Price Dispersion, Private Uncertainty and Endogenous Nominal Rigidities

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Abstract

This paper shows that, when agents learn from prices, large private uncertainty may result from a small amount of heterogeneity in fundamentals. In an economy divided into islands, final and intermediate productions are affected by global and island-specific shocks. I study the consequences of final producers being informed about aggregate conditions only through the equilibrium prices of their local inputs. I show that a small dispersion in shocks to intermediate production, which blur the inference of final producers, generates two types of equilibrium: one that inherits the properties of the unique perfect-information equilibrium, and another where prices are rigid to aggregate shocks and transmit partial information. Moreover, only the latter has strong stability properties. Finally, I use this insight as a microfoundation for price rigidity in an otherwise frictionless monetary model. When producers learn from input prices, even a tiny amount of heterogeneity can lead to a sizable non-neutrality of money.

Keywords: price signals, expectational coordination, dispersed information.

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"The mere fact that there is one price for any commodity - or rather that local prices are connected in a manner determined by the cost of transport, etc. - brings about the solution which (it is just conceptually possible) might have been arrived at by one single mind possessing all the information which is in fact dispersed among all the people involved in the process.” Hayek (1945)

1 Introduction

Overview. Can private uncertainty be a major cause of the business cycle? Phelps (1970) and Lucas (1972, 1973, 1975) introduced learning imperfections to explain the non-neutrality of money. In their models, prices are rigid because producers are privately uncertain about aggregate conditions. Most of the following literature replicated their setting assuming dispersed signals with exogenous precision. This simplification, although useful in a number of contexts, fed criticisms about the non-fundamental nature of the information structure.

To defy skepticism, it would be natural to derive agents’ knowledge from market principles. Nevertheless, the market is likely to work against private uncertainty. Unless markets are severely fragmented, prices would aggregate information and dissipate uncertainty, as Hayek (1945) famously argued. In the end, private uncertainty would hardly survive without sizable dispersion in the primitives of the economy: whether such dispersion concerns signals rather than markets, seems of little help.

This paper contradicts that view: I show that the information revealed by prices about aggregate conditions can remain very small, even in the limit where the noise induced by local disturbances goes to zero. Thus, a tiny amount of dispersion in market fundamentals can originate large private uncertainty and have major consequences.

The result is due to a key innovation in the classical Phelps-Lucas island benchmark. Like in the traditional framework, producers are located on islands and learn about aggregate conditions through the equilibrium prices of their local inputs. The novelty is that local inputs are produced by intermediate firms that compete for the same endowment. The price of this common factor enters each local input price jointly with other local disturbances. Thus, in contrast to the usual setting, changes in local prices are endogenously correlated across islands via market interactions. That is, the precision of information received by final producers is now endogenous to their average input demand. In fact, if final producers’ demand reacts little to the aggregate shock, then intermediate producers’ demand also moves little, and so does the price of endowment, which feeds back into local input prices, making them poorly informative. In turn, poorly informative local prices make final producers’ demand react little, closing the loop. This mechanism allows for the existence of equilibria where not only prices are rigid because producers are uncertain, but also producers are uncertain because prices are rigid. Thereby, equilibria
with price rigidities survive no matter how small the dispersion of fundamentals is across islands.

The findings rely neither on price setting nor on the presence of money, although workhorse monetary models constitute a natural application. To highlight this, I present two models. Section 2 illustrates the core mechanism in the context of a stylized real economy with producers setting quantities. Section 3 introduces a textbook monetary model, where price rigidity originates with an arbitrary small deviation from the frictionless representative economy. The key conditions uncovered by the stylized model always hold in the monetary model. The latter serves as a test-bed for further robustness checks.

Stylized economy. The stylized economy consists of a global market and a continuum of island-specific markets. In the global market, intermediate firms, located on different islands, compete for the same endowment, which is in fixed supply. Intermediate firms produce an island-specific intermediate good that is employed by final firms on the same island. Final and intermediate production are subject to the aggregate productivity shock, whereas only intermediate production is hit by idiosyncratic productivity shocks. Hence, final producers are unsure of the extent to which the prices of their local inputs reflect local rather than global productivity conditions. That is, the price of the local input provides final producers on a given island with an endogenous private signal of aggregate productivity.

I prove that, with vanishingly small dispersion of productivity shocks, two types of equilibrium exist. One equilibrium inherits all the properties of the perfect information equilibrium: input prices are fully reactive to aggregate shocks and input markets are homogeneous. The other, which I tag dispersed information limit equilibrium, exhibits features in stark contrast: i) input prices are almost unreactive to the aggregate shocks, and ii) there is a large dispersion of traded quantities across local capital markets.

Intuition. To get some intuition of the mechanism, consider the effects of a positive aggregate productivity shock. Intermediate producers see higher productivity, so their supply schedule shifts right. Final producers instead need to infer higher productivity from their local input price. Their demand schedule moves right depending on the precision of information received: more confidence on higher productivity induces stronger demand.

When the variance of idiosyncratic shocks is sufficiently large, any movement in local input prices is interpreted as strictly local, i.e. the input price does not transmit any information about aggregate conditions. Hence, the demand schedule of final producers does not move and, in equilibrium, input prices fall to clear a larger supply.

Take now the variance of idiosyncratic shocks at the limit of zero. By continuity an equilibrium close to perfect information exists where prices transmit information with infinite precision. In this case, if the demand effect is sufficiently strong to overcome the supply effect, input prices increase instead of decreasing. When this last condition

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1This is different from the literature on information acquisition where the precision of the information is endogenous to an individual choice as in Angeletos and La’O (2013) and Colombo et al. (2014).
is satisfied (in the monetary model it is always the case), a finite level of precision must also exist for which the demand effect just offsets the supply effect, leaving input prices unaffected. At a dispersed information equilibrium this level of precision is achieved by input prices being almost unreactive to aggregate shocks, no matter how small the variance of idiosyncratic shocks is.

In addition, whenever dispersed information equilibria exist, they are the only stable outcomes. In fact, if final firms weight the input price higher than that which is required by a dispersed information equilibrium, the precision of the price signal decreases, so that it is optimal to put a lower weight instead. This comparative statics is peculiar of dispersed information equilibria. Financial applications of learning from prices (from Grossman (1976) onward) typically feature the opposite: the more agents weight the signal, the higher the precision. In these frameworks, the sign of the correlation between price and the fundamental never changes. Conversely, in the present setting, it can, depending on the precision of information; the existence of dispersed information equilibria relies precisely on this possibility.

Monetary model. I finally show the mechanism at work in the context of an otherwise frictionless version of workhorse monetary models, similar to Christiano et al. (2005). On each island, producers set a price before observing the demand for their differentiated good, on the basis of the prices of their inputs: local labor and local capital. A representative household provides local labor, whereas intermediate producers transform homogeneous raw capital held by the household into local capital. The economy is hit by an aggregate shock to the value of money and two types of idiosyncratic shock, one to the disutility of local labor and another to the productivity of the intermediate technology. The presence of idiosyncratic productivity shocks blurs the local capital price, which is informationally equivalent to an endogenous signal (as in the stylized economy). The presence of idiosyncratic preference shocks prevents the local wage, which is informationally equivalent to an exogenous signal, from revealing the aggregate shock.

Without idiosyncratic productivity shocks the prices for local capital are homogeneous, the aggregate shock is revealed and the neutrality of money holds. Nevertheless, at the limit of no idiosyncratic productivity shocks, a dispersed information limit outcome also exists where: i) aggregate monetary shocks have real effects comparable to the medium-term impact of VAR estimates in Woodford (2003), Christiano et al. (2005) and Smets and Wouters (2007) among others, and ii) final goods prices are highly reactive to idiosyncratic shocks, in line with the empirical findings of Boivin et al. (2009). The sole condition for the multiplicity result is that the precision of the exogenous information, i.e. information inferred from the local wage, is below a certain threshold. Exactly like in the stylized economy, also in this case, whenever dispersed information limit equilibria exist, they are the only stable outcomes.

Literature review. The existence of dispersed information limit equilibria sheds new light on the nature of price rigidities. Previous work typically relied on some form of friction in the availability or use of information about marginal costs to explain why
producers do not readily adjust their prices after a shock. Calvo pricing (Calvo, 1983), menu costs (Sheshinski and Weiss, 1977) and more recently Inattentiveness (Reis, 2006) and Rational Inattention (Mackowiak and Wiederhold, 2009) are a few popular examples of a vast literature. Here, in contrast, producers perceive their marginal costs precisely and are not constrained in their price setting; nevertheless, the signaling role of input prices lead to nominal rigidity as an equilibrium outcome, with strong stability properties.

Angeletos and La’O (2010) provide an insightful analysis of the effects of imperfect information over the business cycle, although their information structure is not micro-founded and the precision of the signals is exogenous. Amador and Weill (2010) present a fully microfounded model to study the welfare consequences of learning from prices. Given the endogenous nature of their information structure, they also find the possibility of a multiplicity that, contrary to that characterized in this paper, vanishes for sufficiently small dispersion of prices or with low enough pay-off complementarities.2

The relationship between private uncertainty, prices and multiplicity has been discussed in the global games literature. Morris and Shin (1998) demonstrate that arbitrarily small private uncertainty over fundamentals leads to a unique equilibrium in an otherwise multiple-equilibrium model; Angeletos and Werning (2006), and Hellwig et al. (2006) clarify that the result holds only in the absence of sufficiently informative public prices. The current paper, in contrast, demonstrates that arbitrarily small price dispersion can cause multiplicity in an otherwise unique-equilibrium model.

Benhabib et al. (2015) show that partly revealing equilibria - called sentiments - can coexist with a fully revealing equilibrium because of the endogenous nature of information. Nonetheless, their multiplicity collapses on the perfect-information outcome in the limit of no dispersion. Moreover, Chahrour and Gaballo (2015) show that sentiment equilibria can also be characterized as dispersed information limit equilibria pushing the variance of a common - rather than an idiosyncratic - informational shock to the limit of zero.

In the game-theoretic language of Bergemann and Morris (2013), dispersed information equilibria are Bayes Nash equilibria, i.e. Bayes correlated equilibria decentralized by a particular information structure. Bergemann et al. (2015) explore the stochastic properties of the set of Bayes correlated equilibria by means of signals with exogenous precision. This paper contributes to that agenda showing how Bayes correlated equilibria (i.e. dispersed information limit equilibria) can be decentralized via an endogenous information structure which is microfounded within a system of competitive prices.

2Many authors found the possibility of multiple REE due to imperfect information with endogenous precision (not necessarily learning from prices). They either restrict the coefficient region to focus on a unique equilibrium or characterize a multiplicity that vanishes with small enough dispersion of signals. A non-exhaustive list includes: Angeletos et al. (2010), Angeletos and La’O (2008), Ganguli and Yang (2009), Manzano and Vives (2011), Vives (2014) and Desgranges and Rochon (2013). In the CARA asset pricing literature, numerous examples of multiple noisy REE exist that rely on the presence of risk aversion to the conditional variance of assets returns (see Walker and Whiteman (2007)).
2 A stylized real economy

This section provides a simple illustration of the core results of the paper. I present a real economy where final producers are noisily informed about aggregate productivity by the prices of their local inputs, which also embed local disturbances. Local inputs are produced by intermediate firms using a homogeneous endowment, which is freely traded across islands. I will provide conditions for the existence and stability of dispersed information limit equilibria.

2.1 A problem of decentralized cross-sectional allocation

Consider a one-period economy inhabited by atomistic agents who appreciate consumption according to the same linear utility function. Agents are of two types: intermediate and final producers. Both are evenly distributed across a continuum of islands indexed in the unit interval. The representative intermediate producer in island \( i \in (0,1) \) can transform a quantity \(^3 Z(i) \) of homogeneous endowment into a quantity

\[
K^*_i \equiv e^{\theta + \eta_i} Z(i),
\]

of an island-specific local input.\(^4\) The transformation is subject to normally and independently distributed productivity shocks \( \theta \sim \mathcal{N}(0,1) \) and \( \eta_i \sim \mathcal{N}(0,\sigma^2) \), accounting for variations in aggregate and island-specific conditions, respectively.\(^5\) The intermediate producer on island \( i \), chooses \( Z(i) \) to maximize profits

\[
K^*_i R_i - Z(i) Q,
\]

where \( R_i \) is the real price of the intermediate good type \( i \) and \( Q \) is the real price of the endowment, which is traded across islands.\(^6\) The endowment is available in a total quantity \( Z > 0 \) and is owned in equal shares by the intermediate producers. The intermediate good is demanded by final producers on the same island. Final producers type \( i \) can transform a quantity \( K_i \) into a quantity

\[
Y(i) \equiv e^{\mu \theta} (K_i)^{1-\alpha},
\]

of homogeneous consumption good. Final production has decreasing returns to scale as \( \alpha \in (0,1) \), and it is only affected by an aggregate productivity shock with an intensity \( \mu \geq 0 \). The final producer on island \( i \) chooses \( K_i \) to maximize expected profits:

\[
E[Y(i)|R_i] - K_i R_i,
\]

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\(^3\)From here onward, subscripts in brackets denote the quantity of a homogeneous good acquired on an island, whereas simple subscripts are used for the differentiated good produced on an island.

\(^4\)In appendix C.1 I discuss the case with decreasing returns to scale. Decreasing returns restrict the conditions for the existence of dispersed information limit equilibria; this feature emerges here due to the specific microfoundations of this simple model. See forward note 14.

\(^5\)\( \mathcal{N}(m,\text{var}) \) conventionally denotes a Normal distribution with mean \( m \) and variance \( \text{var} \).

\(^6\)The price of the homogeneous consumption good, i.e the slope of the utility function, is taken as a numeraire. Hence, \( Q \) and \( R_i \) are prices in units of the consumption good.
where \( E[Y_{(i)}|R_i] \) is the expectation of \( Y_{(i)} \) conditional on the realization of \( R_i \). Final producers need to forecast production as they do not observe the aggregate shock when they trade the local input.

Timing is as follows. First, the markets for the endowment and all local inputs open and close simultaneously. At this stage, final producers observe the equilibrium price of their local input, but not the aggregate shock. Then, the consumption good is actually produced and the aggregate shock finally unfolds to final producers. At the end of the period, consumption occurs.

Notice that \( \sigma^2 \) introduces a metric of the fundamental heterogeneity across islands. With \( \sigma^2 = 0 \) the economy can be represented as an homogeneous one where the law of one price holds. However, I will show that, at the limit \( \sigma^2 \to 0 \), a discontinuity in the set of equilibria emerges.

### 2.2 Allocative vs. informative role of local prices

The distribution of local prices provides the incentives for the cross-sectional allocation of the endowment. In equilibrium, the endowment should be allocated such that it equalizes the marginal productivity of final producers across islands. Nevertheless, local prices depend on final producers’ expectations, which are in turn formed conditionally on local prices.

As usual in the literature on noisy rational expectations from Grossman (1976) and Hellwig (1980) onward, I restrict my analysis to equilibria with a log-linear representation. A formal definition of an equilibrium follows.

**Definition 1.** A rational expectation equilibrium is a collection of prices \((Q, \{R_i\})_{i=0}^1\), quantities \(\{Y_{(i)}, K_i, K_i^s, Z_{(i)}\}_{i=0}^1\) and island-specific expectations \(\{E[Y_{(i)}|R_i]\}_{i=0}^1\) contingent on the stochastic realizations \((\theta, \{\eta_i\}_{i=0}^1)\), such that:

- (optimality) agents optimize their actions according to the prices they observe;
- (market clearing) all markets clear, i.e. \(\int_0^1 Z_{(i)}di = Z\), \(K_i^s = K_i \) in each \(i \in (0, 1)\);
- (log-linearity) prices and quantities are log-linearly distributed.

Producers operating in island \(i\) maximize their profits when prices and quantities satisfy the first-order conditions of their problems:

\[
\begin{align*}
    r_i &= q - \theta - \eta_i, \\
    r_i &= \mu E[\theta | ri] - \alpha k_i,
\end{align*}
\]

where I use lower case to denote stochastic log-deviations.\(^7\) According to (5), an intermediate producer supplies whatever quantity of the local input, provided its price covers its

\(^7\)In particular, given log-linearity, \(r_i\) is defined as \(r_i \equiv \log(R_i) - \log(\bar{R}_i)\) where \(R_i = \bar{R}_ie^{\eta_i}\) with \(\bar{R}_i\) and \(e^{\eta_i}\) being, respectively, the deterministic and the stochastic (log-normal) component of \(R_i\). The other variables are defined in analogy to \(r_i\). For more details see log-linear algebra in appendix B.1.
marginal cost. According to (6), a final producer demands a quantity of the local input that is increasing in the expected productivity $E[\theta|r_i]$ and decreasing in the price $r_i$. As final producers observe only input prices in their own island, expected productivity type $i$ depends on the realization of the local price type $i$. Therefore, $r_i$ has a twofold role: allocative as it induces market clearing, and informative as it informs about the stochastic productivity of the local asset. The effect of the latter is measured by $\mu$, whereas the effect of the former is given by $\alpha$. Ceteris paribus, the higher the value of $\mu$, the higher the elasticity of prices to expectations, whereas, the higher the value of $\alpha$, the steeper the demand schedule.

### 2.3 Dispersed information limit equilibria

Let us first look at the average price over input markets, i.e. the correlated component of local input prices. We can integrate both (5) and (6) to establish

$$ r \equiv \int_0^1 r_i \, di = q - \theta = \int_0^1 (\mu E[\theta|r_i] - \alpha k_i) \, di. \tag{7} $$

Then, using (7) to substitute $q - \theta$ into (5), we obtain the local market clearing price as

$$ r_i = \mu \int_0^1 E[\theta|r_i] \, di - \alpha \theta - \eta_i. \tag{8} $$

after imposing $k_i = k^*_i$ for each $i$ and $\int_0^1 z(i) \, di = 0$, so that the average supply is given by $k \equiv \int_0^1 k^*_i \, di = \int_0^1 (z(i) + \theta + \eta_i) \, di = \theta$.

From the point of view of the final producer in island $i$, the local price of her input constitutes a private signal of an underlying aggregate endogenous state, that is a linear combination of the average expected productivity and its actual realization. The presence of private noise $\eta_i$ generates confusion about the nature, local rather than global, of price fluctuations.\footnote{Note that $k_i$ is known by the final producer, but since $k_i$ is a function of $r_i$ it does not convey additional information. Specifically, suppose that $k_i$ provides some additional information, so that $E[\theta|r_i, k_i]$ is a linear combination of $r_i$ and $k_i$. Then, given (6), $k_i$ is collinear with $r_i$. Thus, focusing on $E[\theta|r_i]$ only does not result in a loss of generality.}

To find the set of equilibria, one needs to solve the fixed point problem which is implicit in (8) and write down the profile of final producers’ expectations as a function of shocks. There are many ways to tackle the signal extraction problem; I will choose the one that provides a well defined out-of-equilibrium characterization.

Let us start from the observation that, when random variables are normally distributed, the optimal forecasting rule is linear in the realization of the signal. Hence,\footnote{In contrast to the popular framework introduced by Grossman (1976), the input price responds to the actual fundamental realization also when all producers are uninformed. This happens because the price embodies information from the intermediate producers who actually observe the realization at the time of trade. This avoids the problems of implementability of the perfect information equilibrium, often discussed in literature.}
agent $i$’s forecast is written as $E[\theta | r_i] = b_i r_i$, that is

$$E[\theta | r_i] = b_i \left( \mu \int_0^1 E[\theta | r_i] \, di - \alpha \theta - \eta_i \right),$$

where $b_i$ is a fixed coefficient weighting the signal type $i$. Given that all agents use the rule above, then by definition the aggregate expectation is

$$\int_0^1 E[\theta | r_i] \, di = -\frac{b\alpha}{1 - b\mu} \theta,$$

provided by $b \neq 1$ where $b \equiv \int_0^1 b_i \, di$ denotes the average weight across agents.\(^{10}\) Therefore, the price signal can be rewritten as

$$r_i = -\frac{\alpha}{1 - \mu b} \theta - \eta_i.$$  

(11)

The private signal $r_i$ provides agents with information about $\theta$ of a relative precision

$$\tau(b) = \frac{\alpha^2}{(1 - b\mu)^2 \sigma^2},$$

which depends non-linearly on the average weight. This is a property of private endogenous signals that does not arise in the case of a public endogenous signal, i.e. with a common $\eta_i$ instead.\(^{11}\) In particular, $\tau(b)$ approaches zero as $b$ gets sufficiently large in absolute value. Finally, notice that relations (9)-(12) hold for whatever given profile of individual weights $\{b_i\}_0^1$.

Expectations comply with Bayesian updating when each weight $b_i$ satisfies the requirement $E[r_i(\theta - b_i r_i)] = 0$, that is, the forecast error is orthogonal to the signal. The popular OLS formula gives the individual best weight function

$$b_i(b) = \frac{\text{cov}(\theta, r_i)}{\text{var}(r_i)} = -\frac{\alpha(1 - \mu b)}{\alpha^2 + (1 - \mu b)^2 \sigma^2}.$$  

(13)

In game theoretic-terms, $b_i(b)$ pins down the unique strictly-dominant action in response to $b$, a sufficient statistics of the profile of others’ actions. Hence, a REE is characterized by a symmetric profile of weights $b_i(b) = b$ for each $i \in (0, 1)$.

Panels (a) and (b) in figure 1 plot the optimal individual weight function (13) for different values of $\sigma^2$ in two different cases: $\mu > \alpha$ and $\mu < \alpha$, respectively. Equilibria lie at the intersection with the bisector. In both cases, when $\sigma^2 \to \infty$, the curve (in solid light gray) approaches the x-axis yielding a unique equilibrium. As $\sigma^2$ decreases the curve approaches the perfect-information line (in dotted black), which is obtained precisely

\(^{10}\)Note that $\int_0^1 b_i \eta_i \, di = 0$ given that $\int_0^1 b_i \eta_i \, di \neq 0$ violates linearity in the realization of the signal.

\(^{11}\)The interested reader can easily go through the previous steps and check that in the case of a public signal its relative precision is independent of the average weight. In fact, given an homogeneous information set, agents would be able to predict the average expectation and disentangle the endogenous component from the public signal.

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when \( \sigma^2 = 0 \). The perfect information line cuts the bisector once at \( b_* = (\mu - \alpha)^{-1} \); it does that from below if and only if \( \mu > \alpha \), as in panel (a) in contrast to panel (b). When \( \sigma^2 \) is close to zero, but not zero, the curve approximates the perfect information line around \( b_* \), although there now exists a sufficiently large absolute value of \( b \) such that the optimal weight \( b_i(b) \) approaches zero. This implies that, if and only if \( \mu > \alpha \), two other intersections must exist by continuity, as the following proposition states.

**Proposition 1.** In the perfect information case \( \sigma^2 = 0 \) a unique equilibrium exists, which is characterized by \( b_* = (\mu - \alpha)^{-1} \). For \( \sigma^2 \to 0 \) instead, provided \( \mu > \alpha \), three equilibria exist: a perfect-information limit equilibrium, characterized by a \( b_o \) arbitrarily close to \( b_* \), and two dispersed information limit equilibria, characterized by \( b_+ > 0 \) and \( b_- < 0 \) respectively, such that \( |b_{\pm}| > M \) where \( M \) is an arbitrarily large finite value.

**Proof.** See appendix A.1.

The perfect-information limit equilibrium inherits the properties of the equilibrium under perfect information: the local prices \( r_i = (\mu - \alpha)\theta \) are perfectly correlated and transmit information with infinite precision, that is \( \text{E}[\theta | r_i] = \theta \). The picture changes dramatically for the dispersed information limit equilibria.

**Proposition 2.** The dispersed information limit equilibria both feature: almost sticky local prices \( r_i \to 0 \) which transmit information with a finite precision

\[
\hat{\tau} \equiv \lim_{\sigma^2 \to 0} \lim_{b \to b_{\pm}} \tau(b) = \frac{\alpha}{\mu - \alpha};
\]

under-reactive aggregate expectation

\[
\lim_{\sigma^2 \to 0} \lim_{b \to b_{\pm}} \int_0^1 \text{E}[\theta | r_i] d\theta = \frac{\alpha}{\mu};
\]
Figure 2: The effect of a positive $\theta$ on the average supply (light grey) and average demand (dark grey) in input markets. For $\sigma^2 \to \infty$: a no-information equilibrium (panel on the left). For $\sigma^2 \to 0$: a perfect information equilibrium (central panel) and a dispersed information limit equilibrium (panel on the right).

and sizable cross-sectional variance of island-specific expectations

$$\lim_{\sigma^2 \to \infty} \lim_{b \to b^\pm} \int_0^1 \left( E[\theta| r_i] - \int_0^1 E[\theta| r_i] \right)^2 di = \frac{\alpha (\mu - \alpha)}{\mu^2}.$$  \hfill (16)

Proof. See appendix A.2. \hfill \Box

To illustrate the forces that sustain dispersed information limit equilibria, it is useful to discuss figure 2, that plots three different equilibria in the case of a positive $\theta$ and $\mu > \alpha$. On the x-axis we measure the average quantity $k \equiv \int_0^1 k_i di$ and on the y-axis the average price $r \equiv \int_0^1 r_i di$. The downward sloping line, with slope $-\alpha$ and intercept $D \equiv \mu \int_0^1 E[\theta| r_i] \, di$, denotes the inverse aggregate demand schedule, i.e. the average of (6). The vertical line denotes the average supply, $k = \theta$. The equilibrium average price $r$ is pinned down at the intersection of demand and supply: the higher the $D$, i.e. the higher the precision of information about $\theta$, the higher the equilibrium $r$.

The no information scenario is illustrated in the panel on the left, where $E[\theta| r_i] = 0$ and $\int_0^1 r_i di = -\alpha \theta$. The local price does not transmit any information, i.e. it has zero precision, when the volatility of idiosyncratic shocks is so large that any movement in local prices is interpreted as being strictly local. As a consequence, market clearing is induced by a lower average local price. In this case, $\theta$ and $r$ are negatively correlated as the allocative role prevails.

The perfect information scenario is illustrated in the central panel, where $E[\theta| r_i] = \theta$ and $\int_0^1 r_i di = (\mu - \alpha) \theta$. The local price fully reveals $\theta$, i.e. it has infinite precision, when the volatility of idiosyncratic shocks is so small that any movement in local prices is interpreted as being strictly global. A higher local price entails a more expensive local
input, but also informs about higher productivity. In particular, provided $\mu > \alpha$, final producers' demand is so high that the average local price increases. In this case, $\theta$ and $r$ are positively correlated as the informative role prevails.

Provided $\mu > \alpha$, the reaction of $r$ to a positive $\theta$ is: negative, in the case of zero precision, and, positive, in case of infinite precision. By continuity, this implies that, for a finite value of precision, exactly (14), the upward shift in demand is just sufficient to offset the supply effect, so that $r$ tends to no reaction. This situation is plotted in the panel on the right, where the average expectation is given by (15). At that point, the reaction of local prices to the aggregate shock is infinitesimally small. Nevertheless, local prices may still transmit information as island-specific shocks also have an infinitesimally small variance. That is, in this case, the precision of the information transmitted by the price signal, namely $\hat{\tau}$, is given by the ratio of two infinitesimal variances.

In the following I will clarify how a dispersed information limit equilibrium can emerge from an out-of-equilibrium initial condition.

### 2.4 Out-of-equilibrium convergence

This section demonstrates that, whenever dispersed information limit equilibria exist, they are the only equilibrium outcomes stable under higher-order belief dynamics and adaptive learning. Therefore, theoretical arguments indicate that dispersed information limit equilibria, in contrast to the perfect information equilibrium, are likely to be observed.

**Convergence in higher-order beliefs**

The following analysis is inspired by the work on Eductive Learning\footnote{Eductive Learning assesses whether or not rational expectation equilibria can be selected as locally unique rationalizable outcome according to the original criterion formulated by Bernheim (1984) and Pearce (1984). The difference here is that I deal with a well-defined probabilistic structure. I consider beliefs on the average weight $b$ characterizing the equilibria, rather than beliefs directly specified in terms of price forecasts, as generally assumed by Guesnerie in settings of perfect information.} by Guesnerie (2005, 1992) and has connections with the usual rationalizability argument used in the Global Games literature (Carlsson and van Damme, 1993; Morris and Shin, 1998).

In the economy, rationality and market clearing are common knowledge among agents, meaning that nobody doubts that individual weights comply with (13). Nevertheless, this is still not enough to determine which equilibrium, if any, is going to prevail. For instance, suppose that possible values of $b$ can be restricted to a neighborhood $\mathcal{I}(\hat{b})$ of an equilibrium characterized by a fixed point $\hat{b}$ of (13). Common knowledge of $\mathcal{I}(\hat{b})$ implies, as first conjecture, that the rational individual weights, and therefore its average, must actually belong to $b_i(\mathcal{I}(\hat{b}))$. But given that $b_i(\mathcal{I}(\hat{b}))$ is also common knowledge, then agents should conclude, in a second-order conjecture, that rational individual weights and therefore its average, must actually belong to $b_2^2(\mathcal{I}(\hat{b}))$. Iterating the argument we have...
that the \( \nu \)th.-order conjecture about \( b \) belongs to \( b'_{\hat{b}}(\Im(\hat{b})) \). Agents can finally conclude that \( \hat{b} \) will prevail if the equilibrium is a locally unique rationalizable outcome as defined below.

**Definition 2.** A REE characterized by \( \hat{b} \in (b_o, b_+, b_-) \) is a locally unique rationalizable outcome if and only if
\[
\lim_{\nu \to \infty} b'_{\hat{b}}(\Im(\hat{b})) = \hat{b},
\]
i.e. \( b \) entails a local contraction around the equilibrium.

From an operational point of view local uniqueness requires that \( |b'(\hat{b})| < 1 \). The following proposition states the result.

**Proposition 3.** Whenever dispersed information limit equilibria exist, they are locally unique rationalizable outcomes whereas the perfect-information limit equilibrium is not. The perfect-information limit equilibrium is a locally unique rationalizable outcome when it is the only limit equilibrium.

**Proof.** See appendix A.3.

To shed some light on the tatonnement process, let us rewrite the best individual weight function
\[
b_i(b) = \frac{\mu b - 1}{\alpha} \frac{\tau(b)}{1 + \tau(b)},
\]
as composed of a scale factor and a precision factor. On the one hand, for a given signal precision, a lower signal variance requires a higher scale factor. On the other hand, for a given signal variance, a higher signal precision requires a higher precision factor. The average weight \( b \) affects both factors.
In a neighborhood of the perfect information equilibrium, the precision factor is always one, so the dynamics of beliefs is governed by the scale factor. Notice that the scale factor entails a first-order divergent effect led by high complementarity, i.e. \( \lim_{b \to b^\circ} b'_i(\hat{b}) = \mu/\alpha > 1 \). In other words, if agents contemplate the possibility that the average weight is higher than the equilibrium value \( b^\circ \), then their best individual weight must be further away from the equilibrium. This happens because, despite the fact that the precision does not change in the neighborhood, the scale factor reacts more than one to one to a change in \( b \). Since the total supply of local inputs is fixed at \( \theta \), this divergence process leads to lower and lower reactions of local prices (see (11)), consistently with a larger and larger scale factor, approaching the configuration (c) in figure 2. This evolution corresponds to the divergent dynamics in higher-order beliefs displayed in figure 3. Hence, the perfect information limit equilibrium is inherently unstable under higher order belief dynamics.13

Nevertheless, as conjectures about \( b \) further diverge from \( b^\circ \), the reaction of local prices will eventually shrink up to the point where they achieve the same order of magnitude of island-specific noise. At that point, a second-order effect driven by the precision factor is activated. The informativeness of the signal finally decreases, so that the divergent effect of the scale factor is overturned by a lower and lower precision effect, up to the point \( (\tau(b) = \hat{\tau} \text{ for } b \to \infty) \) where a new equilibrium emerges. In particular, the sequence of higher-order beliefs finally enters a contracting dynamics, that, as the proposition proves, converges towards a dispersed information limit outcome.

Convergence under adaptive learning

Consider now an infinite-horizon economy that consists of independent and identical realizations of the one-period economy described above. Following the approach of Marcet and Sargent (1989a,b) and Evans and Honkapohja (2001), suppose agents behave as econometricians: they learn over time the right calibration of their forecasting rule. In particular, they individually set their weights in accordance with

\[
\begin{align*}
    b_{i,t} &= b_{i,t-1} + t^{-1} S_{i,t-1}^{-1} r_{i,t} \left( \theta_i - b_{i,t-1} r_{i,t} \right) \\
    S_{i,t} &= S_{i,t-1} + (t + 1)^{-1} \left( r_{i,t}^2 - S_{i,t-1} \right),
\end{align*}
\]

which is a recursive expression for a standard OLS regression, where the matrix \( S_{i,t} \) is the estimate variance of the local price. Notice that, in contrast to the previous setting, here agents ignore that learning is an ongoing collective process. In this sense this is a bounded rationality approach. The following formally defines adaptive stability.

13An interesting observation is that the existence of a multiplicity of limit equilibria does not hinge on the unboundedness of the domain of \( b_i \), but just on the slope of the best individual weight function around the perfect-information limit equilibrium. To see this suppose that one arbitrarily restricts the support of feasible individual weights \( b_i \) in a neighborhood of \( b^\circ \), namely \( \Im(b^\circ) \equiv [\underline{b}, \bar{b}] \). Whenever \( \mu > \alpha \) it is easy to check by a simple inspection of figure 3 that two other equilibria beyond \( b^\circ \) would arise as corner solutions. I thank George-Marios Angeletos for driving my attention to this point.
Definition 3. A REE characterized by \( \hat{b} \) is a locally learnable equilibrium if and only if there exists a neighborhood \( \mathcal{F}(\hat{b}) \) of \( \hat{b} \) such that, given an initial estimate \( b_{i,0} \in \mathcal{F}(\hat{b}) \), it is \( \lim_{t \to \infty} b_{i,t} = \hat{b} \) a.s.

In other words, once estimates are close to the equilibrium values of a locally adaptively stable equilibrium, then convergence toward the equilibrium will almost surely occur. The result is stated by the proposition below.

Proposition 4. Whenever dispersed information limit equilibria exist, they are locally adaptively stable whereas the perfect-information limit equilibrium is not. The perfect-information limit equilibrium is locally adaptively stable when it is the unique limit equilibrium.

Proof. See appendix A.4.

The asymptotic behavior of statistical learning algorithms can be studied by stochastic approximation techniques. The proof uses well known results to show that the asymptotic stability of the system around an equilibrium \( \hat{b} \) is governed by the differential equation

\[
\frac{db}{dt} = b_i(b) - b,
\]

which is stable if and only if \( b'_i(\hat{b}) < 1 \). This condition is known as the E-stability principle where \( b_i(b) \) corresponds to the "projected T-map" in the adaptive learning literature. Therefore, adaptive stability implies eductive stability. Referring to figure 3, the slope of the curves at the intersection with the bisector determines the stable or unstable nature of the equilibrium. It is straightforward to assess adaptive stability of dispersed information limit equilibria: their existence relies precisely on the condition \( b'_i(b_\pm) > 1 \), that, since \( b_i(b) \) is a cubic, necessarily implies \( b'_i(b_\pm) < 1 \).

3 The Fragile Neutrality of Money

This section presents a fully microfounded monetary economy replicating the flexible price version of workhorse DSGE models, like Christiano et al. (2005).

I will show that the mechanism that generates dispersed information limit equilibria can be naturally grafted in this model. As a result, large non-neutralities can originate with an arbitrary small deviation from the frictionless representative economy.

From a technical point of view, the new setting extends the previous analysis to cases where agents forecast an endogenous variable and additional information is also available.

3.1 Model

Preferences

Consider an economy composed of a continuum of islands indexed by \( i \in (0, 1) \). On each island, a differentiated consumption good is produced by competitive producers who
Employ island-specific inputs: local labor and local capital. Local labor is supplied by a representative household, who also consumes a bundle of differentiated goods, appreciates the services of money and sells raw capital to intermediate producers of local capital.

At each period $t$, the household decides a level of composite consumption $C_t$, the supply of labor $L_{s,i,t}$ of each type $i \in (0, 1)$, and future money holdings $M_{t+1}$, in order to maximize a stochastic utility function

$$E_t \left[ \sum_{t=0}^{\infty} \delta^t \left( e^{\theta_t} \left( \frac{C^1 - \psi - 1}{1 - \psi} - \int_0^1 e^{\xi_{i,t}} L^s_{i,t} \, di \right) + \log \frac{M_t}{P_t} \right) \right], \quad (19)$$

subject to a budget constraint for each period

$$\frac{R_t}{P_t} + \int_0^1 \frac{W_{i,t}}{P_t} L^s_{i,t} \, di + \frac{M_t}{P_t} = C_t + \frac{M_{t+1}}{P_t}, \quad (20)$$

where: $\delta \in (0, 1)$ is the discount factor; $W_{i,t}$ is hourly nominal wage type $i$;

$$C_t \equiv \left( \int_0^1 C_{i,t} \, di \right)^{1/\epsilon} \quad \text{and} \quad P_t \equiv \left( \int_0^1 P_{i,t} \, di \right)^{1/\epsilon}$$

are a composite consumption good and its price, respectively; $C_{i,t}$ is the consumption of local good type $i$ whose nominal per-unit price is $P_{i,t}$; and $\epsilon > 1$ is a CES parameter measuring the degree of substitutability of local goods. The lack of convexity in labor disutility greatly simplifies the exposition; however, I consider the convex case in the main proofs and explore its impact in the numerical analysis of section 3.3. The operator $E_t[\cdot]$ denotes the mathematical expectation conditional on the current realizations of the preference shocks $\theta_t$ and $\xi_{i,t}$, which are discussed below. The supply of money is fixed at $M$. Finally $R_t$ is the nominal price of one unit of an endowment in raw capital, which expires and renews in each period in the hands of the household. In a richer version of this model, raw capital can be modeled as the legacy of a capital accumulation choice made by the household at the end of the previous period, as in Christiano et al. (2005). Nevertheless, what matters to the main results of this paper is that the quantity of raw capital is predetermined at the beginning of each period. Therefore, for the sake of simplicity, I will interpret raw capital as being an endowment of a renewable resource.

**Technology**

Raw capital is acquired in an inter-island market in order to be transformed into island-specific capital $K^s_{i,t}$. The transformation is operated by competitive intermediate producers maximizing profits

$$R_{i,t} K^s_{i,t} - R_t Z_{(i),t}, \quad (21)$$

under the constraint of the following linear technology

$$K^s_{i,t} \equiv e^{\eta_{i,t}} Z_{(i),t}, \quad (22)$$
where \( e^{\bar{\eta}_{i,t}} \) is the stochastic island-specific productivity factor, which is discussed below. The local capital \( K_{s,i,t} \) is produced using \( Z_{i(t)} \) units of the endowment which are acquired in a global market at price \( R_t \). The absence of decreasing returns to scale is adopted for the sake of simplicity\(^{14}\); however, I consider the general case in the main proofs and discuss its effects in the numerical analysis of section 3.3.

Local capital and local labor are used by the representative final producer on island \( i \) to produce a differentiated consumption good type \( i \). Competitive firms set prices to maximize profits

\[
P_{i,t} Y_{i,t} - W_{i,t} L_{i,t} - R_{i,t} K_{i,t},
\]

under the constraint of a Cobb-Douglas technology with constant returns to scale

\[
Y_{i,t} (K_{i,t}, L_{i,t}) \equiv K_{i,t}^\alpha L_{i,t}^{1-\alpha},
\]

with \( \alpha \in (0,1) \), where \( K_i, L_i, \) and \( Y_i \) denote respectively the demand for local capital, the demand for local labor and the produced quantity of the consumption good, relative to island \( i \). Notice that final production is island-specific, that is, each final producer hires labor and capital from her own island only. Input markets are segmented and there is one price for each input on each island.

**Shocks**

At each date \( t \), the economy is hit by i.i.d. aggregate and island-specific disturbances. An aggregate source of randomness

\[
\theta_t \sim \mathcal{N}(0,1),
\]

equally concerns the utility of consumption and leisure across islands. Actually, \( \theta \) has the features of a shock to the value of money. An increase in \( \theta \) decreases the marginal utility of real money in terms of marginal utility of consumption and labor, without altering the ratio between the latter two. In particular, given a positive (resp. negative) \( \theta \), an appropriate increase (resp. decrease) in the general price level leaves the allocation unaffected. In this respect, the aggregate shock is isomorphic to a velocity shock. Nevertheless, this formulation allows an exact log-transformation of FOCs without altering the economic effect of the shock.\(^{15}\)

The disutility of working hours type \( i \) varies according to

\[
\xi_{i,t} \sim \mathcal{N}(0, \sigma^2_\xi),
\]

\(^{14}\)In contrast to the model of section 2, considering decreasing returns to scale in the intermediate sector of this economy does not alter the conditions for the existence of dispersed information limit equilibria.

\(^{15}\)An exact log-transformation is also possible in the case of a velocity disturbance assuming permanent shocks, as in Amador and Weill (2010).
which introduces differences in labor supply across islands. The productivity of the intermediate sector type $i$ is affected by idiosyncratic productivity shocks

$$\eta_{i,t} \sim \mathcal{N}(0, \sigma^2),$$  \hspace{1cm} (27)

where $\eta$ is an i.i.d. realization across islands. The size of $\sigma^2$ is a measure of the cross-section heterogeneity in intermediate production.

**Timing of actions and information acquisition**

Each period $t$ is divided into three stages.

i) In the first stage the shocks hit. The household observes the preference shocks and each intermediate producer observes her own productivity shock.

ii) In the second stage, the markets for raw capital and local capitals open and clear simultaneously. On each island, final producers observe the equilibrium price of their local inputs, and fix their selling price before their demand realizes.\(^{16}\)

iii) In the last stage, final producers observe the demand for their product at the fixed price, and hire the quantity of labor needed to clear the market.

The absence of convexity in the disutility of labor allows a local equilibrium wage, which are observable in the second stage, to emerge irrespective of the quantity of working hours, which are traded in the third stage. To allow quantity adjustments otherwise, one can assume that producers observe the labor supply schedule posted by the household, i.e. how the local wage changes with labor quantities.\(^{17}\) This is informationally equivalent to observing the equilibrium wage in the current specification. The most general case is fully worked out in the main proofs of appendix B.1.

### 3.2 Equilibria

**Definition of equilibrium and first-order conditions**

As in the previous section, I restrict the analysis to equilibria with a log-linear representation that, also in this model, obtains with no approximation. A formal definition of an equilibrium follows below.

**Definition 4.** A log-linear rational expectation equilibrium is a distribution of prices $\{\{R_{i,t}, W_{i,t}, P_{i,t}\}_{(0,1)}, R\}$, quantities $\{C_{i,t}, Y_{i,t}, M_t, L_{i,t}, L_{s,t}, K_{i,t}, K_{s,t}, Z_{i,t}\}_{i=0}^1$ and expectations $\{E[Y_{i,t}|R_{i,t}, W_{i,t}]\}_{i=0}^1$, contingent on the stochastic realizations $(\theta_t, \{\xi_{i,t}, \eta_{i,t}\}_{(0,1)})$, such that:

\(^{16}\)If producers were able to observe the demand for their differentiated good when they set prices, then the irrelevance of dispersed information would obtain as explained by Hellwig and Venkateswaran (2014).

\(^{17}\)A more cumbersome solution is to introduce a third production factor whose market opens and clears at the last stage, and let labor be traded in the second stage.
- (optimality) agents optimize their actions according to the prices they observe;

- (market clearing) the market for the raw capital clears, \( \int Z(t) d\tau = 1 \); demand and supply in local markets match, \( L_{i,t} = L_{i,t}^* \) and \( K_{i,t} = K_{i,t}^* \); final markets clear, \( Y_{i,t} = C_{i,t} \); and money demand equals supply, \( M_t = M \);

- (log-linearity) prices and quantities are log-linearly distributed.

The first condition ensures that agents optimize their actions using rationally the information conveyed by the equilibrium prices that they observe. The requirement of a log-linear equilibrium allows the tractability of aggregate and island-specific relations.

The household chooses a level of aggregate and island-specific consumption, island-specific working hours and future money holdings such that:

\[
\Lambda_t = e^{\theta_t C_{i,t}^{-\psi}},
\]

\[
C_{i,t} = \left( \frac{P_{i,t}}{P_t} \right)^{-\epsilon} C_t,
\]

\[
e^{\theta_t e^{\xi_{i,t}}} = W_{i,t} \frac{\Lambda_t}{P_t},
\]

\[
\frac{\Lambda_t}{P_t} = \delta E_t \left[ \frac{\Lambda_{t+1}}{P_{t+1}} \right] + \delta \frac{1}{M_{t+1}},
\]

respectively, where \( \Lambda_t \) is the lagrangian multiplier associated with the budget constraint of the representative household at time \( t \). Intermediate producers supply any quantity of local capital provided

\[
R_{i,t} = e^{-\eta_{i,t}} R_{i,t},
\]

that is, the price of the local capital equals the cost of the raw capital augmented by the productivity disturbance. Final producers hire local capital and fix a price according to

\[
K_{i,t} = \left( \frac{\alpha W_{i,t}}{(1 - \alpha) R_{i,t}} \right)^{1-\alpha} E [Y_{i,t} | R_{i,t}, W_{i,t}],
\]

\[
P_{i,t} = \frac{\epsilon}{\epsilon - 1} \frac{R_{i,t}^\alpha W_{i,t}^{1-\alpha}}{(1 - \alpha)^{1-\alpha} \alpha^\alpha},
\]

representing the optimal demand of local capital for a given production level and the optimal pricing function, respectively. The presence of \( E [Y_{i,t} | R_{i,t}, W_{i,t}] \) emphasizes that final producers fix \( K_{i,t} \) and \( P_{i,t} \) before knowing \( Y_{i,t} \) while observing \( R_{i,t} \) and \( W_{i,t} \). In fact, local labor is determined in the third stage of the period so that \( C_{i,t} = Y_{i,t} = K_{i,t}^\alpha L_{i,t}^{1-\alpha} \), i.e. final good markets clear.

**Learning from prices**

Here, I spell out the signal extraction problem faced by final producers. To start with, notice that producers’ uncertainty is resolved at the end of the period, once final
producers observe their own demand. Therefore, at the end of each period, all agents in the economy have the same information about the next period. Moreover, given that shocks are i.i.d. and the supply of money is fixed, agents’ expectations at time $t$ over the future course of the economy is the unique stochastic steady state at each future date. Hence, as in Amador and Weill (2010), the only intertemporal first-order condition - the one for money - collapses to the one-period equilibrium relation

$$\frac{\Lambda_t}{P_t} = \frac{\delta}{1 - \delta} \frac{1}{M}.$$  \hspace{1cm} (35)

where $M_t = M$ at each $t$. This means that final producers face a one-period signal extraction problem that renews each time. From here onward, I will omit time indexes as the following relationships are all simultaneous.

After plugging (35) into (30), we conclude that observing a log-deviation of the equilibrium wage type $i$ from its deterministic component,

$$w_i = \theta + \xi_i,$$  \hspace{1cm} (36)

provides information about the preference shocks occurred on island $i$. In particular, (36) constitutes a private exogenous signal about $\theta$ with precision $\sigma_\xi^{-2}$.

Using (29) with $Y_{i,t} = C_{i,t}$, one can write the expectation of producers on island $i$ as

$$E[Y_i|R_i, W_i] = E[X^\epsilon|R_i, W_i] P_i^{-\epsilon}$$  \hspace{1cm} (37)

where $P_i$ is the price fixed by final producers (and so known by them) according to (34) and

$$X \equiv Y^{1/\epsilon} P,$$  \hspace{1cm} (38)

is the endogenous common component of each island-specific demand, that producers do not observe at the time of their price choice. Combining (33),(34) and (37), one can express the local capital demand in log-deviations terms as

$$k_i = \epsilon E[x|r_i, w_i] - (\epsilon - 1)(1 - \alpha)w_i - (1 + (\epsilon - 1)\alpha)r_i.$$  \hspace{1cm} (39)

It is instructive to contrast the demand schedule (39) with (6). Given that $k_i = z(\iota) - \eta_i$, and $\int_0^1 z(\iota) d\iota = 0$, here the aggregate log-variation in local capital demand is fixed at zero, i.e. $k \equiv \int_0^1 k_i d\iota = 0$. Nevertheless, now wages contain information about the aggregate shock, in particular $\int w_i d\iota = \theta$. Therefore, integrating (39) on both sides, one gets an expression for the average price of capital $r$ as depending on $\int_0^1 E[x|r_i, w_i] d\iota$ and $\theta$. One can use this relation to substitute $r$ into $r_i = r - \eta_i$ (derived from (32)), to get

$$r_i = \phi \int_0^1 E[x|r_i, w_i] d\iota + (1 - \phi)\theta - \eta_i,$$  \hspace{1cm} (40)

where

$$\phi \equiv \frac{\epsilon}{1 + (\epsilon - 1)\alpha} > 1.$$  \hspace{1cm} (41)
Like in the stylized economy (see (8)), \( r_i \) constitutes a private endogenous signal that reacts positively to the average expectation and negatively to the aggregate fundamental. In contrast, in this economy, \( r_i \) reacts always more to the average expectation than to the aggregate fundamental (i.e. \( \phi > 1 \) occurs for all feasible calibrations). Moreover, here final producers form an expectation about an endogenous aggregate state \( x \), which is a function of \( \theta \), and not about \( \theta \) directly.

The price of the local capital \( r_i \) exhibits opposite reactions to an aggregate shock \( \theta \) in the cases of perfect information (\( \sigma^2 = 0 \)) or no information (\( \sigma^2 \to \infty \) and \( \sigma_x^2 \to \infty \)). In the latter, all movements in \( r_i \) are interpreted as movements in \( \eta_i \) or \( \xi_i \), so we have \( \text{E}[x|r_i, w_i] = 0 \) and hence \( r = (1 - \phi) \theta \); in the former, all movements in \( r_i \) are interpreted as movements in \( \theta \), given that \( \text{E}[x|r_i, w_i] = x = p = \theta \), which implies \( r_i = \theta \).

The underlying economic intuition is simple. Suppose a positive shock \( \theta \) hits. Under perfect information, the shock produces a neutral inflationary effect: all prices go up at the same average rate \( \theta \), whereas quantities remain unchanged. In particular, the price of local capital goes up because, despite more expensive labor, producers demand more capital as they expect a higher nominal value of their production. Conversely, when final producers miss the global nature of a rise in local wages, they tend to reduce production because they anticipate a loss of competitiveness of their own product. As a consequence, the average price of capital goes down to clear the aggregate supply of capital, which is fixed by the predetermined quantity of raw capital.

The actual law of motion of the aggregate state \( x \) can be obtained as follows. Use (34) to get an expression for the aggregate price \( p \) as a function of the average \( r_i \) and \( w_i \), and hence, as a function of \( \int_0^1 \text{E}[x|r_i, w_i] \text{d}i \) and \( \theta \). Then, use the latter into (28) – where \( \lambda = p \) because of (35) – to obtain the actual aggregate demand. Finally plug the actual aggregate demand and the aggregate price jointly into (38). The final expression for \( x \) reads as

\[
x = \beta \int_0^1 \text{E}[x|r_i, w_i] \text{d}i + (1 - \beta)\theta,
\]

where

\[
\beta \equiv \frac{\alpha(\epsilon - \psi^{-1})}{1 + (\epsilon - 1)\alpha} < 1.
\]

Note that (42) has a structure similar to (40), except for the strength of the expectational feedback, which, in contrast to (40), is smaller than one. This feature prevents a switch in the sign of the correlation between \( x \) and \( \theta \), which is positive for whatever precision of the signals. In particular, in the two scenarios of perfect information (\( \sigma^2 = 0 \)) or no information (\( \sigma^2 \to \infty \) and \( \sigma_x^2 \to \infty \)) one gets: \( \text{E}[x|r_i, w_i] = x = p = \theta \), and \( \text{E}[x|r_i, w_i] = 0 \) with \( x = (1 - \beta)\theta \), respectively.\(^{18}\)

\(^{18}\)Most of the literature about endogenous information structures investigates the effect of private signals about the average action, that, in this context, equal to private signals about \( x \). By construction, these signals would not exhibit a sufficient reaction to the average expectation to generate the multiplicity that I document in this paper.
Characterization and existence of an equilibrium

All first order conditions in the model have a multiplicative form, so they can be log-linearized and solved without any approximation. In particular, the requirement of a log-rational equilibrium implies that the aggregate state $X$, as with any other variable in the model, is distributed log-normally according to $X = \bar{X} e^x$, where $x \sim \mathcal{N}(0, \text{var}(x))$ is the stochastic log-deviation of $X$ from its deterministic component $\bar{X}$. The stochastic steady state of $X$ is given by $\bar{X}^{1/\text{var}(x)}$. To better enlighten the origins of the multiplicity, let me state the following proposition that characterizes an equilibrium in terms of a profile of final producers’ expectations about $x$.

**Proposition 5.** Given a profile of weights $\{e_\theta, e_\xi, e_\eta\}$ such that log-linear expectations for final producers are described by

$$E[X|R_i, W_i] = \bar{E} e^{E[x|r_i, w_i]}$$

where $\bar{E} = \bar{X}^{1/\text{var}(x|r_i, w_i)}$ with

$$E[x|r_i, w_i] = e_\theta \theta + e_\xi \xi_i + e_\eta \eta_i,$$

(44) then there exists a unique log-linear conditional deviation and a unique steady state for each variable in the model.

**Proof.** See appendix B.1. 

In practice, an equilibrium is characterized by a distribution of producers’ expectations about $x$, the stochastic aggregate component of local demands. Each individual expectation type $i$ is conditional on the observation of $w_i$ and $r_i$, denoting the stochastic log-components of $W_i$ and $R_i$, respectively. Both are log-linear functions of the shocks, so producers’ expectations must be a linear function of the two signals. In particular, all agents use the rule

$$E[x|r_i, w_i] = b_i \left( \int_0^1 E[x|\tau, w_i] d\tau + (1 - \phi) \theta - \eta \right) + a_i (\theta + \xi_i)$$

(46)

where $a_i$ and $b_i$ denote the weights put by agent $i$ on the local wage and price for local capital, respectively. Let us denote by $a$ and $b$ the average weights. As in section 2, a log-linear equilibrium has to be symmetric. Hence, a profile of the optimal weights given to these two pieces of information maps into a profile of weights $\{e_\theta, e_\xi, e_\eta\}$. The characterization of an equilibrium follows straightaway once the requirement of rational expectations is imposed.

**Definition 3** A log-linear rational expectation equilibrium is characterized by a profile of weights $\{e_\theta, e_\xi, e_\eta\}$ such that (46) are rational expectations of (42) conditional to (36) and (40), with

$$e_\theta = \frac{(1 - \phi)b + a}{1 - b \phi}, \quad e_\xi = a, \quad e_\eta = b$$
being the corresponding weights in (45).

In other words, the number of equilibria of the model corresponds to the number of solutions of the signal extraction problem. In the cases of no information ($\sigma^2 \to \infty$ and $\sigma^2_\xi \to \infty$) and full information ($\sigma^2 = 0$) the economy has a unique equilibrium characterized by $(a, b) = (0, 0)$ and $(a, b) = (0, 1)$, respectively. The following proposition establishes the existence of multiple equilibria at the limit of a vanishing dispersion of local prices for capital, i.e. when $\sigma^2 \to 0$.

**Proposition 6.** Consider the problem of agents forecasting (42) conditionally on the information set $\{r_i, w_i\}$, given by (36) and (40). If the variance of preference shocks $\sigma^2_\xi$ satisfies

$$\sigma^2_\xi < \hat{\tau} = \frac{\phi - 1}{1 - \beta},$$

then in the limit of no productivity shocks, $\sigma^2 \to 0$, there exist:

- a unique perfect-information limit equilibrium, characterized by $a = 0$ and $b = 1$; where neutrality of money holds, i.e.

$$r = p = \theta \text{ and } y = 0,$$

for any realization of $\theta$;

- and two dispersed information limit equilibria, characterized by $a = (1 - \beta)(\phi \sigma^2_\xi)^{-1}$ and $b = b_\pm$ with $\lim_{\sigma^2 \to 0} b_\pm^2 \sigma^2 = \hat{\tau}^{-1}(\sigma^2_\xi - \hat{\tau}^{-2}) \sigma^2_\xi$, where shocks to the value of money yield real effects, in particular,

$$r \to 0, \quad p = (1 - \alpha)\theta \text{ and } y = \alpha\psi^{-1}\theta,$$

for any realization of $\theta$.

Otherwise, only the perfect information limit equilibrium exists.

**Proof.** See appendix B.2. \qed

The intuition for the new condition (47) is simple. A dispersed information limit equilibrium originates when the precision of the overall information about $\theta$ is precisely $\hat{\tau}$. The possibility of achieving this value is prevented when the precision of information is already fixed above this threshold; if producers receive information with exogenous precision exceeding $\hat{\tau}$, nothing can let them lose this information.

In appendix C.2 I prove that (47) is still the relevant condition when considering extended versions of the additional signal (36), which can include correlated noise and/or have endogenous precision (i.e. the signal also reacts to the average expectation). In fact, the existence of dispersed information limit equilibria only hinges on the possibility that the variance of the common component of the endogenous signals shrinks to a sufficiently small number, regardless the presence of common noise into it. Otherwise, the common component of both signals fully reveals and the common noise can be disentangled from the fundamental.
Condition (47) also enlightens the role of preference and technological parameters. Notice that
\[
\hat{\tau} = \frac{(\epsilon - 1)(1 - \alpha)}{1 - \alpha + \alpha \psi^{-1}},
\]
is defined as a simple combination of the CES parameters. In particular, the restriction for the existence of dispersed information limit equilibria relaxes with higher substitutability across goods, i.e. a higher \( \epsilon \), lower returns to scale, i.e. a lower \( \alpha \), and a higher relative risk aversion, i.e. a higher \( \psi \).

In a dispersed information limit equilibrium, a positive \( \theta \), which is equivalent to an increase in the stock of money, has expansionary real effects in line with the original Phelps-Lucas’ island models. A general discussion of the impact of shocks to the value of money is postponed to section 3.3. Finally, the following proposition demonstrates the great amplification of private uncertainty.

**Proposition 7.** In contrast to the perfect-information equilibrium, the dispersed information limit equilibria feature sizable cross-sectional variance of expectations:

\[
\lim_{\sigma^2 \to 0} \lim_{b \to b} \int_0^1 \left( E[x|r_i, w_i] - \int_0^1 E[x|r_i, w_i] \, di \right)^2 \, di = \frac{(1 - \beta)(\phi - 1)}{\phi^2},
\]

(48)

*Proof.* See appendix B.3.

At the dispersed information limit equilibria, idiosyncratic productivity shocks are largely amplified, so that, although infinitesimally small, they account for a fraction of the overall cross-section volatility of expectations. Moreover, the idiosyncratic preference shocks also play a role in pinning down producers’ expectations. Specifically, the contribution of idiosyncratic preference shocks to overall cross-sectional volatility of expectations is given by

\[
a^2 \sigma^2_\xi = (1 - \beta)^2 (\phi^{-2} \sigma^{-2}_\xi) \]

which converges toward (48) when (47) is binding. The contribution of idiosyncratic productivity shocks is instead given by (48) minus \( a^2 \sigma^2_\xi \). The relation between the cross-sectional variance of expectations and the cross-sectional variance of prices and quantities is further discussed in section 3.3.

**Out-of-equilibrium selection**

In this section, I investigate the stability properties of the equilibria. The following analysis extends that in section 2.4 to a case where agents forecast an endogenous state and exogenous information is also available.

Let us start by assessing the iterative process of higher-order beliefs. Suppose that it is common knowledge that the individual weights given to the signals \( \{(a_i, b_i)\}_i \) lie in a neighborhood \( \mathcal{S}(\hat{a}, \hat{b}) \) of the equilibrium characterized by \( (\hat{a}, \hat{b}) \). Common knowledge of \( (a_{i,0}, b_{i,0}) \in \mathcal{S}(\hat{a}, \hat{b}) \) for each \( i \) implies that the average weights \( (a_0, b_0) \) belong to \( \mathcal{S}(\hat{a}, \hat{b}) \). Call \( B : (a, b) \to (a_i, b_i) \) the mapping\(^{19}\) that gives, for a given couple of average weights

\(^{19}\)Given by (63)-(64), see appendix B.4
A couple of individual optimal weights \((a_i, b_i)\). Since \(B(a, b)\) is common knowledge then \((a_0, b_0) \in \mathcal{Z}(\hat{a}, \hat{b})\) implies that a second-order belief is rationally justified for which \((a_{i,1}, b_{i,1}) = B(a_0, b_0)\) for each \(i\), so that \((a_{i,1}, b_{i,1}) \in B(\mathcal{Z}(\hat{a}, \hat{b}))\) and as a consequence \((a_1, b_1) \in B(\mathcal{Z}(\hat{a}, \hat{b}))\). Iterating the argument we have that \((a_\nu, b_\nu) \in B^\nu(\mathcal{Z}(\hat{a}, \hat{b}))\).

**Definition 4.** A REE characterized by \((\hat{a}, \hat{b})\) is a locally unique rationalizable outcome if and only if

\[
\lim_{\nu \to \infty} B^\nu(\mathcal{Z}(\hat{a}, \hat{b})) = (\hat{a}, \hat{b}).
\]

From an operational point of view local uniqueness requires that

\[
||J(B(\hat{a}, \hat{b}))|| < 1,
\]

where \(J(B(\hat{a}, \hat{b}))\) is the Jacobian of the map \(B\) calculated at the equilibrium values \((\hat{a}, \hat{b})\). The following proposition states a result that holds for all the cases investigated in this section.

**Proposition 8.** Whenever dispersed information limit equilibria exist, they are locally unique rationalizable outcomes, whereas the perfect-information limit equilibrium is never.

**Proof.** See appendix B.4.

The intuition for the result is similar to the one discussed for the unidimensional case in section 2.4. The local instability of the perfect information outcome relies on the high elasticity of price signals to expectation, which is maximal when close to the perfect information scenario. This generates a circle of high complementarity between input demand and price signal, that pushes the signal toward steady state, as the aggregate supply of inputs is predetermined. At the point where the variance of the common component of the price signal is of the same order of magnitude as the one of the idiosyncratic component, the price signal loses informativeness. Thus, the circle of high complementarity eventually dampens and the dynamics converges to a fixed point featuring a dispersed information equilibrium.

Let us turn attention now to adaptive learning. At the end of each period producers see the realized demand, so they can eventually revise their beliefs in light of the realized forecast error. Following the approach described in section 2.4, suppose that, at time \(t\), agents set their weights \(\chi_{i,t} = [a_{i,t} \ b_{i,t}]'\) in accordance with the bivariate recursive OLS expression:

\[
\begin{align*}
\chi_{i,t} &= \chi_{i,t-1} + t^{-1} S_{i,t-1}^{-1} \omega_{i,t} (x_t - \chi_{i,t-1}' \omega_{i,t}) \\
S_{i,t} &= S_{i,t-1} + (t+1)^{-1} (\omega_{i,t} \omega_{i,t}' - S_{i,t-1}),
\end{align*}
\]

with \(\omega_{i,t} \equiv [w_{i,t} \ r_{i,t}]'\), where \(S_{i,t}\) is the current estimate of the variance-covariance matrix of price signals. The following proposition formally defines adaptive stability.

**Definition 5.** A REE characterized by \((\hat{a}, \hat{b})\) is a locally learnable equilibrium if and only if there exists a neighborhood \(F(\hat{a}, \hat{b})\) of \((\hat{a}, \hat{b})\) such that, given initial estimates \((a_{i,0}, b_{i,0}) \in F(\hat{a}, \hat{b})\), it is \(\lim_{t \to \infty} (a_{i,t}, b_{i,t}) \overset{a.s.}{=} (\hat{a}, \hat{b})\).
In other words, once estimates are close to the equilibrium values of a locally adaptively stable equilibrium, then convergence toward the equilibrium will almost surely occur.

**Proposition 9.** Whenever dispersed information limit equilibria exist, they are locally adaptively stable, whereas the perfect-information limit equilibrium is never.

*Proof.* See appendix B.5.

As before, the proof uses well known results to show that the asymptotic stability of the system around an equilibrium \((\hat{a}, \hat{b})\) obtains if and only if

\[
J(B)_{(\hat{a}, \hat{b})} < 1,
\]

which is a bi-dimensional transposition of the E-stability principle mentioned previously. Therefore, also in this case, equilibria that are locally unique rationalizable outcomes are also adaptively stable.

### 3.3 Extension and out-of-the-limit exploration

In this section, I explore an extended version of the model beyond the limit of zero cross-sectional variance of productivity shocks. The aim is threefold:

1) to demonstrate that the findings are robust to natural generalizations of the economic environment;

2) to show that the qualitative properties of equilibria at the limit are inherited, by continuity, by equilibria outside the limit;

3) to argue that the economic implications of the model are consistent with widely accepted empirical findings about the effects of monetary shocks.

For the sake of generality, I will focus on an extended version of the model that includes: a more general specification of the utility function,

\[
U_t \equiv E_t \left[ \sum_{t=0}^{\infty} \delta^t \left( e^{\theta_t} \left( \frac{C_t^{1-\psi}}{1-\psi} - \int_0^1 e^{\xi_{i,t}} \frac{(L_{i,t})^{1+\gamma}}{1+\gamma} di \right) + \log \frac{M_t}{P_t} \right) \right],
\]

instead of (19), where \(\gamma > 0\) is the inverse of the Firsh elasticity of labor; and a more general specification of the intermediate technology,

\[
K_{i,t}^{s} \equiv e^{\eta_t}Z_{(i),t}^{1-\zeta},
\]

instead of (22), where \(\zeta \in (0, 1)\) features decreasing returns to scale in intermediate production. The values \(\gamma = \zeta = 0\) entail the case studied above. The proof of proposition 5 in appendix B.1 already considers these more general specifications.
Equilibrium weights and cross-sectional variance of expectations

Figure 4 plots the equilibrium weights and the variance of the individual expectation, as functions of \( \sigma^2 \) for a baseline calibration \( \epsilon = 8, \alpha = 0.33, \psi = 2, \gamma = 0.75, \zeta = 0 \) and two different values of \( \sigma^2 \), namely 0.4 (solid line) and 1.4 (dotted line). The three weights \( e_\theta, e_\xi \) and \( e_\eta \) can be interpreted as the response of the expectation type \( i \) to a unitary realization of \( \theta, \xi_i, \) and \( \eta_i \), respectively.

For a sufficiently low \( \sigma^2 \), three equilibria values exist. In the first two panels, dispersed information limit equilibria correspond to the two equilibria values converging toward the tangency point on the y-axis as \( \sigma^2 \) approaches 0, whereas the unique perfect information limit equilibrium is represented by the intersection with the y-axis. The third panel shows that, as \( \sigma^2 \) approaches 0, \( e_\eta \) goes to either plus or minus infinity: bear in mind that this is a reaction to a unitary realization of a shock whose variance gets infinitesimal.

The last panel concerns the cross-sectional variance of conditional expectations, which is zero in the perfect information limit equilibrium and is given by (48) in the dispersed information limit equilibria. Notice the large multiplier effect of the price feedback on private uncertainty: the introduction of infinitesimal idiosyncratic shocks can generate a large cross-sectional variance of beliefs.

Three remarks are in order here. First, in analogy with the insights of figure 1, only one equilibrium out of three survives for any positive \( \sigma^2 \). This equilibrium becomes a dispersed information limit equilibrium as \( \sigma^2 \) approaches 0.

Second, at the dispersed information limit equilibria, changes in \( \sigma^2 \) do not affect producers’ reaction to an aggregate shock as long as (47) holds. Nevertheless, a higher \( \sigma^2 \) decreases the parameter region where a multiplicity exists for a positive \( \sigma^2 \).

Third, the second panel shows that the equilibrium value of \( e_\xi \) is inversely proportional to \( \sigma^2 \). This effect occurs because, the higher the \( \sigma^2 \), the lower the information transmitted by the local wage. This is an essential feature of learning from prices. Notice that rational inattention (Mackowiak and Wiederholt, 2009) implies the opposite comparative statics: the higher the variance of idiosyncratic shocks, the higher the attention agents are willing to pay to them, and so the higher the sensitivity of expectations to idiosyncratic conditions.

The impact of an aggregate shock on aggregate variables

Let us look now to the impact of an aggregate shock on aggregate variables. Figure 5 illustrates the equilibrium reactions of \( p, y, r \) and \( w \) induced by a unitary realization of \( \theta \). As in figure 4, the analysis is extended to values of \( \sigma^2 \) strictly above zero. In particular, the picture sheds light on the role of the elasticity of labor supply. The plot is obtained for the baseline calibration: \( \epsilon = 8, \psi = 2, \alpha = 0.33 \) and \( \sigma^2 = 0.4 \) and two different values of \( \gamma \), namely 0 (solid line) and 0.75 (dotted line).

\[^{20}\text{Note that } \zeta \text{ does not affect the determination of the impact of the aggregate shock on aggregate variables due to market clearing condition in the market for raw capital.}\]
Figure 4: Equilibrium weights and cross-sectional variance of expectations for $\epsilon = 8$, $\alpha = 0.33$, $\psi = 2$, $\gamma = 0.75$, $\zeta = 0$ for: $\sigma_\xi^2 = 0.4$ (solid) and $\sigma_\xi^2 = 1.4$ (dashed).

Figure 5: The reaction of aggregate variables to $\theta = 1$ for $\epsilon = 8$, $\alpha = 0.33$, $\psi = 2$, $\sigma_\xi^2 = 0.4$, $\zeta = 0$ for: $\gamma = 0$ (solid) and $\gamma = 0.75$ (dashed).
Figure 6: The cross sectional dispersion of $p_i, y_i, r_i$ and $w_i$ for $\epsilon = 8$, $\alpha = 0.33$, $\psi = 2$, $\sigma_\xi^2 = 0.4$, $\gamma = 0.75$ and $\zeta = 0$. The dotted line denotes the contribution of idiosyncratic preference shocks to overall dispersion.

Figure 7: Cross sectional dispersion of local prices and produced quantities for $\epsilon = 8$, $\alpha = 0.33$, $\psi = 2$, $\sigma_\xi^2 = 0.4$, $\gamma = 0.75$ for: $\zeta = 0$ (solid) and $\zeta = 0.2$ (dashed).
At the perfect-information equilibrium, all prices react one to one to a unitary realization of the aggregate shock, whereas average real quantities remain at steady state, that is, $p = w = r = \theta$ and $l = y = 0$. On the contrary, at the dispersed information equilibria, shocks to the value of money feature real effects: aggregate price and output go up together, although the price level rises less than with money neutrality. The temporary effect of money arises because, as local prices lose precision about aggregate conditions, firms are induced to overestimate adverse local conditions; thus, final producers set lower than optimal prices, which boost actual demand. Notice that, at the dispersed information equilibrium, the model predicts a rise in the real average wage $(w - p)$ and a decrease in the real average return on capital $(r - p)$. This is because producers are rather pessimistic when they buy capital, but then, once actual demand realizes, they hire more labor than expected.

All these effects are consistent with the original insights of the Lucas island model (Lucas, 1973) and the medium-term impact of monetary shocks discussed in seminal works like Woodford (2003), Christiano et al. (2005) and Smets and Wouters (2007), among others. In contrast to traditional models of price rigidities, in the current setting, the non-neutrality of money only hinges on an arbitrarily small dispersion in fundamentals.

Finally, a higher convexity in labor disutility, i.e. a higher $\gamma$, magnifies the reaction of $p, y$ and $w$, whereas dampens the one of $r$. A higher $\gamma$ also decreases the parameter region where a multiplicity exists for a positive $\sigma^2$.

The impact of idiosyncratic shocks on island-specific variables

In figure 6, a solid line denotes the cross-sectional volatility of $p_i, y_i, r_i$ and $w_i$, as a function of $\sigma^2$ for the baseline calibration: $\epsilon = 8, \alpha = 0.33, \psi = 2, \sigma^2_\xi = 0.4, \gamma = 0.75$ and $\zeta = 0$. The dotted line plots the fraction of overall cross-sectional volatility that is due to idiosyncratic preference shocks only. At the perfect information limit equilibrium, which corresponds to the y-intercept, idiosyncratic preference shocks explain all the dispersion in the economy. At the dispersed information limit equilibria, which correspond to the point of tangency of the curve with the y-axis, idiosyncratic preference shocks do not explain all the dispersion in the economy, although their impact increases. Hence, on the one hand, infinitesimally small productivity shocks are responsible for a sizable fraction of price dispersion; on the other hand, the importance of preference shocks is also magnified.

This finding is in line with the evidence put forward by Boivin et al. (2009) that, although prices respond little to aggregate shocks, they respond largely to idiosyncratic shocks, so that, the latter explain most overall price volatility in the final markets. The same picture holds for the cross-sectional variance of local output and local wage, whereas the dispersion of the price for local capital is exogenously fixed by $\sigma^2$.

Figure 7 explores the role of decreasing returns to scale in intermediate production. The solid line plots the same solid line in figure 6, whereas the dashed line plots the curve obtained for $\zeta = 0.2$ holding everything else unchanged. The amount of cross-sectional variance generated by a vanishing $\sigma^2$ is measured by the distance between the intercept
and the point of tangency on the y-axis, which identify the perfect information limit equilibrium and the dispersed information limit equilibria, respectively. A higher $\zeta$ tends to increase the cross-sectional volatility of $p_i, y_i, r_i$, and $w_i$, although this is not always the case (look for instance at the dispersion of $p_i$ and $y_i$ close to the perfect information limit equilibrium). In particular, note that the dispersion of $r_i$, which are now also a function of traded quantities, is higher in the perfect information limit equilibrium than in the dispersed information limit equilibria.

The figures demonstrate that, in the current setting, a tiny cross-sectional variance of productivity shocks can generate sizable dispersion of prices and quantities in the economy. In this case, an external observer looking for the source of firms’ disagreement could easily miss its fundamental nature. This results offers a different interpretation of the price dispersion puzzles documented, for example, by Eden (2013) and Kaplan and Menzio (2015), for which search and matching models (see Burdett and Judd (1983)) generally offer an alternative explanation. In particular, non-measurable dispersion in fundamentals can generate sizable price dispersion in final markets without assuming any search friction.

4 Conclusion

This paper set out the conditions under which the introduction of an arbitrarily small degree of dispersion in fundamentals can feature price rigidity as an equilibrium outcome. When agents learn from local prices, no matter how small their dispersion, the tension between the allocative and informative role of prices can dramatically alter their functioning. I have showed that this mechanism is naturally reproduced by a typical input demand function, and so potentially pertains to a large class of macro models. In particular, I presented a standard monetary model with flexible prices with a unique equilibrium under perfect information where money neutrality holds. I have showed that the introduction of vanishingly small heterogeneity in input markets can generate price rigidity to aggregate monetary shocks and overreaction to idiosyncratic shocks, as an equilibrium outcome. This equilibrium, which I tagged as a dispersed information limit equilibrium, has strong stability properties, in contrast to the perfect-information equilibrium.

I leave some important issues for future research. The present model is essentially static in nature, as current realizations are not informative about the future course of the economy. When this is not the case, agents accumulate correlated information over time. The extent to which non-neutralities can persist in a economy where agents learn from current and past prices is a question that hopefully this paper can help to address in a near future. Finally, despite the fact that the model can naturally generate nominal rigidities and even price stickiness in the limit, it cannot reproduce the frequency of price adjustments, which is the focus of most of the micro econometric literature on price setting.
Appendix

A  A stylized economy

A.1 Proof. of proposition 1

With $\sigma^2 = 0$, $b_i(b)$ is linear, so it is trivial to prove $b_* = (\mu - \alpha)^{-1}$ to be the unique solution of the fixed point equation $b_i(b) = b$. No matter how small is $\sigma^2$, the fixed point equation $b_i(b) = b$ is instead a cubic, so that it has at most three real roots. By continuity there must exist a positive solution $b_o$ such that $\lim_{\sigma^2 \to 0} b_i(b_o) = b_o = b_*$. The condition $\mu > \alpha$ is necessary and sufficient to have

$$\lim_{\sigma^2 \to 0} b'_i(b_*) = \lim_{\sigma^2 \to 0} b'_i(b_o) = \frac{\mu}{\alpha} > 1.$$  

In such a case, since for any non-null $\sigma^2$ we have $\lim_{b \to +\infty} b_i(b) = 0$, then by continuity at least another intersection of $b_i(b)$ with the bisector at a positive value $b_+$ must exist too. Notice also that $b_i(0) < 0$. This, jointly with the fact that for $\sigma^2 \neq 0$ it is $\lim_{b \to -\infty} b_i(b) = 0$, implies that by continuity at least an intersection of $b_i(b)$ with the bisector at a negative value $b_-$ exists too. Therefore $(b_-,b_o,b_+)$ are the three solutions we were looking for.

Suppose now that $b_-$ and $b_+$ take finite values - that is $|b_\pm| \leq M$ with $M$ being an arbitrarily large finite number - then $\lim_{\sigma^2 \to 0} \sigma^2 (1 - b_\pm^2) = 0$ and so necessarily $b_\pm = b_o > 0$. Nevertheless $b_- < 0$ so a first contradiction arises. Moreover if $b_+$ is arbitrarily close to $b_o$ then by continuity $\lim_{\sigma^2 \to 0} b'_i(b_+) > 1$ which, for the same argument above, would imply the existence of a fourth root; a second contradiction arises. Hence we conclude $|b_\pm| > M$ with $M$ arbitrarily large.

Finally let me prove that, if and only if $\mu > \alpha$, a multiplicity arises for a $\sigma^2$ small enough. For $\mu < \alpha$, two cases are possible, either $0 < \lim_{\sigma^2 \to 0} b'_i(b_0) < 1$ or $\lim_{\sigma^2 \to 0} b'_i(b_0) < 0$. The case $0 < \lim_{\sigma^2 \to 0} b'_i(b_0) < 1$ implies that for a sufficiently small $\sigma^2$ at least one intersection of $b_i(b)$ with the bisector must exist by continuity. Nevertheless, in this case, this intersection is unique as the existence of a second one would require $\max b'_i(b) > 1$ (given that $\lim_{b \to \pm \infty} b_i(b) = 0$), but $\sup b \lim_{\sigma^2 \to 0} b'_i(b) = \lim_{\sigma^2 \to 0} b'_i(b_o)$.

A.2 Proof. of proposition 2

Taking the limit $\lim_{\sigma^2 \to 0} \lim_{b \to b_\pm}$ (where $|b_\pm| > M$ with $M$ arbitrarily large) to both sides of fixed point (13) with $b_i(b) = b$, we get

$$\lim_{\sigma^2 \to 0} \lim_{b \to b_\pm} (1 - \mu b)^2 \sigma^2 = -\alpha^2 - \lim_{b \to b_\pm} \frac{\alpha(1 - \mu b)}{b} = \alpha(\mu - \alpha),$$

which is positive provided $\mu > \alpha$. Then, given (12), we have (14).

The cross-sectional variance of individual expectations is given by $b^2 \sigma^2$. Notice that

$$\lim_{\sigma^2 \to 0} \lim_{b \to b_\pm} (1 - \mu b)^2 \sigma^2 = \lim_{\sigma^2 \to 0} \lim_{b \to b_\pm} \mu^2 b^2 \sigma^2,$$
and hence,
\[
\lim_{\sigma^2 \to 0} \lim_{b \to b_\pm} b^2 \sigma^2 = \frac{\alpha (\mu - \alpha)}{\mu^2}
\]

Given (10) it is easy to show that
\[
\lim_{\sigma^2 \to 0} \lim_{b \to b_\pm} \int_0^1 E[\theta | r_i] di = -\lim_{\sigma^2 \to 0} \lim_{b \to b_\pm} \frac{\alpha b}{1 - \mu b} \theta = \frac{\alpha}{\mu} \theta,
\]
so that the local price in the limit becomes \( r_i \to 0 \).

### A.3 Proof of proposition 3

The derivative of (13) with respect to \( b \) is given by
\[
b'_i(b) = -\frac{\alpha \mu (\mu^2 b^2 \sigma^2 - 2 \mu b \sigma^2 - \alpha^2 + \sigma^2)}{(\mu^2 b^2 \sigma^2 - 2 \mu b \sigma^2 + \alpha^2 + \sigma^2)^2}.
\]

For the perfect information limit equilibrium, we have
\[
\lim_{\sigma^2 \to 0} \lim_{b \to b_\pm} b'_i(b) = \frac{\mu}{\alpha} > 1.
\]

For the dispersed information limit equilibria notice that
\[
\lim_{\sigma^2 \to 0} \lim_{b \to b_\pm} b'_i(b) = \frac{\alpha \mu (\alpha^2 + \alpha (\alpha - \mu))}{(\alpha^2 - \alpha (\alpha - \mu))^2}
\]

after using (51) and (52) to compute the limit \( \lim_{\sigma^2 \to 0} \lim_{b \to b_\pm} \mu^2 b^2 \sigma^2 \). Finally we can prove
\[
\lim_{\sigma^2 \to 0} \lim_{b \to b_\pm} b'_i(b) < 1,
\]
\[
\lim_{\sigma^2 \to 0} \lim_{b \to b_\pm} b'_i(b) > -1,
\]
given that, respectively,
\[
\alpha \mu (\alpha^2 + \alpha (\alpha - \mu)) - (\alpha^2 - \alpha (\alpha - \mu))^2 = -2 \alpha^2 \mu (\mu - \alpha) < 0,
\]
\[
\alpha \mu (\alpha^2 + \alpha (\alpha - \mu)) + (\alpha^2 - \alpha (\alpha - \mu))^2 = 2 \alpha^3 \mu > 0.
\]

### A.4 Proof of proposition 4

To check local learnability of the REE, suppose we are already close to the rest point of the system. That is, consider the case \( \int \lim_{t \to \infty} b_{i,t} \, di = \hat{b} \) where \( \hat{b} \) is one among the equilibrium points \( \{b_-, b_0, b_+\} \) and so
\[
\lim_{t \to \infty} S_{i,t} = var(r_{i,t}) = \frac{\alpha^2}{(1 - \mu \hat{b})^2} + \sigma^2,
\]
according to (11). From standard results in the stochastic approximation theory, one can write
the associated ODE governing the stability around the equilibria as
\[
\frac{db_t}{dt} = \int \lim_{t \to \infty} E \left[ S_{i,t-1}^{-1} r_{i,t} (\theta_t - b_{i,t-1} r_{i,t}) \right] dt = \var(r_{i,t})^{-1} \int E [r_{i,t} (\theta_t - b_{i,t-1} r_{i,t})] dt = \var(r_{i,t})^{-1} (\text{cov}(\theta, r_{i,t}) - b_t \var(r_{i,t})) = b_t (b_t) - b_t,
\]
where we used relation (13).

B The Fragile Neutrality of Money

B.1 Proof of proposition 5

First-order conditions and information in the extended model. The aim here is to show that for each variable in the model there exists a unique steady state and a unique log-linear deviation from a deterministic component implied by fixing a profile of coefficients \((e_\theta, e_\xi, e_\eta)\). We will consider the extended version of the economy with (49) and (50) instead of (19) and (22). The baseline configuration obtain for \(\gamma = \xi = 0\). Let us start providing the full list of first-order conditions and discuss how final producers’ learning is affected in the extended model.

The household optimizes her problem when
\[
Y_i = P_i^{-\epsilon} P^\epsilon Y, \quad (\text{foc1})
\]
\[
P = e^{\theta Y - \psi}, \quad (\text{foc2})
\]
\[
W_i = e^{\theta + \xi L^\gamma_i}, \quad (\text{foc3})
\]
where we already used (35) with \(\delta((1 - \delta) M)^{-1}\) being normalized to one without any loss of generality.

Because of the convexity in the disutility of labor, the wages cannot be determined independently from the quantity of labor hired. In this case, there is a conflict between observing the equilibrium wage at stage two and leaving working hours determined by actual demand in stage three. To circumvent this difficulty we assume that, at the time when final producers make their decision \((K_i, P_i)\), they are informed about the labor supply schedule (foc3) posted by workers, i.e. final producers know the equilibrium wage contingent to each possible quantity traded. As a consequence, knowing the the labor supply schedule is informationally equivalent to observing \(e^{\theta + \xi_i}\), which is consistent with our baseline model.

The intermediate producers maximize their profits when
\[
R_i = (1 - \zeta)^{-1} e^{-\eta} R Z^\zeta(i), \quad (\text{foc4})
\]
Decreasing returns to scale on intermediate production makes \(R_i\) depends on \(Z(i)\). At the same time \(K_i = K_i^* = e^{\eta} Z^\zeta(i)\) is known, as it is final producers’ own action, and does not convey additional information (see note 8 in analogy). Hence, the information that final producers get from the local capital market is summarized by
\( \hat{R}_i \equiv (1 - \zeta) R_i K_i^{-\zeta/(1 - \zeta)} = e^{-\eta_i/(1 - \zeta)} R_i \),

which is identical to (32) up to a re-scaling of the variance of \( \eta_i \). Therefore, final producers optimize their expected profits conditionally to \( (\hat{R}_i, e^{\theta + \xi_i}) \) that constitutes the information transmitted by the local input markets. The first order condition of final producers relative to the choices \( (K_i, P_i) \) that they carry out in the second stage are

\[
\begin{align*}
K_i &= \alpha^{1-\alpha}(1-\alpha)^{\alpha-1} E[W_i^{1-\alpha} R_i^{\alpha - 1} \hat{R}_i | \hat{R}_i, e^{\theta + \xi_i}], \\
P_i &= \epsilon(\epsilon - 1)^{-1} \alpha^{\alpha - 1} E[R_i^{\alpha} W_i^{1-\alpha} | \hat{R}_i, e^{\theta + \xi_i}].
\end{align*}
\]

(foc5) (foc6)

In the third stage final producers hire working hours to satisfy actual demand.

**Log-linear algebra: preliminaries.** This section aims to clarify some issues linked to the manipulation of log-linear relations. The requirement that the equilibrium has a symmetric log-linear representation implies that the individual production \( Y_i \), as any other variable in the model, has the form

\[
Y_i(\theta, \xi_i, \eta_i) = \bar{Y}_i e^{y_i(\theta, \xi_i, \eta_i)},
\]

where \( y_i \equiv \log Y_i - \log \bar{Y}_i \) is defined as the distance of the logarithm of the particular realization \( Y_i \) from a constant deterministic component \( \bar{Y}_i \). In particular, the log deviation \( y_i \) is a linear combination of the shocks

\[
y_i(\theta, \xi_i, \eta_i) = y_i,\theta + y_i,\xi_i + y_i,\eta_i,
\]

where we use roman characters to index the relative weight of a shock in the combination. Notice that the stochastic steady state of individual production is given by its unconditional mean, \( \bar{Y}_i e^{\text{var}(y_i)/2} \), with \( \text{var}(y_i) \) being the unconditional variance of \( y_i \). In analogy, let us denote by \( y_{a,\theta}, y_{a,\xi} \) and \( y_{a,\eta} \) the log-deviations of the aggregate \( Y \) relative to a unitary increase in \( \theta, \xi \) and \( \eta \), respectively. Of course, by definition we have

\[
Y(\theta) = \bar{Y} e^{y_{a,\theta}},
\]

that is \( y_{a,\xi} = y_{a,\eta} = 0 \). At the same time, the aggregate production can be obtained by aggregating (55) across agents

\[
\left( \int Y_i^{\theta-1} \text{d}i \right)^{\frac{\theta}{\theta - 1}} = \bar{Y} e^{y_{i,\theta} + y_{i,\xi_i} + y_{i,\eta_i}},
\]

so that, contrasting (55) and (58), we conclude \( y_{i,\theta} = y_{a,\theta} \) for any \( \theta > 1 \). In other words, the average log deviation of the individual production is equal to the log deviation of the aggregate production. This is true for whatever CES aggregation.

Therefore, in the following we are going to drop the distinction between individual and aggregate focusing on \( (y_{\theta}, y_{\xi}, y_{\eta}) \) with the understanding that \( y_{\theta} = y_{i,\theta} = y_{a,\theta}, \ y_{\xi} = y_{i,\xi} \) and \( y_{\eta} = y_{i,\eta} \). Regarding the deterministic component of (57) and (58), notice that

\[
\bar{Y} = \bar{Y} e^{-\text{var}(y_{i,\xi_i} + y_{i,\eta_i})},
\]

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that is, both coincide only in the deterministic scenario, in which case they denote the deterministic steady state.

**Uniqueness.** Under the restriction of log linearity the logarithm of each variable is determined by four components, which lie in the four independent complementary subspaces spanned by \((1, \theta, \xi, \eta_i)\). We can therefore study the reaction of the system to each component separately.

First, we are going to consider the reaction to an unitary aggregate shock \(\theta\). Let us determine the variables fixed at stage 2 for a given \(e_\theta\), that represents the weight on \(\theta\) in the stochastic component of the individual expectation about \(X = PY^{1/4}\). To do that in the extended model we need to introduce expectations of final producers on the equilibrium wage and labor, which in turn will depend on expected demand for the fixed price. This corresponds to the following system of relations derived from (foc1), (foc3),(foc4),(foc5), (foc6), the market clearing condition \(\int z(i) di = 0\) and the final technology (24):

\[
\begin{align*}
y^e_\theta &= \epsilon(e_\theta - p_\theta) \\
w^e_\theta &= \gamma l^e_\theta + 1 \\
k_\theta &= 0 = (1 - \alpha)(w^e_\theta - r_\theta) + y^e_\theta \\
p_\theta &= \alpha r_\theta + (1 - \alpha)w^e_\theta, \\
(1 - \alpha)l^e_\theta &= y^e_\theta,
\end{align*}
\]

where \(w^e_\theta, l^e_\theta\) and \(y^e_\theta\) denote the impact of an unitary increase in \(\theta\) on final producers’ expectations about \(W_i, L_i\) and \(Y_i\), respectively. The expectation about \(L_i\) is needed only with \(\gamma \neq 0\) as the producers need to forecast the equilibrium wage to determine the optimal price. The system yields a unique solution

\[
\begin{align*}
y^e_\theta &= \Delta_\theta(\epsilon(1 - \alpha)(e_\theta - 1)) \\
w^e_\theta &= \Delta_\theta(\epsilon\gamma e_\theta + (1 - \alpha + \epsilon\alpha)) \\
r_\theta &= \Delta_\theta(\epsilon (1 + \gamma) e_\theta - (\epsilon - 1)(1 - \alpha)) \\
p_\theta &= \Delta_\theta(\epsilon (\alpha + \gamma) e_\theta + 1 - \alpha), \\
l^e_\theta &= \Delta_\theta(\epsilon\epsilon e_\theta - 1)
\end{align*}
\]

with \(\Delta_\theta \equiv (1 - \alpha + \epsilon(\alpha + \gamma))^{-1}\). Then, we can determine the equilibrium variables fixed at the third stage. The relations (foc2),(foc3), the final technology (24) and the definition of \(X (38)\) determine, for a given \(p_\theta\), the actual impact on \(x_\theta, y_\theta, l_\theta\) and \(w_\theta\) in the last stage, so that

\[
\begin{align*}
1 - \psi y_\theta &= p_\theta, \\
w_\theta &= \gamma l_\theta + 1, \\
y_\theta &= (1 - \alpha)l_\theta, \\
x_\theta &= \epsilon^{-1}y_\theta + p_\theta,
\end{align*}
\]

In particular, note that

\[
x_\theta = ((\epsilon - \psi^{-1})(\alpha + \gamma)e_\theta + 1 - \alpha + \psi^{-1}(\alpha + \gamma))\Delta_\theta
\]

\footnote{foc2 is needed only if one uses expectations about either aggregate price \(P\) or aggregate demand \(Y\), rather than expectation on the combination of the two, \(X\), as I do here.}
which is consistent with (42). Moreover, for \( e_\theta = 1 \) we have \( p_\theta = r_\theta = w_\theta = w^e_\theta = 1 \) and \( y_\theta = y^e_\theta = l_\theta = l^e_\theta = 0 \), which entail the neutrality-of-money outcome.

Let turn attention now to the impact of idiosyncratic deviations, which only concern island-specific variables. Let us start with the variables fixed at the second stage. The system relative to \( \xi \) and \( \eta \) is determined by (foc1),(foc3),(foc4),(foc5),(foc6) and technologies (24) and (50), yielding:

\[
\begin{align*}
y^e_{\xi,\eta} &= \epsilon(e_{\xi,\eta} - p_{\xi,\eta}), \\
w^e_{\xi,\eta} &= \gamma l^e_{\xi,\eta} + 1, \\
r^e_{\xi,\eta} &= \zeta z_{\xi,\eta} - 1, \\
(1 - \zeta)z_{\xi,\eta} &= k_{\xi,\eta} - 1, \\
k_{\xi,\eta} &= y^e_{\xi,\eta} + (1 - \alpha)(w^e_{\xi,\eta} - r_{\xi,\eta}), \\
p_{\xi,\eta} &= \alpha r_{\xi,\eta} + (1 - \alpha)w^e_{\xi,\eta} \\
(1 - \alpha)l^e_{\xi,\eta} &= y^e_{\xi,\eta} - \alpha k_{\xi,\eta}.
\end{align*}
\]

where I index the direct impact of the two shocks \( \xi_i \) and \( \eta_i \) on an individual expectation about \( X \) with \( 1_\xi \) and \( 1_\eta \), respectively. There is a unique solution to the system given by

\[
\begin{align*}
y^e_\xi &= \Delta(\epsilon(\alpha \gamma - (1 + \gamma)\alpha \zeta + 1)e_\xi - \epsilon(1 - \alpha)) \\
r^e_\xi &= \Delta(\epsilon \alpha \gamma + 1)e_\xi - \zeta(1 - \alpha)(\epsilon - 1)) \\
w^e_\xi &= \Delta(\epsilon \alpha \gamma e_\xi + 1 + \zeta(1 - \alpha)) \\
p^e_\xi &= \Delta(\epsilon \alpha \gamma e_\xi + 1 - \zeta(1 - \alpha)(\epsilon - 1)) \\
l^e_\xi &= \Delta(\epsilon \alpha \gamma e_\xi - \epsilon(1 - \alpha) - \zeta(1 - \alpha)(\epsilon - 1)) \\
z^e_\xi &= \Delta(\epsilon \alpha \gamma e_\xi + 1 - \zeta(1 - \alpha)(\epsilon - 1)) \\
k^e_\xi &= \Delta(\epsilon (\alpha \gamma + 1)(1 - \zeta)e_\xi - (1 - \alpha)(\epsilon - 1)(1 - \zeta)).
\end{align*}
\]

and

\[
\begin{align*}
y^e_\eta &= \Delta((\epsilon + \gamma \alpha \epsilon - \zeta(1 + \gamma)e_\eta + \alpha \epsilon(\gamma + 1)) \\
r^e_\eta &= \Delta(\epsilon \alpha \gamma(1 + 1) + e_\eta - 1 - \gamma(\alpha + \epsilon(1 - \alpha)) \\
w^e_\eta &= \Delta(\epsilon \alpha \gamma e_\eta + \alpha \gamma(\epsilon - 1)) \\
p^e_\eta &= \Delta(\epsilon \gamma(1 - \alpha) + \alpha \zeta(1 + \gamma)(1 + \gamma))e_\eta - \alpha(1 + \gamma)) \\
l^e_\eta &= \Delta(\gamma e_\eta + \alpha(\epsilon - 1)) \\
z^e_\eta &= \Delta(\epsilon(1 + \gamma)e_\eta + \alpha(1 + \gamma)(\epsilon - 1)) \\
k^e_\eta &= \Delta(\epsilon(1 + \gamma)(1 - \zeta)e_\eta + 1 + (\epsilon - 1)\alpha + \gamma(\epsilon - 1))
\end{align*}
\]

with \( \Delta \equiv (1 + \gamma(\alpha(1 - \epsilon) + \epsilon) + \zeta(\epsilon - 1)(1 + \gamma))^{-1} \). Finally, the relations (foc1),(foc3) and (24) determine the actual cross section of \( y_{\xi_i}, l_i \) and \( w_i \) as

\[
\begin{align*}
y_{\xi,\eta} &= -\epsilon p_{\xi,\eta}, \\
w_{\xi,\eta} &= \gamma l_{\xi,\eta} + 1, \\
y_{\xi,\eta} &= \alpha k_{\xi,\eta} + (1 - \alpha)l_{\xi,\eta}.
\end{align*}
\]

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for given \( p_{\xi,\eta} \) and \( k_{\xi,\eta} \).

Finally, to pin down the deterministic component we can easily solve the model in the deterministic case. The height relations (foc2), (foc3), (foc4), (foc5), (foc6), (24), (50) and the market clearing condition \( \int z_i \, di = 0 \), form a linear system in height unknown, namely \( \log \bar{Y} \), \( \log \bar{P} \), \( \log \bar{R}_i \), \( \log \bar{W}_i \), \( \log \bar{R} \), \( \log \bar{L}_i \), \( \log \bar{K}_i \), \( \log \bar{Z}_i \) (implicitly, expectations on deterministic components are trivially correct) which define a unique deterministic steady state. The unique stochastic steady state of \( Y \), as of any other variable in the model, is easily given by \( \bar{Y} e^{\text{var}(\varphi)/2} \).

Therefore, also the stochastic steady state is uniquely determined, with island-specific and aggregate steady state variables differing for a second-order constant term.

B.2 Proof of proposition 6

Given that all agents use the rule (46), then the average expectation is

\[
\int_0^1 E[x|w_i, r_i] \, di = \frac{a + \left(1 - \phi\right) b}{1 - \phi b} \theta.
\]

Hence, an individual expectation can be rewritten as

\[
E[x|w_i, r_i] = b \left( \phi \frac{a + \left(1 - \phi\right) b}{1 - \phi b} \theta + \left(1 - \phi\right) \theta + \eta_i \right) + a \left( \theta + \xi_i \right),
\]

\[
E[x|w_i, r_i] = b \left( \frac{1 - \phi + \phi a}{1 - \phi b} \theta + \eta_i \right) + a \left( \theta + \xi_i \right).
\]

The actual law of motion of the aggregate component of island-specific demand (42) is given by

\[
x = (1 - \beta) \theta + \beta \frac{a + \left(1 - \phi\right) b}{1 - \phi b} \theta,
\]

which is a linear combination of exogenous shocks only. Now we can compute a fixed point equation working out the following orthogonal restrictions

\[
E[w_i(x - b_i r_i - a_i w_i)] = 0,
\]

\[
E[r_i(x - b_i r_i - a_i w_i)] = 0.
\]

The bidimensional individual best weight function reads as

\[
a_i (a, b) = \frac{1 - \beta + \beta \frac{a + \left(1 - \phi\right) b}{1 - \phi b} - \beta \frac{1 - \phi + \phi a}{1 - \phi b}}{1 + \sigma_\xi^2}
\]

\[
b_i (a, b) = \frac{(1 - \beta) \frac{1 - \phi b}{1 - \phi + \phi a} + \beta \left( \frac{a + \left(1 - \phi\right) b}{1 - \phi + \phi a} \right) - \beta \frac{1 - \phi b}{1 - \phi + \phi a} \frac{1 - \phi + \phi a}{\sigma_\xi^2}}{1 + \left( \frac{1 - \phi b}{1 - \phi + \phi a} \right)^2 \sigma_\xi^2}
\]

provided \( b \neq \phi \). From (63) we can recover the expression for the equilibrium \( a = a_i (a, b) \), that is

\[
a = \frac{(1 - \beta) (1 - b)}{1 - \beta + (1 - b \phi) \sigma_\xi^2}.
\]
Notice that, with $\sigma_2^2 \neq 0$, $a$ takes finite values for any $b$. Plugging the expression above at the numerator of (64) we obtain a fixed point equation in $b$:

$$b_i (b) = \frac{(\phi - \beta) \sigma_2^2 - (1 - \beta)}{(\phi - 1) \sigma_2^2 - (1 - \beta)} b - \frac{(1 - \beta) \sigma_2^2}{(\phi - 1) \sigma_2^2 - (1 - \beta)}$$

which is a cubic continuous function. With $\sigma_2^2 = 0$, $b_i (b)$ is linear, so it is trivial to prove $(a_*, b_*) = (0, 1)$ to be the unique solution of the fixed point equation (66). No matter how small is $\sigma_2^2$, the fixed point equation (66) is instead a cubic which can be satisfied at most by three real roots. By continuity there must exist a positive solution $b_0$ such that $\lim_{\sigma_2 \to 0} b_i (b_0) = b_0 = b_*$. The condition

$$\sigma_2^2 > \frac{1 - \beta}{\phi - 1} \text{ with } \phi > 1$$

is necessary and sufficient to have

$$\lim_{\sigma_2 \to 0} b_i' (b_*) = \frac{(\phi - \beta) \sigma_2^2 - (1 - \beta)}{(\phi - 1) \sigma_2^2 - (1 - \beta)} > 1$$

for any $\beta < 1$. In such a case given that

$$\lim_{\sigma_2 \to 0} b_i' (b_0) = \lim_{\sigma_2 \to 0} b_i' (b_*)$$

and that for $\sigma_2^2 \neq 0$ we have $\lim_{b \to +\infty} b_i (b) = 0$, then by continuity at least an other intersection of $b_i (b)$ with the bisector at a positive value $b_+$ must exist too. Moreover, (67) also implies $b_i (0) < 0$. This, jointly with the fact that for $\sigma_2^2 \neq 0$ it is $\lim_{b \to -\infty} b_i (b) = 0$, implies that, by continuity, at least an intersection of $b_i (b)$ with the bisector at a negative value $b_-$ also exists. Therefore $(b_- b_0, b_*)$ are the three solutions we were looking for.

Now suppose that $b_-$ and $b_+$ take finite values - that is $|b_\pm| \leq M$ with $M$ being an arbitrarily large finite number - then, since $a$ is finite too, $\lim_{\sigma_2 \to 0} \sigma_2^2 ((1 - \phi b_+) / (1 - \phi + \phi a))^2 = 0$ and so necessarily $b_\pm = b_0 > 0$. Nevertheless $b_- < 0$ so a first contradiction arises. Furthermore, if $b_+$ is arbitrarily close to $b_0$ then by continuity $\lim_{\sigma_2 \to 0} b_i' (b_+) > 1$ that, for the same argument above, would imply the existence of a fourth root; a second contradiction arises. Hence we can conclude $|b_\pm| > M$ with $M$ arbitrarily large.

Finally let me prove that a multiplicity arises for a $\sigma_2^2$ small enough if and only if (67) holds. Out of these conditions two cases are possible, either $0 < \lim_{\sigma_2 \to 0} b_i' (b_0) < 1$ or $\lim_{\sigma_2 \to 0} b_i' (b_0) < 0$. The case $0 < \lim_{\sigma_2 \to 0} b_i' (b_0) < 1$ yields that, for a $\sigma_2^2$ small enough, at least one intersection of $b_i (b)$ with the bisector must exist by continuity. Nevertheless, in this case, such intersection is unique as the existence of a second one would require $\sup_b \lim_{\sigma_2 \to 0} b_i' (b) > 1$, given that $\lim_{b \to +\infty} b_i (b) = 0_\pm$ but $\sup_b \lim_{\sigma_2 \to 0} b_i' (b) = \lim_{\sigma_2 \to 0} b_i' (b_0)$. With $\lim_{\sigma_2 \to 0} b_i' (b_0) < 0$ instead - for a $\sigma_2^2$ small enough - the curve $b_i (b)$ is either strictly decreasing in the first quadrant ($b_i (b) > 0, b > 0$) and never lies in the fourth quadrant ($b_i (b) < 0, b < 0$), or strictly decreasing in the fourth quadrant and never lies in the first quadrant. Hence $\lim_{\sigma_2 \to 0} b_i (b)$ can only have one intersection with the bisector.
B.3 Proof of proposition 7

i) From (65), we get

$$\lim_{b \to \pm \infty} a = \frac{1 - \beta}{\phi \sigma^2_\xi}.$$  

Using (66), we have

$$\lim_{\sigma^2 \to 0} \lim_{b \to b_{\pm}} \left( \frac{1 - \phi b}{1 - \phi + \phi a} \right)^2 \sigma^2 = \frac{(\phi - \beta) \sigma^2_\xi - (1 - \beta)}{(\phi - 1) \sigma^2_\xi - (1 - \beta)} = 1 = \frac{(1 - \beta) \sigma^2_\xi}{(\phi - 1) \sigma^2_\xi - (1 - \beta)}. \quad (70)$$

Using the relation above, we can write the total relative precision of private information about $\theta$ as

$$\lim_{\sigma^2 \to 0} \lim_{b \to b_{\pm}} \tau = \lim_{\sigma^2 \to 0} \lim_{b \to b_{\pm}} \left( \sigma^2_\xi + \left( \frac{1 - \phi b}{1 - \phi + \phi a} \right)^2 \sigma^2 \right) = \sigma^2_\xi + \frac{(\phi - 1) \sigma^2_\xi - (1 - \beta)}{(1 - \beta) \sigma^2_\xi} = \frac{\phi - 1}{1 - \beta},$$

that is positive provided $\phi > 1$.

ii) The cross-sectional variance of individual expectations is given by $a^2 \sigma^2_\xi + b^2 \sigma^2$. Note that

$$\lim_{\sigma^2 \to 0} \lim_{b \to b_{\pm}} \left( \frac{1 - \phi b}{1 - \phi + \phi a} \right)^2 \sigma^2 (1 - \phi + \phi a)^2 = \lim_{\sigma^2 \to 0} \lim_{b \to b_{\pm}} (1 - \phi b)^2 \sigma^2 = \frac{(1 - \beta) \sigma^2_\xi}{(\phi - 1) \sigma^2_\xi - (1 - \beta)} \left( 1 - \phi + \phi \frac{1 - \beta}{\phi \sigma^2_\xi} \right)^2 = \frac{1 - \beta}{\sigma^2_\xi} \frac{((\phi - 1) \sigma^2_\xi - (1 - \beta))}{(1 - \beta) \sigma^2_\xi}.$$  

We also have

$$\lim_{\sigma^2 \to 0} \lim_{b \to b_{\pm}} \frac{(1 - \phi b)^2}{\phi^2} \sigma^2 = \lim_{\sigma^2 \to 0} \lim_{b \to b_{\pm}} b^2 \sigma^2.$$

Using the relations above, we get

$$\lim_{\sigma^2 \to 0} \lim_{b \to b_{\pm}} \left( a^2 \sigma^2_\xi + b^2 \sigma^2 \right) = \frac{1 - \beta}{\sigma^2_\xi \phi^2} \left( (\phi - 1) \sigma^2_\xi - (1 - \beta) \right) + \frac{(1 - \beta)^2}{\phi^2 \sigma^2_\xi} = \frac{1 - \beta}{\phi} \frac{\phi - 1}{\phi^2},$$

that is positive provided $\phi > 1$.

B.4 Proof of proposition 8

To check convergence in higher order beliefs we need to build up the Jacobian computed around the equilibria using (63)-(64). It is

$$\lim_{\sigma^2 \to 0} \lim_{b \to b_{\pm}} J = \left( \begin{array}{cc} \frac{\partial a_\xi}{\partial \alpha} & \frac{\partial a_\xi}{\partial \beta} \\ \frac{\partial b_\xi}{\partial \alpha} & \frac{\partial b_\xi}{\partial \beta} \end{array} \right).$$

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with

\[ J_{11} = \frac{\beta - \phi}{1 - \phi + \sigma^2}, \]
\[ J_{12} = \frac{(\beta - 1)(a\phi + 1 - \phi)}{(1 + \sigma^2)(1 - b\phi)^2}, \]
\[ J_{21} = \frac{-\frac{(1-\beta)(1-b\phi)}{(a\phi-\phi+1)^2}}{1 + \left(\frac{1-\phi}{1-\phi+\phi\alpha}\right)^2 \sigma^2} + \frac{2\phi \left(1 - \beta\right) \frac{1-\phi}{1-\phi+\phi\alpha} + \beta \frac{a + (1-\phi)b}{1-\phi+\phi\alpha} - a \left(\frac{1-\phi}{1-\phi+\phi\alpha}\right)^2}{\left(1 + \left(\frac{1-\phi}{1-\phi+\phi\alpha}\right)^2 \sigma^2\right)^2}, \]
\[ J_{22} = \frac{-\frac{(1-\beta)(1-b\phi)}{(a\phi-\phi+1)^2}}{1 + \left(\frac{1-\phi}{1-\phi+\phi\alpha}\right)^2 \sigma^2} + \frac{2\phi \left(1 - \beta\right) \frac{1-\phi}{1-\phi+\phi\alpha} + \beta \frac{a + (1-\phi)b}{1-\phi+\phi\alpha} - a \left(\frac{1-\phi}{1-\phi+\phi\alpha}\right)^2}{\left(1 + \left(\frac{1-\phi}{1-\phi+\phi\alpha}\right)^2 \sigma^2\right)^2}. \]

The latter terms become

\[ J_{21} = \frac{-\frac{(1-\beta)(1-b\phi)}{(a\phi-\phi+1)^2}}{1 + \left(\frac{1-\phi}{1-\phi+\phi\alpha}\right)^2 \sigma^2} + \frac{2\phi \frac{b(1-\phi)\beta}{(1-\phi+\phi\alpha)^2} \sigma^2}{1 + \left(\frac{1-\phi}{1-\phi+\phi\alpha}\right)^2 \sigma^2}, \]
\[ J_{22} = \frac{-\frac{(1-\beta)(1-b\phi)}{(a\phi-\phi+1)^2}}{1 + \left(\frac{1-\phi}{1-\phi+\phi\alpha}\right)^2 \sigma^2} + \frac{2\phi \frac{b(1-\phi)\beta}{(1-\phi+\phi\alpha)^2} \sigma^2}{1 + \left(\frac{1-\phi}{1-\phi+\phi\alpha}\right)^2 \sigma^2}. \]

after substituting for (64). Notice that

\[ \lim_{\sigma^2 \to 0} \lim_{b \to b_\perp} J_{21} J_{12} = 0 \]

since \( a \) is always finite, \( J_{21} \) goes to infinity of order \( b_\perp \) whereas \( J_{12} \) goes to an infinitesimal of order \( b_\perp^{-2} \) (as remember \( \lim_{\sigma^2 \to 0} b_\perp \alpha \sigma^2 \) is a finite value). Therefore the eigenvalues of the Jacobian are

\[ \lim_{\sigma^2 \to 0} \lim_{b \to b_\perp} J_{11} = \frac{1}{1 + \sigma^2} \in (0, 1) \]

and

\[ \lim_{\sigma^2 \to 0} \lim_{b \to b_\perp} J_{22} = 1 - 2 \frac{\Gamma}{1 + \Gamma} \in (-1, 1) \]

where (see also (70))

\[ \Gamma \equiv \lim_{\sigma^2 \to 0} \lim_{b \to b_\perp} \left( \frac{1 - \phi b}{1 - \phi + \phi a} \right)^2 \sigma^2 = \frac{(1 - \beta) \sigma^2}{\phi - 1} \frac{\alpha^2}{\alpha^2} > 0 \]

if and only if (67) holds. Therefore, whenever dispersed information limit equilibria exist they are stable outcomes of a convergent process in higher order beliefs.

At the perfect-information limit equilibrium instead the Jacobian is given by

\[ \lim_{\sigma^2 \to 0} \lim_{b \to 1} J = \begin{pmatrix} \frac{\beta - \phi}{(1+\sigma^2)(1-\phi)} - \frac{1-\beta}{\beta \frac{1-\phi}{1-\phi}} \\ \frac{1-\beta}{1-\phi} - \frac{1-\beta}{1-\phi} \end{pmatrix}. \]
The product of its eigenvalues (the determinant of $J$) is
\[
\Delta(J) = \frac{1 - 2\beta + \phi}{(1 + \sigma_\xi^2)} (\phi - 1),
\]
and their sum (the trace of $J$) is
\[
\text{Tr}(J) = \frac{(2 + \sigma_\xi^2)(\phi - \beta)}{(1 + \sigma_\xi^2)} (\phi - 1).
\]
Provided $\text{Tr}(J)^2 > 4\Delta(J)$, the largest real eigenvalue is greater than one whenever
\[
\frac{1}{2}\text{Tr}(J) + \frac{1}{2} \sqrt{\text{Tr}(J)^2 - 4\Delta(J)} > 1
\]
that is, $\text{Tr}(J) > 1 + \Delta(J)$, which requires $\sigma_\xi^2 (1 - \beta) > 0$ given that $\phi > 1$ (this also implies $\Delta(J) > 0$). Let us prove now $\text{Tr}(J)^2 > 4\Delta(J)$, that is
\[
\frac{(2 + \sigma_\xi^2)^2 (\phi - \beta)^2}{(1 + \sigma_\xi^2)^2 (\phi - 1)} > 4 (1 - 2\beta + \phi),
\]
that is equivalent to
\[
(\beta - \phi)^2 \sigma_\xi^2 + \left(4 (\beta - \phi)^2 - 4 (\phi - 1)(\phi - 2\beta + 1)\right) \sigma_\xi^2 +
+ \left(4 (\beta - \phi)^2 - 4 (\phi - 1)(\phi - 2\beta + 1)\right) > 0
\]
which holds whenever $(\phi - 1)(\phi - 2\beta + 1) < 0$ that is always true for $\beta < 1$ and $\phi > 1$. Hence, the perfect information limit equilibrium entails a divergence in higher order belief dynamics.

Finally, notice that $J_{22}$ alone characterizes the univariate case when fixing $a = 0$, which obtains at the limit $\sigma_\xi^2 \to \infty$

### B.5 Proof of proposition 9

To check local learnability of the REE, suppose we are already close to the rest point of the system. That is, consider the case $\int \lim_{t \to \infty} a_{i,t} \ di = \bar{a}$ and $\int \lim_{t \to \infty} b_{i,t} \ di = \bar{b}$ where $(\bar{a}, \bar{b})$ characterizes an equilibrium, so that
\[
\lim_{t \to \infty} S_{i,t} = S = \begin{pmatrix}
1 + \sigma_\xi^2 + \frac{1 - \phi + a \phi}{1 - b \phi} \\
\frac{1 - \phi + a \phi}{1 - b \phi} \left(\frac{1 - \phi + a \phi}{1 - b \phi}\right)^2 + \sigma_\xi^2
\end{pmatrix}.
\]
From standard results in the stochastic approximation theory, we can write the associated ODE governing the stability around the equilibria as
\[
\begin{align*}
\frac{d\chi_{i,t}}{dt} &= \int \lim_{t \to \infty} E \left[ S_{i,t-1}^{-1} \omega_{i,t} \left( x_t - \omega_{i,t} \chi_{i,t-1} \right) \right] \ di = \\
&= S^{-1} \int E \left[ \omega_{i,t} \left( x_t - \omega_{i,t} \chi_{i,t-1} \right) \right] \ di = \\
&= S^{-1} (\text{cov}(x, \omega_{i,t}) - \chi_{i,t-1} = \\
&= B \left( a_{i,t-1}, b_{i,t-1} \right) - [a_{i,t-1} b_{i,t-1}].
\end{align*}
\]
where we used relation (63)-(64). The proposition directly follows from the result in B.4.

C Other Robustness Checks

C.1 A stylized economy: decreasing returns to scale in intermediate production

Consider the problem of intermediate producers in island $i$ being:

$$\max_{Z(i)} \{ e^{\theta + \eta Z} - Z R - Z Q \},$$

where $\vartheta \in (0, 1)$ measures the returns to scale on the intermediate production. The problem (72) generalizes (1)-(2), which obtain for $\vartheta = 0$. The first order condition of (3) is

$$(1 - \vartheta) e^{\theta + \eta Z} R = Q,$$

which can be expressed in log-deviations from the steady state as

$$r_i = q + \vartheta z(i) - \theta - \eta_i.$$ 

The latter combined with (6) and market clearing in the endowment market $\int_0^1 z(i) \, di = 0$ gives

$$r_i = \mu \int_0^1 E[\theta | r_i] \, di - \alpha \theta + \vartheta z(i) - \eta_i.$$ 

Given that the quantity traded in equilibrium $k_i = \theta + \eta_i + (1 - \vartheta) z(i)$ is known by the final producer, then $r_i$ is observationally equivalent to

$$\tilde{r}_i = r_i - \frac{\vartheta}{1 - \vartheta} k_i = \mu \int_0^1 E[\theta | \tilde{r}_i] \, di - \tilde{\alpha} \theta - \tilde{\eta}_i,$$

where

$$\tilde{\alpha} \equiv \frac{(1 - \vartheta) \alpha + \vartheta}{1 - \vartheta}, \quad \tilde{\eta}_i \equiv \frac{1}{1 - \vartheta} \eta_i,$$

and $E[\theta | r_i] = E[\theta | \tilde{r}_i]$. Thus, the analysis in the text can be exactly replicated. A multiplicity arises in this case when $\mu > \tilde{\alpha}$, that is,

$$\mu - \alpha > \frac{\vartheta}{1 - \vartheta},$$

which becomes more and more restrictive as $\vartheta$ approaches 1. At that point, the term $\tilde{\alpha}$, which measures the strength of the allocative effect, goes to infinity so that $\theta$ is fully revealed irrespective of any finite value of $\mu$. Notice that for $\vartheta = 1$ it is $k_i = \theta$, meaning that each producer knows the fundamental.

C.2 The fragile neutrality of money: Extensions of the signal extraction problem

Here, we reconsider the signal extraction problem solved by proposition 6 assuming a more general specification of the additional signal, which was provided, in the original model, by the
local wage. Suppose agents forecast (42) while observing (40) and an additional signal\textsuperscript{22}:

\[ \hat{s}_i = \theta + \theta_e \int_0^1 E[x|\hat{s}_i, r_i] di + \hat{\phi} + \hat{\xi}_i \]

where \( \hat{\phi} \) is normally distributed shock orthogonal to the fundamental \( \theta \), and \( \theta_e \) is a constant. Combining linearly\textsuperscript{23} \( \hat{s}_i \) and \( r_i \) we can obtain:

\[ s_i = \left( 1 - \phi - \frac{\phi}{\theta_e} \right)^{-1} \left( r_i - \frac{\phi}{\theta_e} \hat{s}_i \right) = \theta + \theta \eta_i + \varphi + \xi_i, \tag{75} \]

where \( \varphi \) and \( \xi_i \) are re-scaled realizations of \( \hat{\phi} \) and \( \hat{\xi}_i \), respectively, and \( \theta \) is a constant. In the case \( \theta_e = 0 \), this transformation is not needed and we will consider (75) with \( \theta = 0 \) instead, that is, \( s_i = \hat{s}_i \).

An equilibrium is characterized by the two coefficients \((\hat{a}, \hat{b})\) that weight \( \hat{s}_i \) and \( r_i \), respectively, such that the individual expectation expressed as a linear combination of these two signals is a rational expectation. Then, for each couple \((\hat{a}, \hat{b})\) there must exist another couple \((a, b)\) where: \( b \) weights \( r_i \), and \( a \) weights \( s_i \), such that

\[ E[x|r_i, s_i] = b \left( \phi \int_0^1 E[x|r_i, s_i] di + (1 - \phi) \theta + \eta_i \right) + a \left( \theta + \theta \eta_i + \varphi + \xi_i \right) \tag{76} \]

is still a rational expectation. Let us denote by \( \sigma^2_\varphi \) the variance of the common stochastic component \( \varphi \) which measures the commonality of the information structure. This specification nests several cases:

i) \( \sigma^2_\varphi = 0, \ \theta = 0 \) is when \( \hat{s}_i \) is the wage signal studied in proposition 6;

ii) \( \sigma^2_\varphi \neq 0, \ \theta = 0 \) is when \( \hat{s}_i \) is an exogenous private signal with common noise;

iii) \( \sigma^2_\varphi = 0, \ \theta \neq 0 \) is when \( \hat{s}_i \) is an endogenous private signal like \( r_i \);

iv) \( \sigma^2_\varphi \neq 0, \ \theta \neq 0 \) is when \( \hat{s}_i \) is an endogenous private signal with common noise.

We will show here that, in all these cases, the conditions for the existence of dispersed limit equilibria - characterized as \( b \to b_\pm \) where \( b^2_\pm \sigma^2 = b_\pm \) with \( \kappa \) finite - are identical to the ones uncovered in proposition 6.

Given that all agents use the linear rule above the average expectation is

\[ \int_0^1 E[x|r_i, s_i] di = \frac{a + (1 - \phi)}{1 - \phi b} \theta + \frac{a}{1 - \phi b} \phi. \tag{77} \]

Hence, an individual expectation can be rewritten as

\[ E[x|r_i, s_i] = b \left( \frac{1 - \phi + \phi a}{1 - \phi b} \theta + \frac{\phi a}{1 - \phi b} \varphi + \eta_i \right) + a \left( \theta + \theta \eta_i + \varphi + \xi_i \right). \]

\textsuperscript{22}The signal can be interpreted as a sufficient statistics produced by the existence of separated additional signals, some being purely private and some other being purely public.

\textsuperscript{23}Of course, the expectation operator in \( r_i \) should be written conditional to \( \hat{s}_i \) instead of \( w_i \).
The actual law of motion of the aggregate component of island-specific demand (42) is given by

\[ x = (1 - \beta) \theta + \beta \left( \frac{a + (1 - \phi) b}{1 - \phi b} \theta + \frac{a}{1 - \phi b} \varphi \right), \tag{78} \]
as functions of weights and exogenous shocks only. The fixed point equation is defined by

\[
\begin{align*}
E[s_i(x - br_i - as_i)] &= 0, \\
E[r_i(x - br_i - as_i)] &= 0,
\end{align*}
\]
that give,

\[
a = -\frac{1 - \beta + \beta \left( \frac{a + (1 - \phi) b + a}{1 - \phi b} \sigma_\varphi^2 \right) - b \left( \frac{1 - \phi + \phi a}{1 - \phi b} + \frac{\varphi a}{1 - \phi b} \sigma_\varphi^2 + \varphi \sigma^2 \right)}{1 + \varphi^2 \sigma^2 + \sigma_\varphi^2 + \sigma_\xi^2} \tag{79}
\]

\[
b = \frac{(1 - \beta)(1 - \phi b)}{1 - \phi + \phi a} + \beta \left( \frac{a + (1 - \phi) b + \varphi a^2}{1 - \phi + \phi a} \sigma_\varphi^2 \right) - a \left( \frac{1 - \phi b + \varphi a(1 - \phi b)}{1 - \phi + \phi a} \sigma_\varphi^2 + \left( \frac{1 - \phi b}{1 - \phi + \phi a} \right)^2 \varphi \sigma^2 \right)
\]

\[
1 + \left( \frac{1 - \phi b}{1 - \phi + \phi a} \right)^2 \sigma_\varphi^2 + \left( \frac{\varphi a}{1 - \phi + \phi a} \right)^2 \sigma_\xi^2 \tag{80}
\]
provided \( b \neq \phi^{-1} \). It is easy to prove by substitution that for \( \sigma^2 \to 0 \), \((a, b) = (0, 1)\) is a solution.

Let now prove that the condition for the existence of dispersed information limit equilibria are the same in this extended version. We can solve (79) for \( a \) to get

\[
a = \frac{(1 - \beta)(1 - b) - b(1 - b\phi) \sigma^2 \varphi}{(1 - \beta)(1 + \sigma_\varphi^2) + (1 - b\phi) \left( \sigma_\xi^2 + \sigma^2 \varphi^2 \right)}
\]

Let us look now for the solution at the limit \( \sigma^2 \to 0 \) such that \( b \to b_\pm \) where \( \lim_{\sigma^2 \to 0, b \to b_\pm} b^2 \sigma^2 = \kappa \) with \( \kappa \) being finite (so that \( b_\pm \to \pm \infty \)). In such a case, the equilibrium value of \( a \) takes finite values for any \( b \), provided \( \sigma_\xi^2 \neq 0 \), in particular:

\[
\lim_{\sigma^2 \to 0, b \to b_\pm} a = \frac{1 - \beta}{\varphi \sigma_\xi^2}
\]

which is identical to the dispersed information limit equilibrium value found in proposition 6.

Taking the same limit \( \lim_{\sigma^2 \to 0, b \to b_\pm} \) on both the right-hand side and left-hand side of (80) we obtain:

\[
b_\pm = \left( \frac{(1 - \beta)\phi}{1 - \phi + \phi a} + \beta \left( \frac{\phi}{1 - \phi + \phi a} \right) - a \left( \frac{\phi}{1 - \phi + \phi a} - \frac{a \phi^2}{(1 - \phi + \phi a)^2} \varphi \sigma_\varphi^2 \right) \right) b_\pm
\]

\[
1 + \left( \frac{\phi}{1 - \phi + \phi a} \right)^2 \kappa + \left( \frac{\varphi a}{1 - \phi + \phi a} \right)^2 \sigma_\xi^2 \varphi^2 \tag{81}
\]

that finally pins down the finite value of \( \kappa \)

\[
\kappa = \frac{\phi - 1 - a \phi}{\phi^2} = \frac{(\phi - 1) \sigma_\xi^2 - (1 - \beta)}{\phi^2 \sigma_\xi^2},
\]

and a condition for its non negativity \( \sigma_\xi^2 > (1 - \beta)(\phi - 1)^{-1} \), which is (47). Note in particular, that the condition is independent of \( \varphi \) or \( \sigma_\varphi^2 \), i.e. whether or not the signal embeds the average expectation or a common noise term, respectively.
References


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