mathcal{B}(x)\right)\right] d\Lambda(x)} E\left[\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} | x \geq \underline{x}\right] \\
=\ & \left(1+\frac{\int_{\underline{x}}^{\overline{x}} I(x)\, d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}} \left[I(x)\left(1+\mathcal{B}(x)\right)\right] d\Lambda(x)}\right) E\left[\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} | x \geq \underline{x}\right] \\
\leq\ & \left(1+\frac{\int_{\underline{x}}^{\overline{x}} I(x)\, d\Lambda(x)}{\left(1+\mathcal{B}(\underline{x})\right)\int_{\underline{x}}^{\overline{x}} I(x)\, d\Lambda(x)}\right) E\left[\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} | x \geq \underline{x}\right] \\
=\ & \left(1+\frac{1}{1+\mathcal{B}(\underline{x})}\right) E\left[\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} | x \geq \underline{x}\right].
\end{aligned}
$$

So, if

$$
l_{\text{sup}}^{\sigma_\nu} > \left(1+\frac{1}{1+\mathcal{B}(\underline{x})}\right) E\left[\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} | x \geq \underline{x}\right],
$$

we have

$$
l_{\text{sup}}^{\sigma_\nu} > \left(1+\frac{1}{1+\mathcal{B}(\underline{x})}\right) E\left[\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} | x \geq \underline{x}\right] \geq \frac{\int_{\underline{x}}^{\overline{x}} I(x)\left(2+\mathcal{B}(x)\right) \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}} \left[I(x)\left(1+\mathcal{B}(x)\right)\right] d\Lambda(x)}.
$$

Then, from Condition 2 in Lemma A.3 we can get

$$
l_{\text{sup}}^{\sigma_\nu} = \frac{\partial\sigma(\underline{x})/\sigma(\underline{x})}{\partial\sigma_\nu/\sigma_\nu} > \frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}.
$$

■

We now show conditions under which participation increases, i.e. under which the cutoff level of expertise for participation $\underline{x}$ declines, as total volatility increases. In particular, we show that participation increases with total volatility in Cases 1 and 2 of Proposition 3.4, but only under a tight restriction in Case 3. In Case 3, participation only increases if the elasticity of the effective volatility of the lowest expertise investor is not too different from that of the average participating investor. In other words, participation increases if there is very little difference across expertise levels in the effect of changes in total volatility on effective volatility, so that elasticities are nearly constant, as in Case 1, in which participation always increases as total volatility increases. Notice that the condition restricting the differences in elasticities across investors is the same as Condition 1 in Lemma A.4, which bounds the change in α from below. Thus, participation will increase only if the change in α is large enough, which will be the case if all participating investors face similar changes to their effective volatility as total volatility changes. We discuss the more empirically relevant case for Case 3 of Proposition 3.4, in which participation declines as total volatility and asset complexity increase, in the text. Recall that the condition for declining participation requires that elasticities vary enough across high expertise and low expertise agents, so that market clearing does not require the participation of lower expertise agents.

Proposition A.1 *Define the entry cutoff* $\underline{x}$*,*

$$
\underline{x} = \sigma^{-1}\left(\frac{\alpha}{\sqrt{2\gamma f_{xx}}}\right),
$$

where $\sigma^{-1}(\cdot)$ is the inverse function of $\sigma(x)$. We have that participation increases with total volatility,

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} < 0$$

if the following conditions hold

1. $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} \geq 0$, *(Proposition 3.4 Cases 1 and 2) or*

2. $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} < 0$, *(Proposition 3.4 Case 3) and* $l^{\sigma_\nu}_{\text{sup}} < \left(1 + \frac{1}{1+\mathcal{B}(\bar{x})}\right) l^{\sigma_\nu}_{\text{inf}}$.

Proposition A.1 shows that participation increases in Cases 1 and 2 as total volatility increases. The reason is that demand for the complex asset by incumbent experts declines, and new wealth must be brought into the market to clear the fixed supply. However, in Case 3, a tight restriction is required for participation to increase. The restriction is that there is a very small difference between the highest and lowest elasticities. The restriction is tight since β is the tail parameter of a pareto distribution, so that $\beta \approx 1\, \forall x$. Then, given the definition of $\mathcal{B}(x)$, this implies that $\mathcal{B}(\bar{x}) \to \infty$. Intuitively, if the increase in total volatility adversely impacts low expertise agents much more than high expertise agents, then high expertise agents can clear the market without the demand from low expertise investors with substantially deteriorated individual Sharpe ratios.

Proof. Proof of Proposition A.1 First,

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} < 0 \text{ iff } \frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\nu} > 0.$$

We have

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\nu} = 2\left(\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \sigma_\nu/\sigma_\nu}\right).$$

Therefore

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\nu} > 0 \text{ iff } \frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} > \frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \sigma_\nu/\sigma_\nu}.$$

If $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} \geq 0$, from Proposition A.2 we have

$$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} > l^{\sigma_\nu}_{\text{inf}} = \frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \sigma_\nu/\sigma_\nu}.$$

If $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} < 0$ and $l^{\sigma_\nu}_{\text{sup}} < \left(1 + \frac{1}{1+\mathcal{B}(\bar{x})}\right) l^{\sigma_\nu}_{\text{inf}}$, from Lemma A.3, we know

$$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} > l^{\sigma_\nu}_{\text{sup}} = \frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \sigma_\nu/\sigma_\nu}.$$

■

Proof. Proof of Proposition 3.5 First,

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} > 0 \text{ iff } \frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\nu} < 0.$$

We have

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\nu} = 2\left(\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \sigma_\nu/\sigma_\nu}\right).$$

Therefore

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\nu} < 0 \text{ iff } \frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} < l_{\text{sup}}^{\sigma_\nu} = \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_\nu / \sigma_\nu}.$$

If $\frac{\partial \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}}{\partial x} < 0$ and $l_{\text{sup}}^{\sigma_\nu} > \left(1 + \frac{1}{1+\mathcal{B}(\underline{x})}\right) E\left[\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} | x \geq \underline{x}\right]$, from Lemma A.3, we know

$$\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} < l_{\text{sup}}^{\sigma_\nu} = \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_\nu / \sigma_\nu}.$$

■

We note that the conditions in Proposition A.1 and Proposition 3.5 are sufficient, but not necessary. As discussed in the main text, we use the tail parameters for the highest and lowest expertise levels since the entire wealth distribution is a mixture of Pareto distributions (a complicated object). We also note that the conditions for increasing vs. decreasing participation in Case 3 are not overlapping, because

$$\left(1 + \frac{1}{1+\mathcal{B}(\overline{x})}\right) l_{\text{inf}}^{\sigma_\nu} \leq \left(1 + \frac{1}{1+\mathcal{B}(\underline{x})}\right) E\left[\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} | x \geq \underline{x}\right].$$

Proof. Proof of Proposition 3.6

For convenience, we consider the partial derivative of the positive transformation $\log SR^{ew}$, namely, the log of the integral over all participants' Sharpe ratios divided by the measure of participants:

$$\begin{aligned} \log SR^{ew} &= \log \frac{\int_{\underline{x}}^{\overline{x}} \frac{\alpha}{\sigma(x)} d\Lambda(x)}{1 - \Lambda(\underline{x})} \\ &= \log \int_{\underline{x}}^{\overline{x}} \frac{\alpha}{\sigma(x)} d\Lambda(x) - \log[1 - \Lambda(\underline{x})] \end{aligned}$$

As σ_ν changes, the equilibrium equally weighted market-level Sharpe ratio will change from several effects. First, each individual effective volatility will increase, according to the elasticity of effective volatility with respect to total volatility at each expertise level. To clear the market, the equilibrium α will increase. Finally, participation will change. Taking these effects together, the change in the equally weighted market-level Sharpe ratio will be the change in the individual Sharpe ratios of each expertise level of investors, weighted by their mass in the distribution of expertise, plus the effect on participation. By direct calculation, we have:

$$\begin{aligned} &\frac{\partial \log SR^{ew}}{\partial \sigma_\nu} \\ = &\frac{\frac{\alpha}{\sigma_\nu} \int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} \left(\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}\right) d\Lambda(x) - \frac{\alpha}{\sigma(\underline{x})} \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu}}{\int_{\underline{x}}^{\overline{x}} \frac{\alpha}{\sigma(x)} d\Lambda(x)} + \frac{1}{1 - \Lambda(\underline{x})} \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu}, \end{aligned}$$

where the first term gives the weighted average change in individual Sharpe ratios, and the second term gives the effect on participation. Note the appearance of the expertise-specific differences in the elasticity of alpha and effective volatility with respect to total volatility. Individual Sharpe ratios increase for individuals for which this difference in elasticities is positive. The larger these differences are on average, then, the more likely it is that the market Sharpe ratio will increase. The sign of $\frac{\partial \underline{x}}{\partial \sigma_\nu}$ depends on which case of Proposition 3.4 applies. Collecting terms, and cancelling α in the numerator and denominator of the first term, we have:

$$\frac{\partial \log SR^{ew}}{\partial \sigma_\nu}$$

$$= \frac{1}{\sigma_\nu} \frac{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} \left(\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} \right) d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} + \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu} \left\{ \frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\}. \quad (38)$$

Equation (38) contains two changes in equilibrium outcomes, α and $\underline{x}$. The other variables depend only on parameters, or distribution or functional form assumptions. The market clearing condition can be used to eliminate one of the equilibrium outcomes. In particular, we know that aggregate investment must be unchanged and equated to the aggregate supply of the risky asset. Using Equation (33) for the change in expertise-level investment with total volatility and accounting for the change in participation we then have for aggregate investment it must be that:

$$\begin{aligned} 0 &= \frac{\partial I}{\partial \sigma_\nu} = \int_{\underline{x}}^{\overline{x}} \frac{\partial I(x)}{\partial \sigma_\nu} d\Lambda(x) - I(\underline{x}) \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu} \\ &= \int_{\underline{x}}^{\overline{x}} \left\{ \frac{I(x)}{\sigma_\nu} (1+\mathcal{B}(x)) \left(\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} \right) \right\} d\Lambda(x) \\ &\quad - \int_{\underline{x}}^{\overline{x}} \frac{I(x)}{\sigma_\nu} \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x) - I(\underline{x}) \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu}. \end{aligned} \quad (39)$$

Rearranging terms, we have an expression for the change in the participation threshold, weighted by the mass of agents at the threshold, that must be satisfied in order for the risky asset market to clear:

$$\begin{aligned} \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu} &= \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\overline{x}} \left\{ \frac{I(x)}{\sigma_\nu} (1+\mathcal{B}(x)) \left(\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} \right) \right\} d\Lambda(x) \\ &\quad - \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\overline{x}} \frac{I(x)}{\sigma_\nu} \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x). \end{aligned} \quad (40)$$

Collecting terms with the elasticities with respect to α and effective volatility $\sigma(x)$, we can write this as:

$$\lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu} \quad (41)$$

$$= \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\overline{x}} \frac{I(x)}{\sigma_\nu} (1+\mathcal{B}(x)) \frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x) - \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\overline{x}} \frac{I(x)}{\sigma_\nu} (2+\mathcal{B}(x)) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)$$

Equation (41) essentially states that the change in demand for the complex asset from the change in participation must be met by an offsetting change in demand by the remainder of participants in the complex risky asset market. There are two parts to the change in demand of participants above the threshold. First, there is an increase in demand of 1% for every 1% increase in α, multiplied by the elasticity of wealth with respect to changes in volatility. Second, there is a decrease in demand of 2% for every 1% increase in effective volatility, again weighted by the elasticity of wealth with respect to changes in volatility. Grouping terms, and then plugging this expression for $\lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu}$ into $\frac{\partial \log SR^{ew}}{\partial \sigma_\nu}$, we obtain

$$
\begin{aligned}
&\frac{\partial \log SR^{ew}}{\partial \sigma_\nu} \\
= \quad & \frac{1}{\sigma_\nu} \frac{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} \left(\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} \right) d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} + \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu} \left\{ \frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \\
= \quad & \frac{1}{\sigma_\nu} \left(\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right) + \lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_\nu} \left\{ \frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \\
= \quad & \frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} \left\{ \frac{1}{\sigma_\nu} + \left\{ \frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \frac{1}{\sigma_\nu I(\underline{x})} \int_{\underline{x}}^{\overline{x}} I(x)(1+\mathcal{B}(x)) d\Lambda(x) \right\} \\
& - \left\{ \frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \frac{1}{\sigma_\nu I(\underline{x})} \int_{\underline{x}}^{\overline{x}} I(x)(2+\mathcal{B}(x)) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x) \\
& - \frac{1}{\sigma_\nu} \frac{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)}.
\end{aligned}
$$

Therefore, $\frac{\partial \log SR^{ew}}{\partial \sigma_\nu} > 0$ if and only if

$$
\begin{aligned}
& \frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} \\
> \quad & \frac{\left\{ \frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\overline{x}} I(x)(2+\mathcal{B}(x)) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x) + \frac{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)}}{\left\{ \frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \frac{1}{I(\underline{x})} \int_{\underline{x}}^{\overline{x}} \left[I(x)(1+\mathcal{B}(x)) \right] d\Lambda(x) + 1}.
\end{aligned}
$$

We divide through by $\frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)}$ and rearrange to get an expression that will allow us to more easily bound the elasticity of α. We then have $\frac{\partial \log SR^{ew}}{\partial \sigma_\nu} > 0$ if and only if

$$
\begin{aligned}
& \frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} \\
> \quad & \frac{\int_{\underline{x}}^{\overline{x}} I(x)(2+\mathcal{B}(x)) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x) + \frac{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \frac{1}{\left\{ \frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \frac{1}{I(\underline{x})}}}{\int_{\underline{x}}^{\overline{x}} \left[I(x)(1+\mathcal{B}(x)) \right] d\Lambda(x) + \frac{1}{\left\{ \frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \frac{1}{I(\underline{x})}}}. \qquad (42)
\end{aligned}
$$

Now, in order to derive an exact expression for α as a function of parameters, consider again the term $\frac{\partial \underline{x}}{\partial \sigma_\nu}$. The participation condition in Equation (18) can be used to show that the individual Sharpe ratio of the marginal agent is a constant function of the coefficient of relative risk aversion and the participation maintenance cost. Then, using this condition, $\frac{\alpha^2}{\sigma^2(\underline{x})} = 2\gamma f_{xx}$, we can get another equation relating the change in $\underline{x}$ to the change in α as follows:

$$F(\sigma_\nu, \underline{x}) = 2\log\alpha - 2\log\sigma(\underline{x}) - \log(2\gamma f_{xx}) = 0.$$

Using the implicit function theorem, we obtain another expression for the change in the participation threshold as total volatility changes:

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} = -\frac{\partial F\left(\sigma_\nu, \underline{x}\left(\sigma_\nu\right)\right)/\partial \sigma_\nu}{\partial F\left(\sigma_\nu, \underline{x}\right)/\partial \underline{x}} = \frac{1}{\sigma_\nu} \frac{\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \sigma_\nu/\sigma_\nu}}{\frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \underline{x}}}, \tag{43}$$

where we see the difference in the elasticity of alpha and effective volatility with respect to total volatility for the marginal agent in the numerator. Since effective volatility is decreasing in expertise, the effective volatility for the marginal agent decreases as $\underline{x}$ increases, and the denominator is negative. Thus, participation declines and $\underline{x}$ increases in total volatility the smaller the elasticity of α is relative to the elasticity of the effective volatility of the marginal agent. Then, from Equations (41) and (43), we have another expression for the elasticity of α with respect to changes in total volatility:

$$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} = \frac{\int_{\underline{x}}^{\overline{x}} I\left(x\right)\left(2+\mathcal{B}(x)\right) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda\left(x\right) - \frac{I(\underline{x})\lambda(\underline{x})}{\frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \underline{x}}} \frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \sigma_\nu/\sigma_\nu}}{\int_{\underline{x}}^{\overline{x}} I\left(x\right)\left(1+\mathcal{B}(x)\right) d\Lambda\left(x\right) - \frac{I(\underline{x})\lambda(\underline{x})}{\frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \underline{x}}}}. \tag{44}$$

Now, from Equation (42) we have a lower bound on what the elasticity of α must be in order for $\frac{\partial \log SR^{ew}}{\partial \sigma_\nu} > 0$. From Equation (44) we have an expression for α. Thus, we have that $\frac{\partial \log SR^{ew}}{\partial \sigma_\nu} > 0$ if and only if:

$$\frac{\int_{\underline{x}}^{\overline{x}} I\left(x\right)\left(2+\mathcal{B}(x)\right) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda\left(x\right) - \frac{I(\underline{x})\lambda(\underline{x})}{\frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \underline{x}}} \frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \sigma_\nu/\sigma_\nu}}{\int_{\underline{x}}^{\overline{x}} I\left(x\right)\left(1+\mathcal{B}(x)\right) d\Lambda\left(x\right) - \frac{I(\underline{x})\lambda(\underline{x})}{\frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \underline{x}}}}$$

$$> \frac{\int_{\underline{x}}^{\overline{x}} I\left(x\right)\left(2+\mathcal{B}(x)\right) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda\left(x\right) + \frac{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \frac{1}{\left\{\frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)}\right\} \frac{1}{I(\underline{x})}}}{\int_{\underline{x}}^{\overline{x}} I\left(x\right)\left(1+\mathcal{B}(x)\right) d\Lambda\left(x\right) + \frac{1}{\left\{\frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)}\right\} \frac{1}{I(\underline{x})}}}. \tag{45}$$

To continue the proof, we define a simplified notation for this condition, namely we write that $\frac{\partial \log SR^{ew}}{\partial \sigma_\nu} > 0$ if and only if:

$$\frac{a+c}{b+d} > \frac{a+e}{b+f}, \tag{46}$$

where we define:

$$
\begin{aligned}
a &= \int_{\underline{x}}^{\overline{x}} I(x)\,(2+\mathcal{B}(x))\,\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x)\,,\\
b &= \int_{\underline{x}}^{\overline{x}} I(x)\,(1+\mathcal{B}(x))\,d\Lambda(x)\,,\\
c &= -\frac{I(\underline{x})\,\lambda(\underline{x})}{\frac{\partial\sigma(\underline{x})/\sigma(\underline{x})}{\partial\underline{x}}}\frac{\partial\sigma(\underline{x})/\sigma(\underline{x})}{\partial\sigma_\nu/\sigma_\nu},\\
d &= -\frac{I(\underline{x})\,\lambda(\underline{x})}{\frac{\partial\sigma(\underline{x})/\sigma(\underline{x})}{\partial\underline{x}}},\\
e &= \frac{\int_{\underline{x}}^{\overline{x}}\frac{1}{\sigma(x)}\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}}\frac{1}{\sigma(x)}d\Lambda(x)}\frac{1}{\left\{\frac{1}{1-\Lambda(\underline{x})}-\frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}}\frac{1}{\sigma(x)}d\Lambda(x)}\right\}\frac{1}{I(\underline{x})}},\\
f &= \frac{1}{\left\{\frac{1}{1-\Lambda(\underline{x})}-\frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}}\frac{1}{\sigma(x)}d\Lambda(x)}\right\}\frac{1}{I(\underline{x})}}.
\end{aligned}
$$

Comparing the right and left hand sides of the condition under which the equally weighted market Sharpe ratio is increasing,

$$\frac{a+c}{b+d} > \frac{a+e}{b+f},$$

note that the difference is that the left side essentially includes $\frac{c}{d}$, while the right hand side includes $\frac{e}{f}$, while both sides include $\frac{a}{b}$. Recall from the proof of Lemma A.3, and Equations (36) and (37), that $\frac{a}{b}$ is the ratio of the partial equilibrium change in demand with respect to total volatility (holding α and $\underline{x}$ constant), and the partial equilibrium change in demand with respect to α. To gain further intuition for our condition, we can use the expressions for $\frac{c}{d}$ and $\frac{e}{f}$, and simplify to show that:

$$\frac{c}{d} = \frac{\partial\sigma(\underline{x})/\sigma(\underline{x})}{\partial\sigma_\nu/\sigma_\nu} = \frac{\partial\sigma(\underline{x})/\sigma(\underline{x})}{\partial\sigma_\nu/\sigma_\nu} \text{ and } \frac{e}{f} = \frac{\int_{\underline{x}}^{\overline{x}}\frac{1}{\sigma(x)}\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}}\frac{1}{\sigma(x)}d\Lambda(x)}.$$

Thus, in loose terms what is needed for the equally weighted market Sharpe ratio to increase is that the elasticity of the marginal agent is sufficiently larger than the weighted average over all elasticities, where the weights are increasing in expertise. This condition essentially ensures that α increases by enough to outweigh the increase in effective volatilities of agents with lower expertise levels who still choose to participate. We begin by describing sufficient conditions for

$$\frac{a+c}{b+d} > \frac{a+e}{b+f},$$

in the three cases of Proposition 3.4, and depending on whether participation is increasing or decreasing in total volatility. Then, using these results, we derive a sufficient condition covering all cases.

Case 1 of Proposition 3.4, and Case 3 of Proposition 3.4 with Increasing Participation: First, we have

$$\frac{e}{f} \leq \frac{c}{d} < \frac{a}{b}.$$

To see this, note that in Case 1 of Proposition 3.4, elasticities are constant in x and so $\frac{e}{f} = \frac{c}{d}$. The last inequality follows from the fact that when participation is increasing, Equation (37) from the proof of Lemma A.3 shows that $\frac{a}{b}$ is larger than the elasticity of the marginal agent, and hence larger than the elasticity of all agents. In Case 3, we have declining elasticities, generating the first inequality. The last inequality again follows from Equation (37), which holds when participation increases.

We can show that $\frac{a+c}{b+d} > \frac{a+e}{b+f}$ if

$$d < f.$$

This is true because

$$\begin{aligned}
\frac{a+c}{b+d} &> \frac{a+e}{b+f} \Leftrightarrow (a+c)(b+f) > (a+e)(b+d) \\
&\Leftarrow af + cb > ad + be \\
&\Leftrightarrow a(f-d) > b(e-c) \\
&\Leftarrow f > d \text{ and } \frac{a}{b} > \frac{e-c}{f-d}
\end{aligned}$$

We relate the condition that $d < f$ to underlying parameters and collect the results for all Cases of Proposition 3.4 at the end of the proof.

Case 2 of Proposition 3.4: First, we prove that

$$\frac{a}{b} = \frac{\int_{\underline{x}}^{\overline{x}} I(x)\left(2+\mathcal{B}(x)\right)\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}}\left[I(x)\left(1+\mathcal{B}(x)\right)\right]d\Lambda(x)} > \frac{\int_{\underline{x}}^{\overline{x}}\frac{1}{\sigma(x)}\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}}\frac{1}{\sigma(x)}d\Lambda(x)} = \frac{e}{f}.$$

To show this, Consider the function:

$$\begin{aligned}
H(K(x)) &= \frac{\int_{\underline{x}}^{\overline{x}} K(x)\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}} K(x)\,d\Lambda(x)} \\
&= \int_{\underline{x}}^{\overline{x}}\frac{K(x)\,d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}} K(x)\,d\Lambda(x)}\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}dx.
\end{aligned}$$

This function, $H(K(x))$, represents a "weighted" average of elasticities, with weights given by $K(x)$. Suppose we have

$$K_1(x) = K_2(x)\,K_3(x),$$

where we assume that K_3 is monotonically increasing in x. Because we have that $\frac{\partial\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}}{\partial x} \geq 0$, and K_3 is monotonically increasing in x, we have that the weights $K_1(x)$ underweight lower elasticities and overweight higher elasticities relative to the weights $K_2(x)$. Therefore,

$$H(K_1(x)) \geq H(K_2(x)).$$

Note that we can write:

$$\underbrace{I(x)\left(1+\mathcal{B}(x)\right)}_{K_1(x)} = \overbrace{\frac{1}{\sigma(x)}}^{K_2(x)}\underbrace{\left[I(x)\,\sigma(x)\left(1+\mathcal{B}(x)\right)\right]}_{K_3(x)},$$

and further we have that:

$$I(x)\,\sigma(x)\left(1+\mathcal{B}(x)\right) = \frac{\alpha}{\gamma\sigma(x)}Z(x)\left(1+\mathcal{B}(x)\right),$$

which is monotonic increasing, as assumed for $K_3(x)$. Thus, we have:

$$\frac{\int_{\underline{x}}^{\overline{x}} I(x)\left(2+\mathcal{B}(x)\right)\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}}\left[I(x)\left(1+\mathcal{B}(x)\right)\right]d\Lambda(x)} > \frac{\int_{\underline{x}}^{\overline{x}} I(x)\left(1+\mathcal{B}(x)\right)\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}}\left[I(x)\left(1+\mathcal{B}(x)\right)\right]d\Lambda(x)} > \frac{\int_{\underline{x}}^{\overline{x}}\frac{1}{\sigma(x)}\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}d\Lambda(x)}{\int_{\underline{x}}^{\overline{x}}\frac{1}{\sigma(x)}d\Lambda(x)}, \tag{47}$$

where the first inequality is because we replace a one with a two in the numerator, and the second inequality is because we showed that $H(K_1) > H(K_2)$.

So, we have $\frac{c}{d} < \frac{e}{f} < \frac{a}{b}$. The first inequality is because in Case 2 of Proposition 3.4, because elasticities increase with expertise, the lowest expertise agent has the smallest elasticity with respect to total volatility. Thus, $\frac{c}{d} = l_{\text{inf}}^{\sigma_\nu}$. The second inequality follows from Equation (47).

Using these results, we can derive the following inequalities

$$\frac{a+e}{b+f} < \frac{a}{b}, \text{ and } \frac{a+c}{b+d} < \frac{a}{b},$$

Therefore, we have $\frac{a+c}{b+d} > \frac{a+e}{b+f}$ if

$$d << f.$$

This is true mathematically because if d is small enough relative to f, then the value of $\frac{a+c}{b+d}$ will be closer to $\frac{a}{b}$ than $\frac{a+e}{b+f}$ will be.

Case 3 of Proposition 3.4 with Decreasing Participation. In Case 3 of Proposition 3.4 we have that $\frac{c}{d} > \frac{e}{f}$. Because in Case 3 elasticities decline with expertise, the lowest expertise agent has the largest elasticity with respect to total volatility. Thus, $\frac{c}{d} = l_{\text{sup}}^{\sigma_\nu}$, which will be larger than the weighted average $\frac{e}{f}$. We also have, then, that $\frac{c}{d} > \frac{a}{b}$ from Equation (36). Given these relationships, we proceed to show that a sufficient condition for

$$\frac{a+c}{b+d} > \frac{a+e}{b+f}$$

is that

$$d > f.$$

We consider each case for $\frac{a}{b}$ vs. $\frac{e}{f}$ separately, since we do not know their relative values in Case 3:

1. First, if $\frac{e}{f} \leq \frac{a}{b} < \frac{c}{d}$, then

   $$\frac{a+e}{b+f} \leq \frac{a}{b}, \text{ and } \frac{a+c}{b+d} > \frac{a}{b},$$

   thus

   $$\frac{a+c}{b+d} > \frac{a+e}{b+f},$$

   without any restriction on d relative to f.

2. Second, if $\frac{c}{d} > \frac{e}{f} > \frac{a}{b}$ and $d > f$, we have

   $$\frac{a}{b} < \frac{e}{f} < \frac{c-e}{d-f},$$

   where the second inequality is from the following algebraic argument:

   $$\frac{e}{f} < \frac{c-e}{d-f} \Leftrightarrow ed < cf \Leftrightarrow \frac{c}{d} > \frac{e}{f}.$$

Then, we have, using these results and our assumption that $d > f$:

$$\begin{aligned} & d > f \text{ and } \frac{a}{b} < \frac{c-e}{d-f} \\ \Rightarrow \quad & a(f-d) > b(e-c) \\ \Leftrightarrow \quad & af + cb > ad + be \\ \Rightarrow \quad & af + cb + cf > ad + be + ed \\ \Leftrightarrow \quad & (a+c)(b+f) > (a+e)(b+d) \\ \Leftrightarrow \quad & \frac{a+c}{b+d} > \frac{a+e}{b+f}. \end{aligned}$$

Combining Conditions for Cases 1-3 of Proposition 3.4 In Case 1 and Case 3 with increasing participation, we need

$$d < f.$$

In Case 3 with decreasing participation, we need

$$d > f.$$

Combined with participation effects, we can use one condition as following

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} d > f \frac{\partial \underline{x}}{\partial \sigma_\nu}.$$

That is

$$\begin{aligned} & -\frac{\partial \underline{x}}{\partial \sigma_\nu} \frac{I(\underline{x})\lambda(\underline{x})}{\frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \underline{x}}} > \frac{1}{\left\{ \frac{1}{1-\Lambda(\underline{x})} - \frac{\frac{1}{\sigma(\underline{x})}}{\int_{\underline{x}}^{\overline{x}} \frac{1}{\sigma(x)} d\Lambda(x)} \right\} \frac{1}{I(\underline{x})}} \frac{\partial \underline{x}}{\partial \sigma_\nu} \\ \Leftrightarrow \quad & -\frac{\partial \underline{x}}{\partial \sigma_\nu} \frac{\lambda(\underline{x})}{\frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \underline{x}}} > \frac{1}{\frac{1}{1-\Lambda(\underline{x})} \left\{ 1 - \frac{\frac{1}{\sigma(\underline{x})}}{E\left[\frac{1}{\sigma(x)} | x \geq \underline{x}\right]} \right\}} \frac{\partial \underline{x}}{\partial \sigma_\nu} \\ \Leftrightarrow \quad & \frac{\partial \underline{x}}{\partial \sigma_\nu} \frac{\lambda(\underline{x})}{1-\Lambda(\underline{x})} \left\{ 1 - \frac{1}{E\left[\frac{\sigma(\underline{x})}{\sigma(x)} | x \geq \underline{x}\right]} \right\} > -\frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \underline{x}} \frac{\partial \underline{x}}{\partial \sigma_\nu}. \end{aligned}$$

Next, we can rearrange some terms using the equations below

$$\begin{aligned} \frac{\partial \log[1-\Lambda(\underline{x})]}{\partial \log \sigma_\nu} &= -\frac{\lambda(\underline{x})}{1-\Lambda(\underline{x})} \frac{\partial \underline{x}}{\partial \log \sigma_\nu}, \\ E\left[\frac{\sigma(\underline{x})}{\sigma(x)} | x \geq \underline{x}\right] &= \frac{E\left[\frac{\alpha}{\sigma(x)} | x \geq \underline{x}\right]}{\frac{\alpha}{\sigma(\underline{x})}} = \frac{SR^{ew}}{SR(\underline{x})}, \end{aligned}$$

And from Equation (43), we have

$$\begin{aligned} \frac{\partial \log SR(\underline{x})}{\partial \log \sigma_\nu} &= \frac{\partial \log \alpha}{\partial \log \sigma_\nu} - \frac{\partial \log \sigma(\underline{x})}{\partial \log \sigma_\nu} \\ &= \frac{\partial \underline{x}}{\partial \log \sigma_\nu} \frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \underline{x}}. \end{aligned}$$

Thus,

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} \frac{\lambda(\underline{x})}{1-\Lambda(\underline{x})} \left\{ 1 - \frac{1}{E\left[\frac{\sigma(\underline{x})}{\sigma(x)} | x \geq \underline{x}\right]} \right\} > -\frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \underline{x}} \frac{\partial \underline{x}}{\partial \sigma_\nu}$$

$$\Leftrightarrow \quad \underbrace{-\frac{\partial \log[1-\Lambda(\underline{x})]}{\partial \log \sigma_\nu}}_{\text{elasticity of participation}} \underbrace{\left(1 - \frac{SR(\underline{x})}{SR^{ew}}\right)}_{\text{difference between marginal agent and average}} > - \underbrace{\frac{\partial \log SR(\underline{x})}{\partial \log \sigma_\nu}}_{\text{elasticity of marginal SR}}$$

■

Proof. Proof of Proposition 4.1 Suppose we have two risky assets with different total volatility, σ_ν^H and σ_ν^L. Because each agent can only invest in one risky asset, we only need to compare the value function of investing in σ_ν^H versus the value function of investing in σ_ν^L. Suppressing the notation ν, denote the effective volatilities for each asset as $\sigma^H(x)$ and $\sigma_L(x)$.

From the Proposition 2.1, in stationary equilibrium, we have

$$\begin{aligned} V^n(w,x) &= y^n(x) \frac{w^{1-\gamma}}{1-\gamma}, \\ V_H^x(w,x) &= y_H^x(x) \frac{w^{1-\gamma}}{1-\gamma}, \\ V_L^x(w,x) &= y_L^x(x) \frac{w^{1-\gamma}}{1-\gamma}, \end{aligned}$$

where $y^n(x)$, $y_H^x(x)$ and $y_L^x(x)$ are given by:

$$\begin{aligned} y_H^x(x) &= \left[\frac{(\gamma-1)(r_f - f_{xx}) + \rho}{\gamma} + \frac{(\gamma-1)\alpha_H^2}{2\gamma^2 \sigma_H^2(x)}\right]^{-\gamma} \\ y_L^x(x) &= \left[\frac{(\gamma-1)(r_f - f_{xx}) + \rho}{\gamma} + \frac{(\gamma-1)\alpha_L^2}{2\gamma^2 \sigma_L^2(x)}\right]^{-\gamma} \\ y^n(x) &= \left[\frac{(\gamma-1)r_f + \rho}{\gamma}\right]^{-\gamma}. \end{aligned}$$

Consider an equilibrium in which all higher expertise investors, $x \geq \hat{x}$, invests in σ_ν^H, medium expertise, $\underline{x} \leq x < \hat{x}$, invests in σ_ν^L, and low expertise, $\underline{x} > x$, invests in risk free asset only. Then, the following conditions must be satisfied,

$$\begin{aligned} \frac{\alpha_H^2}{\sigma_H^2(x)} &\geq \frac{\alpha_L^2}{\sigma_L^2(x)}, \text{ for all } x \geq \hat{x}, \\ \frac{\alpha_H^2}{\sigma_H^2(x)} &\leq \frac{\alpha_L^2}{\sigma_L^2(x)}, \text{ for all } \underline{x} \leq x < \hat{x}, \\ \frac{(\gamma-1)\alpha_L^2}{2\gamma^2\sigma^2(\underline{x})} &= f_{xx}. \end{aligned}$$

Since $\hat{x}$ is indifferent between investing in σ_ν^H and σ_ν^L, the equilibrium α's must satisfy

$$\alpha_H = \frac{\sigma_H(\hat{x})}{\sigma_L(\hat{x})} \alpha_L = \left(1 + \frac{\sigma_H(\hat{x}) - \sigma_L(\hat{x})}{\sigma_L(\hat{x})}\right) \alpha_L.$$

It is straightforward to further show that:

$$\begin{aligned} \frac{\alpha_H^2}{\sigma_H^2(x)} &\geq \frac{\alpha_L^2}{\sigma_L^2(x)}, \text{ for all } x \geq \hat{x} \Leftrightarrow \frac{\frac{\sigma_H(x)-\sigma_L(x)}{\sigma_L(x)}}{\sigma_H - \sigma_L} \leq \frac{\frac{\alpha_H-\alpha_L}{\alpha_L}}{\sigma_H - \sigma_L}, \\ \frac{\alpha_H^2}{\sigma_H^2(x)} &\leq \frac{\alpha_L^2}{\sigma_L^2(x)}, \text{ for all } \underline{x} \leq x < \hat{x} \Leftrightarrow \frac{\frac{\sigma_H(x)-\sigma_L(x)}{\sigma_L(x)}}{\sigma_H - \sigma_L} \geq \frac{\frac{\alpha_H-\alpha_L}{\alpha_L}}{\sigma_H - \sigma_L}, \end{aligned}$$

which is true if the elasticity of effective volatility is decreasing, i.e. if

$$\frac{\partial \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}}{\partial x} < 0.$$

The model can be solved in an analogous way to each single asset economy. There are now four equations and four unknowns. The participation conditions are:

$$\begin{aligned} \frac{\alpha_H^2}{\sigma_H^2(\hat{x})} &= \frac{\alpha_L^2}{\sigma_L^2(\hat{x})}, \\ \frac{(\gamma-1)\,\alpha_L^2}{2\gamma^2\sigma^2(\underline{x})} &= f_{xx}, \end{aligned}$$

The equilibrium $\alpha_L, \alpha_H, \hat{x}$,and $\underline{x}$, must satisfy these two conditions, as well as the two market clearing conditions equating total demand to total detrended supply of each asset. ■