Monetary Policy Switching and Indeterminacy

Jean Barthélemy Magali Marx*

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Abstract

This paper assesses the validity of the Taylor principle -the nominal interest rate should respond by more than one-for-one to changes in inflation- to ensure stability when monetary policy switches amongst different monetary policy rules. By excluding certain equilibria, the existing literature underestimates the indeterminacy region, where policy parameters allow for multiple bounded equilibria. When considering all bounded equilibria, we find two main results. First, one monetary policy regime can violate the Taylor principle without implying indeterminacy. Second, two monetary policy regimes with very different responses to inflation may trigger indeterminacy even if both regimes satisfy the Taylor principle.

Keywords: Markov-switching, indeterminacy, monetary policy.

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1 Introduction

Good monetary policy should prevent indeterminacy, i.e. the existence of multiple stable equilibria. When multiple equilibria are consistent with economic agents' expectations,

*Barthélemy: Banque de France, 39 rue Croix des Petits Champs, 75001 Paris. Email: jean.barthelemy@banque-france.fr. Marx: Department of Economics, Sciences Po, 28 rue des Saints Pères, 75007 PARIS. Email: magali.marx@sciences-po.org. We thank Eric Leeper, Dan Waggoner and Tao Zha for stimulating discussions. We are also grateful to Klaus Adam, Pamfili Antipa, Robert Barsky, Jess Benhabib, Francesco Bianchi, Vincent Bignon, Florin Bilbiie, Carlos Carvalho, Seonghoon Cho, Lawrence Christiano, Nicolas Coeurdacier, Filippo Ferroni, Gaetano Gaballo, Christian Hellwig, Michel Juillard, Raphaël Jungers, Bruce McGough, Eric Mengus, François Velde, and Michael Woodford for their very helpful comments on this paper. We are also grateful to seminar participants at the 2012 and 2013 Conference on Computing in Economics and Finance, the 2013 T2M conference, and the 2013 EEA-ESEM conference. An earlier version of this paper circulated under the title "Generalizing the Taylor Principle: New Comment". The views expressed in this paper do not necessarily reflect the opinion of the Banque de France. All remaining errors are ours.

the economy may respond to non-fundamental -sunspot- disturbances, and hence, may be affected by extrinsic volatility. Since limiting inflation volatility is a well-accepted objective for monetary policy, determinacy is a desirable feature of monetary policy.

In the absence of monetary policy switches, the widespread view is that monetary authorities should increase the nominal interest rate by more than one-for-one in response to inflation to ensure determinacy. This condition is known as the Taylor principle.

Policy switching is, however, seldom irreversible. Economic agents observe oscillating policy regimes. They, thus, internalize the possibility of future policy switches, when forming their expectations. Since determinacy depends on economic agents' expectations, expectations of future regime switching affect stability conditions and change the Taylor principle.

In this paper, we extend the Taylor principle to account for monetary policy switching by characterizing stable equilibria in a new-Keynesian economy experiencing switching among multiple monetary policy regimes. We identify both determinacy and indeterminacy regions as a function of two key policy parameters, namely the persistence of the regimes and the responses of the nominal interest rate to inflation in each regime. Our results are threefold.

First, monetary policy can deviate from the Taylor principle in one regime without generating indeterminacy. Intuitively, the expectations of switching to a stabilizing monetary policy regime may suffice to anchor expectations and to rule out multiplicity of equilibria. This result suggests that the too accommodative US monetary policy identified by Clarida et al. (2000) in the 70s does not lead necessarily to indeterminacy as long as a switch to a more aggressive monetary policy regime is anticipated. Based on plausible calibrations for the period under consideration, we show that we cannot settle between indeterminacy and determinacy.

We further show that if the too accommodative monetary policy in the 70s effectively led to indeterminacy, sunspots equilibria emerge even in the monetary policy regime that satisfies the Taylor principle as long as the switch to this regime is not permanent.[1] It, thus, casts doubts on the explanation by Lubik and Schorfheide (2004) of the fall in the volatility in the beginning of the 80s based on indeterminacy and changes in extrinsic volatility.

Second, an economy may suffer from indeterminacy even if the two monetary policy regimes satisfy the Taylor principle. We identify such a configuration when the two monetary policies share the same rule - nominal interest rate reacts proportionally to inflation in both regimes- but with different intensities. This configuration is more likely when the monetary policy regime with the highest reaction to inflation is unfrequent and short-lasting. In other words, the expectations of a future monetary policy that strongly reacts to inflation in the future is destabilizing if the probability of such an event is too low.

[1]There is some evidence that monetary policy switching occurred not only around the nomination of P. Volcker as the chairman of the Fed but also afterwards (see Bianchi, 2013, for instance).

Economic agents' expectations of a short period of high reaction to inflation regimes followed by a long-lasting period of low reaction to inflation regimes are key to explain this indeterminacy. On the one hand, expectations of a positive output-gap generate protracted inflationary pressure as long as monetary policy responds modestly to inflation. On the other hand, an aggressive monetary policy strongly reacts to such inflationary pressure generating a large fall in the output-gap. The overall effect is, thus, a large drop in the output-gap if these two policies are consecutive. Eventually, if such successions of regimes occur with significantly large probabilities, multiple equilibria arise.

Third, in a long-lasting dovish monetary policy regime, the anticipation of a short-lived monetary policy switch destabilizes the economy if the response to inflation in the latter regime is either too high or too low. On the one hand, raising the nominal interest rate when facing inflationary pressure helps reducing the interplay between current and expected inflation. On the other hand, a too large response to inflation in one monetary policy regime increases the sensitivity of the output-gap to expectations which can eventually lead to indeterminacy.

Finally, the technical contribution of this paper is to provide the determinacy condition for any linear and forward-looking Markov-switching rational expectations models. The condition depends on the stability of an infinite product of matrices involving all the future possible regimes. The determinacy region cannot be settled through the computation of a finite number of operations. This complexity arises from the importance of the exact timing of future regime switches in economic agents' expectations. Some equilibria may depend on all the past regimes, possibly in a complex, non-recursive way.

The existing literature (Davig and Leeper, 2007; Farmer et al., 2009b; Cho, 2013) always restricts the class of equilibria, leading to underestimating the size of the indeterminacy region and as a consequence the destabilizing properties of regime switching models. When unique, the equilibrium itself is not impacted by these restrictions. Since the unique equilibrium depends on the current shocks and regime only, it always belongs to the classes of equilibria considered in these papers. However, for certain configurations of policy parameters, these restrictions may lead to conclude to determinacy whereas multiple stable equilibria exist. In other worlds, the existing literature provides sufficient conditions for indeterminacy but underestimates conditions of determinacy.

The remainder of the paper is organized as follows. In Section 2, we present the class of models and provide a determinacy condition. Since this condition is hard to compute in a finite amount of time, we also give practical sufficient conditions for determinacy and for indeterminacy. In Section 3, we generalize the Taylor principle when monetary policy switches. We discuss our results and assumptions in Section 4. We conclude in section 5.

2 Determinacy conditions for Markov-Switching Rational Expectations Models

2.1 The class of models

Most micro-founded macroeconomic models can be summarized by a system of non-linear equations involving structural parameters governing economic agents' preferences, technology, market structures, and economic policies. Allowing these parameters to switch over time results in non-linear regime switching models. When shocks are small enough, the stability of this class of models can be understood by studying the determinacy of linear regime switching models (Barthélemy and Marx, 2011). In this paper, we restrict our attention to linear models of the following form:

$$\Gamma_{s_t} z_t = \mathbb{E}_t z_{t+1} + C_{s_t} \epsilon_t, \tag{1}$$

where the vector z_t is a $(n \times 1)$ real vector of endogenous variables, the vector ε_t is a $(p \times 1)$ real vector of exogenous shocks, and the index s_t indicates the current regime, in $\{1, \cdots, N\}$. For any index $i \in \{1, \cdots, N\}$, the matrices Γ_i and C_i are respectively $(n \times n)$ and $(n \times p)$ real matrices. We assume that the vector of shocks, ϵ_t, is bounded[2] and independent from the current and past regimes.[3] Finally, we assume that the regimes, s_t, follow a Markov-chain with constant transition probabilities:

$$p_{ij} = Pr(s_t = j | s_{t-1} = i). \tag{2}$$

This class of models has two main limitations. First, it precludes pre-determined variables like capital or debt. Second, it restricts to constant transition probabilities cases. In any cases, studying forward-looking models is a necessary fist-step before understanding more complex regime switching models.[4]

We consider that an equilibrium is stable if it is bounded.

Definition 1. *A stable equilibrium is a bounded continuous process satisfying Equation* (1).

This definition of stability is similar to the one used in non-linear rational expectations

[2]To the best of our knowledge, assuming bounded shocks is a prerequisite for linearizing a non-linear rational expectations model. Otherwise, the application of a perturbation approach is questionable. See Woodford (1986) for more details on the perturbation approach.

[3]We can extend the results to first-order autoregressive shocks with regime-dependent autocorrelation and variance.

[4]Cho (2013) has recently made significant progress in understanding MSRE models with backward-looking components. In Barthélemy and Marx (2011), we show how perturbing a MSRE model to study a model with state-dependent transition probabilities

models (e.g. Woodford, 1986; Jin and Judd, 2002), we detail this definition in appendix A. We discuss this stability concept in Section 4.

2.2 Determinacy condition

In this section, we establish necessary and sufficient conditions for the existence and uniqueness of a stable equilibrium. Our strategy is inspired by the forward iteration approach developed in the seminal paper by Blanchard and Kahn (1980). Forward iterations of Equation (1) yields:

$$z_t = \sum_{k=0}^{T} \mathbb{E}_t \left[\prod_{i=0}^{k} \Gamma_{s_{t+i}}^{-1} \right] C_{s_{t+k}} \epsilon_{t+k} + \mathbb{E}_t \left[\prod_{i=0}^{T} \Gamma_{s_{t+i}}^{-1} \right] z_{t+T+1}. \tag{3}$$

The asymptotic behavior of $\mathbb{E}_t \left[\| \prod_{i=0}^{k} \Gamma_{s_{t+i}}^{-1} \| \right]$, where $\| \cdot \|$ is any given matrix norm, measures the asymptotic growth rate of expectations, and hence, their stability. We, thus, introduce the following sequence :

$$u_k = \left(\sum_{(i_1,\cdots,i_k)\in\{1,\cdots,N\}^k} p_{i_1 i_2} \cdots p_{i_{k-1}k} || \Gamma_{i_1}^{-1} \Gamma_{i_2}^{-1} \cdots \Gamma_{i_k}^{-1} || \right)^{1/k} \tag{4}$$

The determinacy condition is then given by:

Proposition 1. *There exists a unique stable equilibrium if and only if the limit of u_k when k tends to infinity is smaller than* 1*.*

We prove Proposition 1 in appendix, section B. We first prove that the sequence u_k converges and admits a limit independent from the chosen norm. If the limit is smaller than 1, Equation (3) proves the existence of a unique stable equilibrium. Second, we prove the reciprocal by showing that if the limit is larger than unity, multiple equilibria exist. When unique, the unique equilibrium only depends on current shocks and regime (see appendix E.1).

Proposition 1 generalizes Blanchard and Kahn (1980) to Markov-switching rational expectations models. When there is no regime switching ($\Gamma_{s_t} = \Gamma$ for any regime), the existence of a unique stable equilibrium depends on the asymptotic behavior of $u_k \sim ||\Gamma^{-k}||$. This sequence behaves as an implosive exponential if and only if all the eigenvalues of Γ^{-1} are less than one. This coincides with the well-known Blanchard and Kahn conditions.

This proposition also extends Farmer et al. (2009a) to multivariate models. When the model is univariate, the matrices Γ_{s_t} are commutative as they are scalars. The limit of u_k hence depends on a simple combination between probabilities and these scalars.

In general, the computation of the limit is, however, challenging as the number of terms to compute is growing exponentially. This complexity comes from the non commutativity of the product of the matrices appearing in the definition of the sequence, u_k. As a consequence, the sequence is not necessarily monotonous and the speed of convergence is unknown. The limit of the sequence u_k shares similar properties with mathematical objects such as the joint spectral radius (e.g. Theys, 2005) and the p-radius.[5]The complexity of these concepts is well known in control theory.

The insight behind this complexity is the following. In some circumstances, economic agents' decisions may depend on the exact order of future regimes and the attached expectations conditional to the considered path. As a consequence, far past shocks and regimes may impact current decision in case of indeterminacy. The existing literature partially rules out this class of equilibria.

2.3 Sufficient conditions of determinacy and indeterminacy

Since computing the exact limit of the sequence u_k is impossible for any models and any parameters, we derive sufficient conditions ensuring determinacy (Proposition 2) and indeterminacy (Proposition 3). From an econometric standpoint, sufficient determinacy conditions are crucial to ensure that parameters in an estimated model are consistent with a unique equilibrium.

Proposition 2. *If there exists k such that $u_k < 1$, then there exists a unique equilibrium.*

The proof of Proposition 2 is in appendix, section D. It suffices to prove that if there exists k such that u_k is smaller than 1, then its limit is smaller than 1.

This result provides sufficient determinacy conditions. Such conditions are absent from the literature (Davig and Leeper, 2007; Farmer et al., 2009b; Cho, 2013). Determinacy conditions are usually given for a subclass of equilibria, and hence, *de facto* correspond to sufficient indeterminacy conditions. We complement these existing conditions by providing new conditions by considering a large (but not exhaustive) class of stable equilibria.

Proposition 3. *If for a certain integer q, there exist N^{q+1} real numbers in the (interior of the) unit disk, $\alpha(i_0, \cdots, i_q)$, such that the highest eigenvalue of the matrix*

$$\left[\sum_{(i_1,\cdots,i_{q-1})\in\{1,N\}^{q-1}} p_{ii_1}p_{i_1i_2}\cdots p_{i_{q-1}j}\Gamma_i^{-1}\cdots\Gamma_{i_{q-1}}^{-1}\alpha(i,i_1,\cdots,i_{q-1},j)\right]_{(i,j)} \tag{5}$$

is larger than 1, then there exist multiple bounded solutions.

We prove this Proposition by constructing a continuum of stable equilibria under assumption of Proposition 3 (see Appendix E.2). The class of sunspots equilibria that we construct

[5] For a two-regime model, if the transition probabilities are symmetric ($p_{11} = p_{22} = 1/2$), the limit ν exactly corresponds to the 1-radius of $\{\Gamma_1^{-1}, \Gamma_2^{-1}\}$. We refer to Jungers and Protasov (2011) for a detailed presentation of this quantity.

is strictly larger than equilibria considered by Davig and Leeper (2007) and Farmer et al. (2009b). Sunspots equilibria put forward by the latter correspond to the specific case of $q = 1$. For a given length q, the sunspot equilibria we consider depend on the q past regimes, shocks and endogenous variables. Every q periods, they belong to and are contracting on a fixed span. Nevertheless, the equilibrium may be locally explosive. Finally, when q increases, the size of the state variables needed to describe the equilibrium increases. When q is infinitively large, the dimension of the state space is infinite. In this case the equilibrium is fully history-dependent.

3 Monetary policy regime switching

Since Taylor (1993), economists are used to model the nominal interest rate as a state-contingent rule with constant parameters. We, however, see at least three reasons to believe that parameters of the monetary policy rule may vary over time. First, if monetary policy is optimal, any structural change in the economy should translate into a change in monetary policy. Second, monetary policy rule results from multiple beliefs on: the structure of the economy, the role of monetary policy and the monetary policy transmission mechanisms. All these beliefs may change over time depending on empirical as well as theoretical advances in macroeconomics. Third, the rule captures economic preferences of the central banker. These preferences have little reasons to be stable over time. Governors of major central banks are chosen by the government depending on its own preference, and hence, depend on political cycles.

Many empirical works contradict the standard assumption of constant monetary policy parameters. Clarida et al. (2000) and Lubik and Schorfheide (2004) prove that macroeconomic instability in the 70s may have been caused by a monetary policy rule that did not react enough to inflation. while the post Volcker would be characterized by a determinate monetary policy. Davig and Leeper (2007) show that these findings are not robust when allowing economic agents to form expectations on regime switching. In their paper, they show that existing estimations of policy parameters do not lead to indeterminacy. They, thus, rule out the indeterminacy-based story for explaining the great inflation period.

These findings have been found by restricting admissible equilibria. Farmer et al. (2010a) have casted doubts on the robustness of these findings when taking into account a more general class of equilibria. We propose to settle this question by applying our results to a canonical monetary model.

3.1 The model

We consider a log-linearized New-Keynesian model in which the monetary policy is described by a Taylor rule with recurring shifts in parameters. The Markov-switching New-Keynesian model can be put into the following form:

$$y_t = \mathbb{E}_t y_{t+1} - \sigma(r_t - \mathbb{E}_t \pi_{t+1} + \epsilon_t^d), \tag{6}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa y_t + \epsilon_t^s, \tag{7}$$

$$r_t = \alpha_{s_t} \pi_t + \gamma_{s_t} y_t + \epsilon_t^r, \tag{8}$$

where the variables y_t, π_t and r_t are the output-gap, inflation (in log) and the nominal interest rate (in deviation around a certain steady state), respectively. Equation (6) is an IS curve linking the output-gap to all the future ex-ante real interest rates and future and current demand shocks, ϵ_t^d. The parameter σ is the risk aversion. Equation (7) is a New-Keynesian Phillips Curve linking inflation to all the future marginal costs summarized by the output-gap. The parameter κ measures the degree of nominal rigidities while β stands for the discount factor. The shock ϵ_t^s denotes a cost-push shock translating the Phillips Curve. Equation (8) is a simplified Taylor rule with a potential shift in the reaction to inflation (α_1 or α_2) and to the output-gap (γ_1 or γ_2). The index $s_t \in \{1, 2\}$ stands for the current observable regime of monetary policy. Finally, the shock ϵ_t^r is a disturbance measuring the unsystematic part of monetary policy.

When there is no regime switching, i.e. $\alpha_1 = \alpha_2 = \alpha$, and no reaction to the output-gap, i.e. $\gamma_1 = \gamma_2 = 0$, the model admits a unique stable equilibrium if and only if the real interest rate increases with inflation, $\alpha > 1$. This condition is often called the Taylor principle. Following Leeper (1991), we say that the regime is active when it satisfies the Taylor principle. Otherwise, the regime is passive. When monetary policy reacts to the output gap, the determinacy condition becomes (see Woodford, 2003) :

$$\alpha + \frac{1-\beta}{\kappa}\gamma > 1.$$

The question we want to address here is how these conditions evolve in a context of regime switching monetary policy. This question has been at the core of the controversy opposing Farmer et al. (2010a) and Davig and Leeper (2010) but remains unsolved.

3.2 Generalizing the Taylor Principle

To establish our three main results, we calibrate the model consistently with Davig and Leeper (2007). The discount factor, β, is set to 0.99 reflecting an annualized steady-state real interest rate equal to 4%. We assume log utility ($\sigma = 1$). The price stickiness is such that the slope of the New-Keynesian Philips Curve, κ, is 0.17. We assume that the persistence of the first regime, p_{11} is relatively low, 0.8, while the persistence of the second regime is high, $p_{22} = 0.95$. These probabilities correspond to average durations of 4 and 19 quarters for the regimes 1 and 2 respectively.

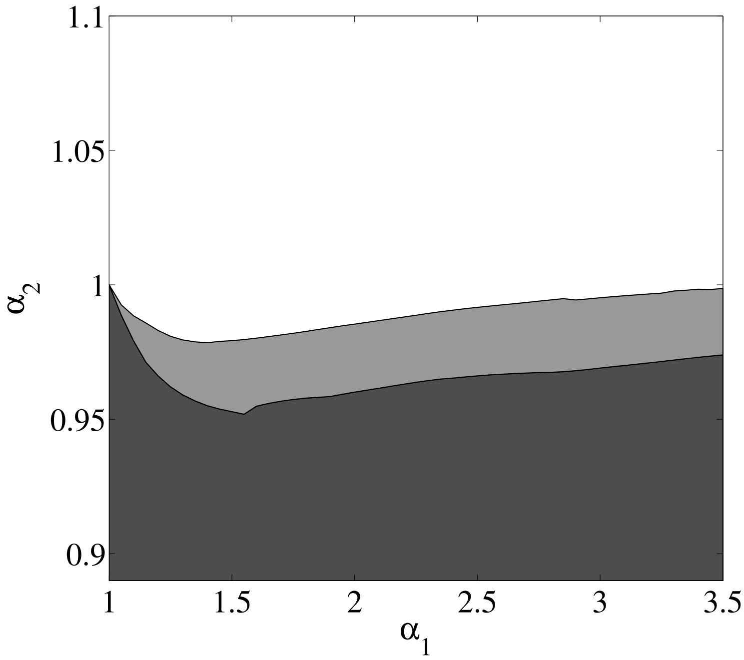

Figure 1: Determinacy regions and policy parameters: new Keynesian model with Markov-switching monetary policy

Note: the white area displays the determinacy region; the light-shaded area represents a region in which we cannot decide whether there is a unique bounded equilibrium or not in a reasonable amount of time; the dark-shaded area represents a region in which there exists multiple stable equilibria. Probability of remaining in regime 1 (2) is 0.8 *(*0.95 *respectively).*

Result 1. *A regime switching economy can be determinate even if one of the two regimes does not satisfy the Taylor principle.*

Figure 1 displays the different regions of (in)determinacy given by Propositions 2 and 3, with respect to the reaction of the nominal interest rate to inflation in each regime, α_1 and α_2. The white area corresponds to the parameters region for which determinacy is guaranteed by Proposition 2.[6] The darkest region represents configuration of policy parameters leading to sunspot equilibria.[7] Finally, the grey area depicts the parameters region for which we cannot conclude whether there exists or not a unique stable equilibrium.

[6]For each set of parameters, we check whether u_k is smaller than 1, for k smaller than 17.

[7]We apply Proposition 3 for length q equal to 6. When conditions of this Proposition are satisfied, we can construct multiple sunspots equilibria depending on past inflation, on the past output-gap, and on the last six regimes. See Appendix E.2

Monetary policy can deviate from the Taylor principle modestly ($\alpha_1 = 0.98$) but relatively persistently ($p_{11} = 0.95$) without implying indeterminacy if monetary policy reacts sufficiently to inflation in the other regime (for instance if $\alpha_2 = 1.5$) as shown by Figure 1. While the first regime is not active enough to ensure determinacy by its own, the expectations of a switch toward a more active regime suffice to anchor expectations, and hence, to rule out sunspots equilibria. Thus, the expectations of a more aggressive monetary policy may be effective in guaranteeing macroeconomic stability.

This result is in line with the contribution of Davig and Leeper (2007). In this paper, the authors argue that "a unique bounded equilibrium does not require the Taylor principle to hold in every period". They find this result by restricting the solutions space (see Section 4 for further details). We prove that this result is unchanged when considering the whole class of equilibria.

Result 2. *An economy may suffer from indeterminacy even if the two monetary policy regimes satisfy the Taylor principle.*

We identify such a configuration when the two monetary policies share the same rule - nominal interest rate reacts proportionally to inflation in both regimes- but with different intensities. The two regimes satisfy the Taylor principle. In the first regime, the dovish one, the central bank reacts modestly to inflation ($\alpha_1 = 1.01$). In the second, the hawkish regime, the central bank reacts (incredibly) strongly to inflation ($\alpha_2 = 6$). When monetary policy switches between prolonged periods of the dovish regime ($p_{11} = 0.95$) and short-lasting periods of the hawkish regime ($p_{22} = 0.5$), Proposition 3 proves that the economy is indeterminate.

Indeterminacy reflects the explosiveness of economic agents expectations. Figure 2 plots the ten largest contributions to the sequence u_k, which measures the explosiveness of expectations. Unsurprisingly, a prolonged dovish central bank significantly contribute to this explosiveness, as monetary policy slowly stabilizes the economy in this regime. But this expectation is not enough to explain indeterminacy by itself as the dovish regime satisfies the Taylor principle, and hence, induces determinacy when taken in isolation. The other largest contributions correspond to the alternation between a short period of the hawkish regime and a protracted period of the dovish regime.

Figure 3 reports the responses of the output-gap and inflation to an expectation 17-quarter ahead conditionally to the future regimes path identified as the largest contributors to expectations explosiveness, u_{16}, in Figure 2. We plot the responses to an expected rise in the output-gap ($E_t y_{t+17} = 1$) or inflation ($E_t \pi_{t+17} = 1$) in 17 quarters. In black line without markers, we plot the responses if the economy remains in the dovish regime. As the Taylor principle is satisfied in this regime, an infinite horizon expectation does not matter for current decision. However, at a finite horizon, inflation expectations are locally explosive when facing a rise in the output-gap expectation and slow to converge in the case of a rise in inflation

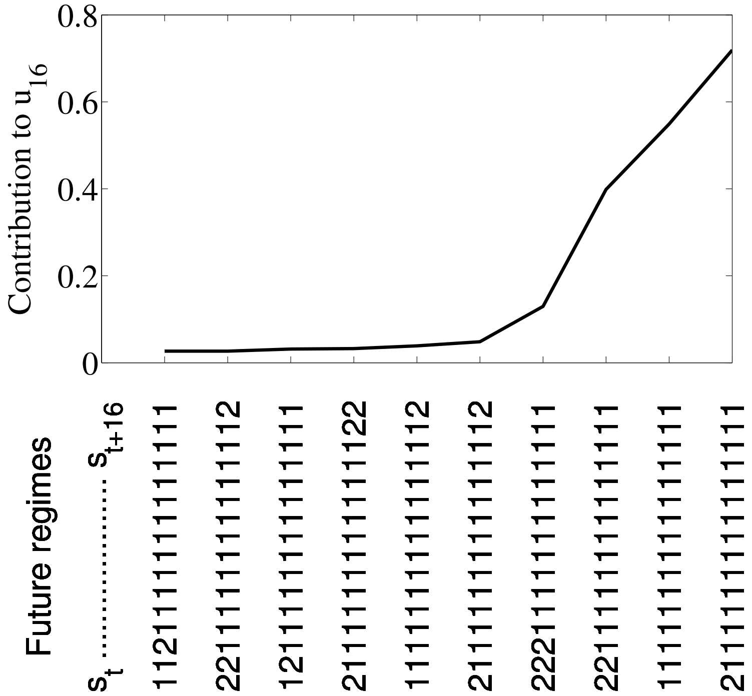

Figure 2: Ten largest contributions to a measure of expectations explosiveness

Note: we report the ten largest contributions -in terms of future regimes trajectory and amongst 2^{36} *possible trajectories- to a measure of expectations explosiveness,* u_{16} *(see Proposition 1). The x-axis represents a particular path of future regimes. The y-axis stands for the associated contribution to* u_{16}*. A higher contribution suggests that the trajectory is more destabilizing. When all the contributions sum to less than one, it proves than the economy is determinate. Otherwise, it suggests (without formally proving) indeterminacy. Policy parameters and probabilities are set to* $\alpha_1 = 1.01$*,* $\gamma_1 = 0$ *and* $p_{11} = 0.95$ *and* $\alpha_2 = 6$*,* $\gamma_2 = 0$ *and* $p_{22} = 0.5$*.*

expectation. In red with crosses, dashed blue with diamonds and dashed green with squares, we plot the trajectories of the succession of one, two, and three periods of the hawkish regime followed by a long-lasting period of the dovish regime.

In these three cases, the explosiveness or the slow convergence of inflation when agents expect a dovish monetary policy (regime 1) generates large fall in the output-gap in the hawkish regime (regime 2). Positive expectations of the output-gap or inflation generate protracted inflationary pressures as long as monetary authority is dovish. Since these expectations induce inflationary pressures, the hawkish monetary authority reacts to these expectations by raising the nominal interest rate, eventually leading to a large fall in the output-gap. The overall effect of positive output-gap or inflation expectations is, thus, a large drop in the output-gap if these two policies are consecutive. Eventually, if such successions of regimes occur with sufficiently large probabilities, multiple equilibria arise.

This configuration is more likely when the hawkish monetary policy regime is unfrequent

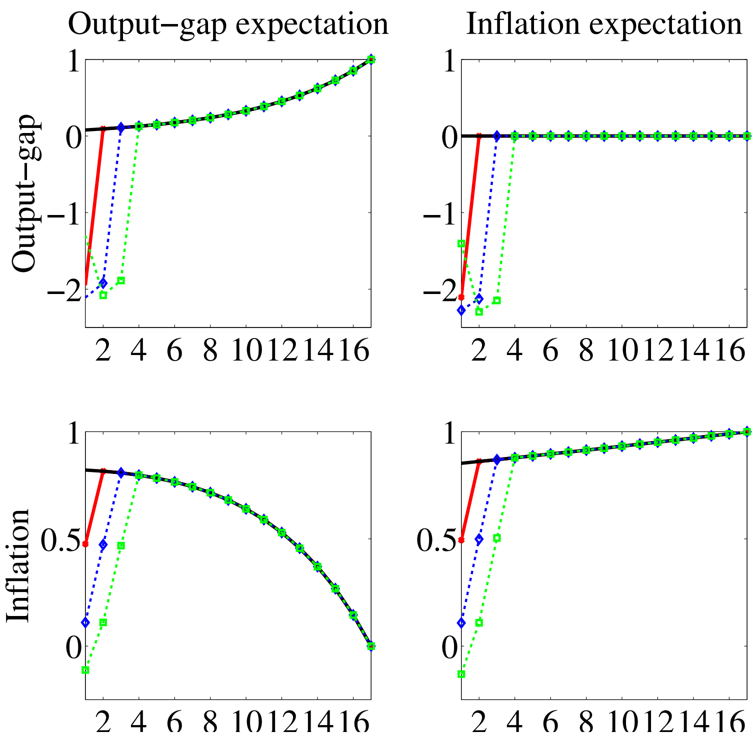

Figure 3: Impact of a 17-period ahead expectation: new Keynesian model with Markov-switching monetary policy

Note: the plots report the responses of the output-gap (upper panel)and inflation (lower panel) to an expectation in 17 periods of a rise in the output-gap (left panel) or in inflation (right panel). We report the four largest contributors to u_{16} (see Figure 2). The thick red line with crosses displays the trajectory if the future regimes are 2 *then only* 1. *The thick black line represents the trajectory conditionally to staying in regime* 1. *In dashed blue line with diamonds, two regimes* 2 *and then regimes* 1 *and in dashed green line with squares, three regimes* 2 *and then regimes* 1. *Policy parameters and probabilities are set to* $\alpha_1 = 1.01$, $\gamma_1 = 0$ *and* $p_{11} = 0.95$ *and* $\alpha_2 = 6$, $\gamma_2 = 0$ *and* $p_{22} = 0.5$.

and short-lasting. On the contrary, indeterminacy occurs when the dovish monetary policy regime is very persistent. Figure 4 displays determinacy regions with respect to the probabilities in each regime. Thus, the expectations of unfrequent very active monetary policy regime can destabilize rather than stabilize economic agents expectations leading to indeterminacy.

This second result stems from price stickiness. In a flexible-price economy ($\kappa \to \infty$), inflation is only determined by inflation expectations since the feedback between output and inflation observed in Figure 3 does not hold anymore. If monetary policy satisfy the Taylor principle in the two regimes, current and future monetary policies (whatever the regime is) stabilize inflation expectations.[8] Hence, inflation is uniquely defined and sunspots equilibria

[8]In this environment, determinacy conditions depend on a simple combination between transition proba-

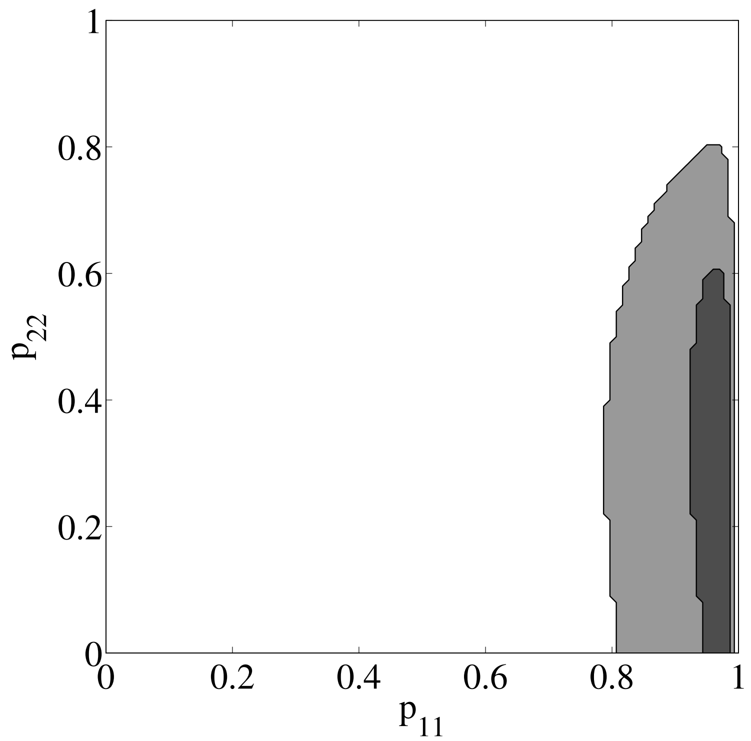

Figure 4: Determinacy regions and persistence of regimes: new Keynesian model with Markov-switching monetary policy

Note: the white area displays the determinacy region; the light-shaded area represents a region in which we cannot decide whether there is a unique bounded equilibrium or not in a reasonable amount of time; the dark-shaded area represents a region in which there exists multiple bounded equilibria. Policy parameters are calibrated such that the two regimes satisfy the Taylor principle. Response to inflation in the first (second) regime is $\alpha_1 = 1.01$ *(*$\alpha_2 = 6$*, respectively).*

are ruled out.

Result 3. *In a long-lasting dovish monetary policy regime, the anticipation of a short-lived monetary policy switch stabilizes the economy if the response to inflation in the latter regime is neither too low nor too high.*

Let us consider a policymaker who is reluctant to durably fight against inflation but only accept to deviate from its policy infrequently to prevent indeterminacy. In such a context, what are the degrees of aggressiveness required to achieve determinacy? To answer this question we calibrate model parameters such that monetary policy modestly reacts to inflation in the most frequent and long-lasting regime ($p_{11} = 0.95$ and $\alpha_1 = 0.99$) while the other regime is short-lasting ($p_{22} = 0.5$).

bilities and inflation reaction in both regimes (see Farmer et al., 2009a).

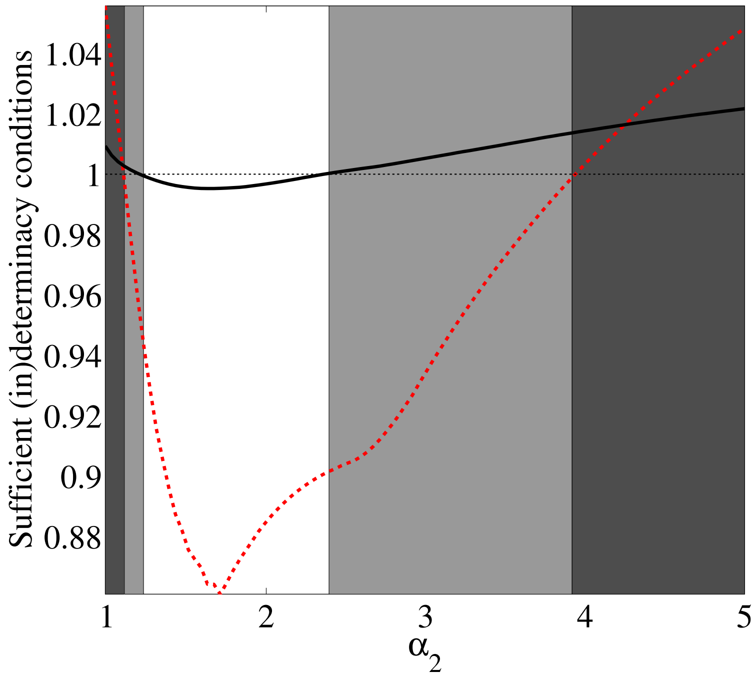

Figure 5: Optimal degrees of activism: new Keynesian model with Markov-switching monetary policy

Note: the white area displays the determinacy region; the light-shaded area represents a region in which we cannot decide whether there is a unique bounded equilibrium or not in a reasonable amount of time; the dark-shaded area represents a region in which there exists multiple bounded equilibria. Policy parameters are calibrated as follows: $\alpha_1 = 0.99$*,* $p_{11} = 0.95$ *and* $p_{22} = 0.5$*. The dark line displays the explosiveness of expectations,* u_{16}*. When this line is below one, the economy is determinate. The dashed red line displays the largest eigenvalue of Proposition 3 for* $q = 6$ *(and for any scalars* α_{ijkl}*). When this largest eigenvalue is larger than one, we can construct sunspots equilibria that depend on the last six regimes.*

Too low or too high response to inflation in the unfrequent regime leads to indeterminacy, whereas an intermediate response to inflation stabilizes the economy. Figure 5 displays determinacy regions with respect to the inflation response in the unfrequent regime. The darkest area represents the indeterminacy region, the lightest grey area, the parameters region for which we cannot conclude and the white area, the determinacy region. Determinacy arises for responses to inflation, α_2, between 1.2 and 2.4 (white area) at least. Indeterminacy emerges at least when the response is smaller than 1.1 or larger than 3.9.

This result stems from the combination of two effects. First, as in the first result, a large enough reaction to inflation helps ruling out sunspots equilibria that emerges because of the dovish monetary policy in the long-lasting regime. By raising the nominal interest rate when facing inflationary pressure, it helps reducing the interplay between inflation and inflation expectations. Second, as explained in details above, a too large response to inflation in one monetary policy regime increases the sensitivity of the output-gap to expectations and can eventually lead to indeterminacy.

Thus, in a regime switching economy, a strong reaction is not necessarily desirable even if the weight of inflation in the monetary policy objective is high. When choosing its level of policy reaction, the central bank should internalize the possibility of a switch toward a passive monetary policy and may be forced to lower its reaction relative to what it would be otherwise. This could explain why we do not observe very large responses to inflation empirically.

4 Discussion

4.1 Indeterminacy hardly explains the Great Inflation

Clarida et al. (2000) and Lubik and Schorfheide (2004) estimate a change in the US monetary policy around 1980. They show that monetary policy in the 70s was too dovish and failed to satisfy the Taylor principle. Lubik and Schorfheide (2004) suggest that this failure caused the high level of volatility experienced by the US in the 70s and called the Great Inflation period by contrast with the Great Moderation that followed. However, such an explanation is based on the Taylor principle that neglects regime switching. Results from Section 3 proves that considering regimes in isolation does not suffice to deal with determinacy. We revisit this explanation in the light of our generalized Taylor principle.

We consider the two-regime new-Keynesian model described in Section 3 and we calibrate policy parameters as in Davig and Leeper (2007).[9] The first regime stands for the pre-Volcker period in which the Taylor principle is violated ($\alpha_1 = 0.89$ and $\gamma_1 = 0.15$). The second regime represents the post-1979 monetary policy in which the Taylor principle is satisfied ($\alpha_2 = 2.19$ and $\gamma_2 = 0.3$). These parameters are consistent with more recent estimates that take formally into account regime switching by Bianchi (2013).

Indeterminacy hardly explains the change in volatility at the end of the 70s. As long as economic agents expect a switching back to pre-Volcker type of policy (regime 1), i.e. if the post-Volcker monetary policy regime is not absorbent, sunspots equilibria may arise in both regimes not only in the regime that fails to satisfy the Taylor principle if the economy is indeterminate. Thus, explaining the fall in volatility at the beginning of the 80s by the disappearance of extrinsic volatility requires to assume that the monetary policy switching was irreversible and perceived as such.

[9] This calibration is based on estimates by Lubik and Schorfheide (2004). However, the model estimated by these authors differs from ours as they (i) do not take formally into account regime switching (ii) include a backward component in the Taylor rule.

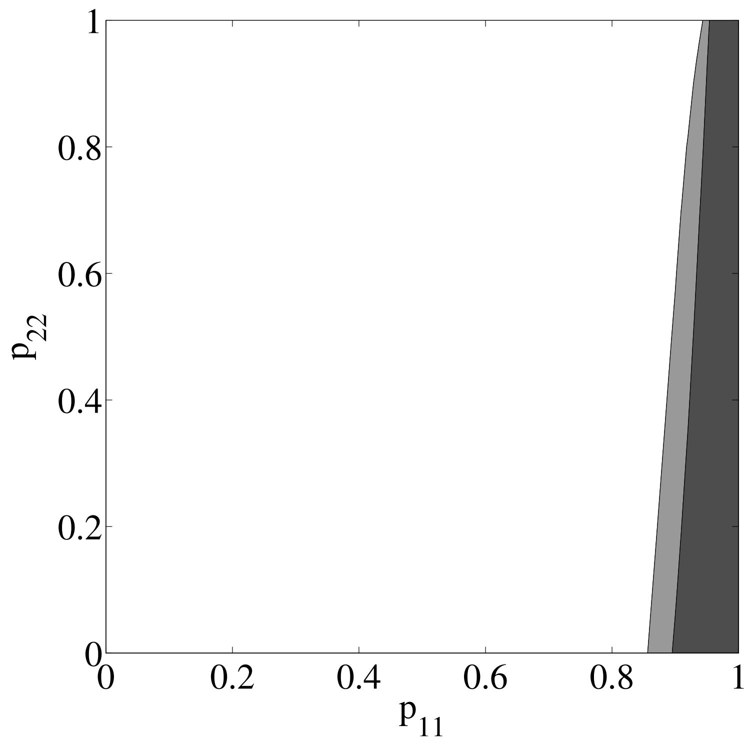

Figure 6: Determinacy regions and persistence of regimes: new Keynesian model with Markov-switching monetary policy

Note: the white area displays the determinacy region; the light-shaded area represents a region in which we cannot decide whether there is a unique bounded equilibrium or not in a reasonable amount of time; the dark-shaded area represents a region in which there exists multiple bounded equilibria. Policy parameters are calibrated as in Davig and Leeper (2007): $\alpha_1 = 0.89$ *and* $\gamma_1 = 0.15$ *in the first regime,* $\alpha_2 = 2.19$ *and* $\gamma_2 = 0.3$ *in the second regime.*

Figure 6 displays the determinacy regions with respect to the persistence of each regime, p_{11} and p_{22}. The higher the probability of remaining in one or the other regime, the more likely indeterminacy. The determinacy region is, however, more sensitive to the persistence of the pre-Volcker regime (see Figure 6).

For plausible probabilities, determinacy and indeterminacy are equally plausible. Bianchi (2013) estimates probabilities of the pre- and post-Volcker regimes between 0.82 and 0.96 and between 0.87 and 0.99 respectively. This region of probabilities encompasses the indeterminacy frontier plotted in Figure 6. Even if Bianchi's estimates are based on a model that slightly differs from the simple new-Keynesian model considered in this paper, this suggests that determinacy is a relevant empirical issue.

To conclude, indeterminacy is unlikely to explain the great inflation as the indeterminacy-based explanation needs the post-Volcker regime to be absorbent. Two scenarios remain possible: either sunspots hit the economy in both regimes or in none of them. Settling between these two scenarios is beyond the scope of this paper and is a natural avenue for future research. This needs to estimate regime switching model without ruling out sunspots equilibria.

4.2 Why does the existing literature underestimate the size of the indeterminacy region?

While the considered class of equilibria does not affect determinacy conditions in the standard linear rational expectations model, determinacy conditions change with the class of equilibria in a regime switching environment. Nonetheless, in both cases (with and without regime switching), there exist history-dependent equilibria among sunspots equilibria when the economy is indeterminate. In the no regime switching environment, the exact stochastic properties of sunspot equilibrium, such as its correlation with past fundamental shocks, do not affect stability. In a regime switching economy, the dependency between sunspots and past regimes and shocks matters for the stability of the sunspots equilibria. By ruling out some classes of sunspots equilibria, the existing literature, thus, underestimates the size of the indeterminacy region.

In a seminal paper, Davig and Leeper (2007) claim that they provide determinacy conditions for linear Markov-switching rational expectations models (MSRE, henceforth). However, using a new-Keynesian model with Markov-switching monetary policy, Farmer et al. (2010a) exhibit two different bounded equilibria in a region of parameters verifying Davig and Leeper's determinacy conditions. They, thus, prove that Davig and Leeper (2007) underestimate the indeterminacy region.

Branch et al. (2007) show that their determinacy conditions only apply for equilibria that depend on the current regime but not on past regimes. Davig and Leeper (2007) transform the Markov-switching model into a linear model with more state variables. We show in Appendix E.1 that this transformation is only valid for equilibria that depend on a finite number of past regimes. We call these latter Markovian equilibria.

In their reply, Davig and Leeper argue that the existence of Farmer et al.'s counterexample stems from mainly two reasons. First, they study different models. Restricting to Markovian equilibria allows them transforming the Markov-switching problem into a linear one which can be considered as a different model. It is, however, more natural to recognize that the definitions of the solutions space are different. Second, they dismiss Farmer et al.'s counterexample based on a non-Markovian equilibrium as contrived and of little economic sense. This second argument relates to the broader issue of the solutions space's proper definition in rational expectations models.

Farmer et al. (2009b) are subject to the same criticism: they also abstract from some stable equilibria. They restrict their attention to first-order autoregressive equilibria whose transition matrix depends on current and one-lagged regimes. This restriction corresponds roughly to imposing $q = 1$ in Proposition 3. In Appendix E.3, we recall and complement the restrictions of Farmer et al. (2009b). For certain parameters values, this restriction leads to miss some stable equilibria (see Appendix E.3). A large literature has built on this restriction (Cho, 2013; Foerster et al., 2011; Benhabib, 2010). Consequently, the existing literature generally underestimates the size of the indeterminacy region, and hence, understates the destabilizing property of regime switching.

4.3 Economically meaningful restrictions on the class of equilibria

Since restrictions on the class of equilibria modify determinacy conditions in MSRE models, it is crucial to understand which restrictions make sense.

Excluding some implausible equilibria may help circumventing the high complexity involved by Proposition 1. The equilibria responsible for this complexity are especially those depending on all the past regimes in a non-recursive way or involving a large number of state variables. These equilibria may appear as unrealistic as they require very large economic agents' brain capacities.

Fully history-dependent equilibria -like those identified in Proposition 3 when q tends to infinity- may be ruled out based on reasonable cognitive limitations.[10] However, focusing only on sunspots equilibria that depend on current regime (like in Davig and Leeper, 2007) or on current and 1-lagged regimes (like in Farmer et al., 2009b) may seem arbitrary and disconnected from assumptions on economic agents information. Sunspots equilibria depending on last two regimes are as realistic as those depending on immediate past regime, s_{t-1}.

A possible avenue for future research could be to find a restriction on the information set that rules out equilibria depending on an unrealistically large number of state variables. Ideally, this should be done without breaking the law of iterated expectations that prevents from the non-recursivity of the anticipations.

In any cases, it is worth noticing that our main results rely on the existence of sunspots equilibria depending on a small number of past regimes and shocks (we apply Proposition 3 at worst for $q = 6$). Thus, reasonable restrictions on informational set might not affect these results.

4.4 Stability concept

Farmer et al. (2009b) claim that the complexity of determinacy conditions in MSRE models

[10]Results by Blondel et al. (2003) suggest that determinacy conditions may be affected by restriction to finite dimensional state space versus infinite dimensional one. Applied to Proposition 3, this means that fully history-dependent equilibria may exist even if there are no sunspots equilibria of the form introduced by Proposition 3 for any finite length, q.

results from the choice of boundedness as stability concept. By contrast, they put forward an alternative stability concept called Mean Square Stability following the influential book by Costa et al. (2005). This latter concept consists in assuming the convergence of the first two conditional moments. Mean square stability appears more convenient when restricting the solutions space to some specific equilibria (Farmer et al., 2009b, 2010b). Adopting this latter concept and restricting to a particular class of equilibria, they find determinacy conditions for purely forward-looking models. Cho (2013) refines and complements their findings for models with pre-determined variables.

However, the Mean Square Stability concept presents no decisive advantage over boundedness when considering the general class of equilibria. Findings by Farmer et al. (2009b); Cho (2013) result from restrictions on the solutions space already mentioned above. These restrictions rather than a different stability concept explain why these authors find verifiable determinacy conditions and smaller indeterminacy region compared to us. Besides, mean square stability allows unbounded support for economic variables, which is, in general, inconsistent with a local approach (see Woodford, 1986; Barthélemy and Marx, 2011). *De facto*, boundedness is the only stability concept whose consistency with a perturbation approach has been proved.

5 Conclusion

From a theoretical standpoint, this paper gives a necessary and sufficient condition of determinacy for purely forward-looking rational expectations models with Markov-switching. This condition depends on the asymptotic behavior of all matrices products raising deep mathematical as well as economic issues. The complexity raised by regime switching models reflects the path-dependency of economic agents' expectations. To overcome this difficulty, we derive verifiable sufficient conditions for indeterminacy and determinacy.

We apply these results to a canonical monetary model in which monetary policy switches amongst different Taylor rules. We, thus, generalizes the Taylor principle that stipulates that in a no regime switching economy, the central banker should rise its nominal interest rate by more than one-for-one in response to inflation to prevent indeterminacy.

We then establish a non trivial relationship between monetary policy regimes and determinacy. An active monetary policy may help anchoring inflation expectations if the other regime fails to satisfy the Taylor principle. However, a too active monetary policy may lead to indeterminacy. This aggressive monetary policy destabilizes the output-gap by reacting too much to inflation expectations due to the other regime. This may happen even if the two regimes satisfy the Taylor principle.

We finally prove that determinacy conditions in the presence of regime switching intimately depend on the restrictions on the solutions space. On the one hand, we argue that the existing literature underestimates the destabilizing property of regime switching models by focusing on too restrictive classes of equilibria. On the other hand, we acknowledge that some stable equilibria can be considered as unrealistic from an economic point of view as they depend on an infinite dimensional space. Thus, a possible avenue for future theoretical research on regime switching is to characterize equilibria in an environment of limited information. We can hope that such restrictions could lead to very tractable determinacy conditions and exclude equilibria depending on an infinite number of state variables.

Another avenue for future applied research is to determine the empirical plausibility of indeterminacy given observed regime switching. To achieve such a goal, it is needed to develop an econometric method to estimate Markov-switching models with and without sunspots equilibria.

APPENDIX

Appendices A to F are for Online Publication only.

A Definition of a solution

We assume that F is a bounded set such that $\varepsilon_t \in F$, and we denote by $\varepsilon^t = \{\varepsilon_t, \cdots, \varepsilon_{-\infty}\}$ and $s^t = \{s_t, \cdots, s_{-\infty}\}$ the history of shocks and regimes. We define a stable equilibrium as follows:

Definition 2. *A stable equilibrium of* (1) *is function* z *on* $\{1, \cdots, N\}^\infty \times F^\infty$*, satisfying equation* (1) *and such that*

$$\|z\|_\infty = \sup_{z^t, \varepsilon^t} \|z(s^t, \varepsilon^t)\| < \infty \tag{9}$$

We denote by $\mathcal{B}$ the set of all the bounded functions on $\{1, \cdots, N\}^\infty \times F^\infty$. The set $\mathcal{B}$, with the norm $\|\cdot\|_\infty$ defined in equation (9) is a Banach space.

B Proof of Proposition 1

In this section, we prove Proposition 1. Assuming that Γ_i is invertible for any $i \in \{1, \cdots, N\}$, we rewrite (1) as:

$$z_t + \Gamma_{s_t}^{-1} \mathbb{E}_t z_{t+1} = -\Gamma_{s_t}^{-1} C_{s_t} \varepsilon_t \tag{10}$$

Then, considering $z_t = z(s^t, \varepsilon^t)$ as a function of all the past shocks $\{\varepsilon_t, \cdots, \varepsilon_{-\infty}\}$ and regimes $\{s_t, \cdots, s_{-\infty}\}$, introducing ψ_0 such that $\psi_0(s^t, \varepsilon^t) = -\Gamma_{s_t}^{-1} C_{s_t} \varepsilon_t$ and defining the operator $\mathcal{R}$ as

$$\mathcal{R} : z \mapsto ((s^t, \varepsilon^t) \mapsto -\Gamma_{s_t}^{-1} \mathbb{E}_t z(s^{t+1}, \varepsilon^{t+1})) \tag{11}$$

Equation (10) is equivalent to the functional equation:

$$(\mathbb{1} - \mathcal{R}) z = \psi_0 \tag{12}$$

This equation admits a unique solution if the operator $\mathbb{1} - \mathcal{R}$ is invertible, and thus, if $1 \notin \sigma(\mathcal{R})$. As a consequence, conditions of existence and uniqueness of a solution of (1) rely on the spectrum of $\mathcal{R}$, this spectrum depending on the space of solutions we consider.

Before characterizing this spectrum, we first show that the sequence (u_k) in equation (4) is convergent.

B.1 Behavior of the sequence u_p

In this section, we prove the following result

Lemma 1. *The sequence (u_k) in equation (4) has the following properties.*

- *The sequence $(u_k)^k$ is sub-multiplicative ($(u_{m+n})^{m+n} \leq u_m^m u_n^n$), and thus, convergent.*
- *The limit, ν, does not depend on the chosen norm.*

We first show that (u_k^k) is sub-multiplicative. By sub-multiplicativity of a matricial norm, u_{m+n}^{m+n} satisfies:

$$\begin{aligned}
&\sum_{(i_1,\cdots,i_m,i_{m+1},\cdots,i_{m+n})\in\{1,\cdots,N\}^{m+n}} p_{i_1i_2}\cdots p_{i_{m-1}i_m}\times \\
&\quad p_{i_mi_{m+1}}\cdots p_{i_{m+n-1}i_{m+n}}\|\Gamma_{i_1}^{-1}\cdots\Gamma_{i_m}^{-1}\Gamma_{i_{m+1}}^{-1}\cdots\Gamma_{i_{m+n}}^{-1}\| \\
\leq &\sum_{(i_1,\cdots,i_m,i_{m+1})\in\{1,\cdots,N\}^{m+1}} p_{i_1i_2}\cdots p_{i_{m-1}i_m}p_{i_mi_{m+1}}\|\Gamma_{i_1}^{-1}\cdots\Gamma_{i_m}^{-1}\| \\
&\times\left(\sum_{(i_{m+2},\cdots,i_{m+n})\in\{1,\cdots,N\}^{n-1}} p_{i_{m+1}i_{m+2}}\cdots p_{i_{m+n-1}i_{m+n}}\|\Gamma_{i_{m+1}}^{-1}\cdots\Gamma_{i_{m+n}}^{-1}\|\right)
\end{aligned}$$

We find an upper bound for the second term by summing on i_{m+1}, as all the terms are positive:

$$\begin{aligned}
&\sum_{(i_{m+2},\cdots,i_{m+n})\in\{1,\cdots,N\}^{n-1}} p_{i_{m+1}i_{m+2}}\cdots p_{i_{m+n-1}i_{m+n}}\|\Gamma_{i_{m+1}}^{-1}\cdots\Gamma_{i_{m+n}}^{-1}\| \\
\leq &\sum_{(i_{m+1},i_{m+2},\cdots,i_{m+n})\in\{1,\cdots,N\}^{n-1}} p_{i_{m+1}i_{m+2}}\cdots p_{i_{m+n-1}i_{m+n}}\|\Gamma_{i_{m+1}}^{-1}\cdots\Gamma_{i_{m+n}}^{-1}\| = (u_n)^n
\end{aligned}$$

Thus,

$$(u_{n+m})^{n+m} \leq u_n^n \sum_{(i_1,\cdots,i_m,i_{m+1})\in\{1,\cdots,N\}^{m+1}} p_{i_1i_2}\cdots p_{i_{m-1}i_m}p_{i_mi_{m+1}}\|\Gamma_{i_1}^{-1}\cdots\Gamma_{i_m}^{-1}\| = u_n^n\times u_m^m$$

since $\sum_{i_{m+1}\in\{1,\cdots,N\}} p_{i_mi_{m+1}} = 1$.

This shows that (u_k^k) is sub-multiplicative.

Besides, if a sequence of non-negative real numbers (v_k) is sub-multiplicative, then $v_k^{1/k}$ is converging and $\lim_{k\to+\infty} v_k^{1/k} = \inf_k v_k^{1/k}$, see for instance Lemma 21 p.8 in Müller (2003). Thus, (u_k) is convergent.

Finally, by equivalence of the norms in $\mathcal{M}_n(\mathbb{R})$, it is immediate that ν does not depend on the chosen norm.

B.2 Characterization of the spectral radius of $\mathcal{R}$

We will prove the following lemma, describing the spectrum of $\mathcal{R}$ in $\mathcal{B}$.

Lemma 2. *The operator $\mathcal{R}$ is bounded in $\mathcal{B}$ and its spectrum is given by:*

$$\sigma(\mathcal{R}) = [-\nu, \nu]$$

First, $\mathcal{R}$ is bounded as the expectation operator is a bounded operator. The rest of the proof is based on two main arguments:

- The spectrum of $\mathcal{R}$ is symmetric convex.
- $$\lim_{k\to+\infty} \|\mathcal{R}^k\|^{1/k} = \nu$$

The second point ensures that $\rho(\mathcal{R}) = \nu$ by applying the Gelfand characterization of the spectral radius for an operator, see for instance Theorem 22 p.8 in Müller (2003), while the first point leads to the equality $\sigma(\mathcal{R}) = [-\nu, \nu]$.

First, we introduce the operators $\mathcal{F}_i$, for $i \in \{1, \cdots N\}$, $\mathcal{F}$ and $\mathcal{L}$ on $\mathcal{B}$ defined by:

$$\mathcal{F}_i : \phi \mapsto ((s^t, \varepsilon^t) \mapsto \int_V \phi(is^t, \varepsilon\varepsilon^t) d\varepsilon$$

$$\mathcal{L} : \phi \mapsto ((s^t, \varepsilon^t) \mapsto \phi(s^{t-1}, \varepsilon^{t-1})$$

$$\mathcal{F}(\phi)(s^t, \varepsilon^t) = (p_{s_t1}\mathcal{F}_1 + p_{s_t2}\mathcal{F}_2)(\phi)(s^t, \varepsilon^t)$$

The operators $\mathcal{F}_i$ and $\mathcal{L}$ have the following straightforward properties.

1. $\mathcal{F}_i\mathcal{L} = \mathbb{1}$, and $\mathcal{F}\mathcal{L} = \mathbb{1}$
2. $|||\mathcal{F}_i||| = 1$ and $|||\mathcal{L}||| = 1$

where $|||\cdot|||$ is the triple norm associated with the infinite norm $\|\cdot\|_\infty$ on $\mathcal{B}$. Then $\mathcal{R}$ can be rewritten as:

$$\mathcal{R}(\phi)(s^t, \varepsilon^t) = \Gamma_{s_t}^{-1}(p_{s_t1}\mathcal{F}_1 + p_{s_t2}\mathcal{F}_2)(\phi)(s^t, \varepsilon^t)$$

We define $\tilde{\mathcal{R}}$ by

$$\tilde{\mathcal{R}} : \phi \mapsto \Gamma_{s_{t-1}}\mathcal{L}(\phi)(s^t, \varepsilon^t)$$

We have that:

$$\tilde{\mathcal{R}}\mathcal{R} = \mathcal{L}\mathcal{F}, \qquad \mathcal{R}\tilde{\mathcal{R}} = \mathbb{1}$$

We mimic techniques used to study the spectrum of isometries in Banach spaces as for instance in Conway (1990). We refer to this book and to Müller (2003) for the different type of spectrum. We know that the spectrum of $\mathcal{R}$ is a closed subset of $[-\|\mathcal{R}\|, \|\mathcal{R}\|]$, and that the boundary $\partial\sigma(\mathcal{R})$ of $\sigma(\mathcal{R})$ is included in the point approximate spectrum, i.e. the set of values λ such that $\mathcal{R} - \lambda\mathbb{1}$ is not injective or not bounded below. We assume that $\sigma(\mathcal{R})$ is not convex, and that there exists $\lambda_0 \in (0, \nu)$ such that $\lambda \in \partial\sigma(\mathcal{R})$. Then, we prove that λ_0 is an eigenvalue. Actually, $\mathcal{R} - \lambda\mathbb{1}$ is bounded below for any $\lambda < \|\mathcal{R}\|$. $\mathcal{R}$ is the composition of an invertible operator and an isometry, and thus, is bounded below. Moreover, we notice that:

$$\|\mathcal{R}\| = \sup_{v \in \mathrm{Im}(\tilde{\mathcal{R}})} \frac{\|\mathcal{R}v\|}{\|v\|} = \left(\inf_u \frac{\|\tilde{\mathcal{R}}u\|}{\|u\|} \right)^{-1}$$

which implies that:

$$\|u - \lambda\tilde{\mathcal{R}}u\| \leq (1 - \frac{\lambda}{\|\mathcal{R}\|})\|u\|$$

We show now that for any α such that $|\alpha| < 1$, then $\lambda\alpha$ belongs to $\sigma(\mathcal{R})$. We know that λ is an eigenvalue of $\mathcal{R}$, let $\phi_0 \in \mathcal{B}$ an eigenvector of $\mathcal{R}$ associated with λ,

$$\mathcal{R}\phi_0 = \lambda\phi_0$$

We define f by:

$$f = \phi_0 - \lambda\tilde{\mathcal{R}}\phi_0$$

We notice that $\mathcal{R}(f) = o$, and that $\|(\lambda\tilde{\mathcal{R}})^k(f)\| \leq \|\phi_0\|$. Fix α such that $|\alpha| < 1$. We define $\tilde{\phi}_0$ by :

$$\tilde{\phi}_0 = \sum_{k=0}^{\infty} \alpha^k (\lambda\tilde{\mathcal{R}})^k(f)$$

We compute:

$$\mathcal{R}(\tilde{\phi}_0) = \sum_{k=0}^{\infty} \alpha^k \mathcal{R}(\lambda\tilde{\mathcal{R}})^k(f)$$

$$\mathcal{R}(\tilde{\phi}_0) = \alpha\lambda \sum_{k=0}^{\infty} \alpha^k (\lambda\tilde{\mathcal{R}})^k(f) = \alpha\lambda\tilde{\phi}_0$$

Thus, $\alpha\lambda$ is an eigenvalue of $\mathcal{R}$, which contradicts $\lambda \in \partial\sigma(\mathcal{R})$, and $\partial\sigma(\mathcal{R}) = \nu$.

Concerning the second point, we first prove that $\lim_{k\to+\infty} \|\mathcal{R}^k\|^{1/k} \leq \nu$. Then we construct, for any k, a function ϕ_k, such that:

$$\|\mathcal{R}^k(\phi_k)\|^{1/k} \geq \rho(S_k)^{1/k}$$

This construction is a generalization to the multivariate cases of Farmer et al. (2009a) and Farmer et al. (2010a).

We compute

$$\mathcal{R}^k(\phi)(s^t) = \sum_{i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} \Gamma_{s_t}^{-1} \Gamma_{i_1}^{-1} A_{i_1} \cdots \Gamma_{i_{k-1}}^{-1} \mathcal{F}_{i_1} \cdots \mathcal{F}_{i_k}(\phi)(s^t)$$

We will find an upper bound and a lower bound for $\|\mathcal{R}^k\|$, in terms of a sequence (u_k) associated to well-chosen norms on $\mathcal{M}_n(\mathbb{R})$. First, we consider the triple norm associated to the infinite norm on $\mathcal{M}_n(\mathbb{R})$ and the associated sequence u_k. For any ϕ such that $\|\phi\|_\infty = 1$, we obtain by sub-additivity of the norm,

$$\|\mathcal{R}^k(\phi)\|_\infty \leq \sum_{i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} |||\Gamma_{s_t}^{-1}\Gamma_{i_1}^{-1}\cdots\Gamma_{i_{k-1}}^{-1}||| = u_k^k$$

which leads to $\lim\limits_{k\to+\infty} \|\mathcal{R}^k\|^{1/k} \leq \nu$.

Reciprocally, we consider on $\mathcal{M}_{r,s}(\mathbb{R})$ the norm $|\cdot|$ defined by:

$$|M| = \sum_{i,j} |m_{i,j}|, \quad \text{where} \quad M = [m_{i,j}]_{(i,j)\in\{1,\cdots,r\}\times\{1,\cdots,s\}}$$

This norm satisfies:

- $|M| \leq r|||M|||_\infty$
- If we write $M = [M_1, M_2, \cdots, M_l]$ by blocks, we notice the following useful property:

$$|M| = \sum_{i=1}^{l} |M_i|$$

Fix $s_t \in \{1,\cdots,N\}$ and let us denote by $\{w_{i_1\cdots i_{k+1}}, \forall(i_1\cdots i_{k+1} \in \{1,\cdots,N\})\}$ a family of $n\times 1$ vectors and rewrite the following sum as a product of matrices by blocks:

$$\sum_{(i_1,\cdots,i_k)\in\{1,\cdots,N\}^k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} A_{s_t} A_{i_1} \cdots A_{i_{k-1}} w_{s_t i_1 \cdots i_{k-1}}$$

$$= \left[\begin{array}{ccc} p_{s_t 1}\cdots p_{11}[\Gamma_{s_t}^{-1}\cdots\Gamma_1^{-1}] & \cdots & |p_{s_t N}\cdots p_{NN}[\Gamma_{s_t}^{-1}\cdots\Gamma_N^{-1}] \end{array}\right] \times \left[\begin{array}{c} w_{s_t 1\cdots 1} \\ \vdots \\ w_{s_t N\cdots N} \end{array}\right]$$

Thus,

$$\sup_{\|w_{i_1\cdots i_p}\|_\infty\leq 1} \|\sum_{i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} A_{s_t} A_{i_1} \cdots A_{i_{k-1}} w_{i_1\cdots i_p}\|_\infty$$

$$= \sup_{\|w_{i_1\cdots i_p}\|_\infty\leq 1} \|\left[\begin{array}{ccc} p_{s_t 1}\cdots p_{11}[\Gamma_{s_t}^{-1}\cdots\Gamma_1^{-1}] & \cdots & |p_{s_t N}\cdots p_{NN}[\Gamma_{s_t}^{-1}\cdots\Gamma_N^{-1}] \end{array}\right] \times \left[\begin{array}{c} w_{s_t 1\cdots 1} \\ \vdots \\ w_{s_t N\cdots N} \end{array}\right]\|_\infty$$

$$= |||\left[\begin{array}{ccc} p_{s_t 1}\cdots p_{11}[\Gamma_{s_t}^{-1}\cdots\Gamma_1^{-1}] & \cdots & |p_{s_t N}\cdots p_{NN}[\Gamma_{s_t}^{-1}\cdots\Gamma_N^{-1}] \end{array}\right]|||_\infty$$

$$\geq \frac{1}{Nn}|\left[\begin{array}{ccc} p_{s_t 1}\cdots p_{11}[\Gamma_{s_t}^{-1}\cdots\Gamma_1^{-1}] & \cdots & |p_{s_t N}\cdots p_{NN}[\Gamma_{s_t}^{-1}\cdots\Gamma_N^{-1}] \end{array}\right]|$$

$$\geq \frac{1}{Nn} \sum_{i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} |A_{s_t} A_{i_1} \cdots A_{i_{k-1}}|$$

Furthermore, as the considered space is a bounded subset of finite-dimensional vectorial space, the supremum is reached and there exist N^k vectors $(w_{s_t i_1 \cdots i_{k-1}})$ for $(i_1, \cdots, i_{k-1}) \in \{1, \cdots, N\}^{k-1}$ such that:

$$\| \sum_{i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} \Gamma_{s_t}^{-1} \Gamma_{i_1}^{-1} \cdots \Gamma_{i_{k-1}}^{-1} w_{i_1 \cdots i_p} \|$$

$$\geq \frac{1}{Nn} \sum_{i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} |\Gamma_{s_t}^{-1} \Gamma_{i_1}^{-1} \cdots \Gamma_{i_{k-1}}^{-1}|$$

We define the function ϕ_0 by:$\phi_0(s^t) = w_{s_t s_{t-1} s_{t-2} \cdots s_{t-k}}$. This function is bounded and of norm 1. Moreover, ϕ_0 satisfies:

$$\sum_{s_t} \|\mathcal{R}^k(\phi_0)(s^t)\| \geq \frac{1}{Nn} \sum_{s_t, i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} |\Gamma_{s_t}^{-1} \Gamma_{i_1}^{-1} \cdots \Gamma_{i_{k-1}}^{-1}| = \frac{1}{Nn} (\tilde{u}_k)^k$$

which leads to:

$$\|\mathcal{R}^k(\phi_0)\|_\infty \geq \frac{1}{N^2 n} (\tilde{u}_k)^k$$

Finally, this implies that:

$$\|\mathcal{R}^k\|^{1/k} \geq (N^2 n)^{-1/k} \tilde{u}_k$$

Taking the limit, we get that $\lim_{k \to +\infty} \|\mathcal{R}^k\|^{1/k} \geq \nu$. This ends the proof of Lemma 2.

B.3 Proof of Proposition 1

A consequence of Lemma 2 is that $1 \in \sigma(\mathcal{R})$ if and only if $\nu \geq 1$, and thus, $(\mathbb{1} - \mathcal{R})$ is invertible if and only if $\nu < 1$, which proves Proposition 1.

C Asymptotic behavior of u_p

For computational reasons, it is often quicker to compute eigenvalues rather than sums with an increasing number of terms.

Let us start with the univariate case. In this case, the sequence u_p^p is the sum of the terms of a matrix S^p where S is defined as follows:

$$S = \left(p_{ij} \|\Gamma_i^{-1}\|\right)_{ij}.$$

Actually, the matrices Γ_i^{-1} reduce to scalars in this case, and hence, are commutative. Thus, for univariate models, u_p^p behaves as $\|S^p\|$ and the limit ν is equal to the spectral radius of S,

$\rho(S)$. Farmer et al. (2009a) find a comparable result in the specific context of the Fisherian model of inflation determination.

In the general case, we introduce the matrix S_k.

$$S_k = \left(\sum_{(i_1,\cdots,i_{k-1})\in\{1,\cdots,N\}^{k-1}} p_{ii_1}\cdots p_{i_{k-1}j} \|\Gamma_i^{-1}\Gamma_{i_1}^{-1}\cdots\Gamma_{i_{k-1}}^{-1}\| \right)_{ij}.$$

For any k, an (i,j) element of the matrix S_k corresponds to an upper bound of the expected impact (in norm) of the endogenous variables along trajectories starting from regime i to regime j in k steps weighted by the probability of each trajectory.

The following result links the behavior of u_p to the one of the spectral radius $\rho(S_p)$.

Lemma 3. *The sequence* $\left(\rho(S_p)^{1/p}\right)$ *is equivalent to* (u_p) *when* p *tends to* ∞.

Now, we consider the norm $|\cdot|_\infty$ on $\mathcal{M}_2(\mathbb{R})$ defined by $|M|_\infty = \sum_{i,j}|m_{ij}|$. One can observe that:

$$|S_p|_\infty = \sum_{i,i_1,\cdots,i_{p-1},j} p_{ii_1}\cdots p_{i_{p-1}j}\|\Gamma_i^{-1}\Gamma_{i_1}^{-1}\cdots\Gamma_{i_{p-1}}^{-1}\| = u_{p-1}^{p-1} \tag{13}$$

As the spectral radius is the infimum of matricial norms, Equation 13 leads to:

$$\rho(S_p) \leq u_{p-1}^{p-1} \tag{14}$$

Furthermore,

$$\left(S_p^q\right)_{ij} = \sum_{i_1,\cdots,i_{p-1},i_p,i_{p+1},\cdots,i_{2p},\cdots,i_{p(q-1)+1},\cdots,i_{pq-1}} p_{ii_1}\cdots p_{i_{p-1}i_p}p_{i_{(q-1)p}i_{(q-1)p+1}}\cdots p_{i_{qp-1}j}\times$$
$$\|\Gamma_i^{-1}\cdots\Gamma_{i_{p-1}}^{-1}\|\cdots\|\Gamma_{i_{(q-1)p}}^{-1}\cdots\Gamma_{i_{qp-1}}^{-1}\|$$

And by sub-multiplicativity of matricial norms:

$$\left(S_p^q\right)_{ij} \geq \sum_{i_1,\cdots,i_{p-1},i_p,i_{p+1},\cdots,i_{2p},\cdots,i_{p(q-1)+1},\cdots,i_{pq-1}} p_{ii_1}\cdots p_{i_{p-1}i_p}p_{i_{(q-1)p}i_{(q-1)p+1}}\cdots p_{i_{qp-1}j}\times$$
$$\|\Gamma_i^{-1}\cdots\Gamma_{i_{p-1}}^{-1}\cdots\Gamma_{i_{(q-1)p}}^{-1}\cdots\Gamma_{i_{qp-1}}^{-1}\|$$

and hence,

$$|S_p^q|_\infty \geq u_{pq-1}^{pq-1} \tag{15}$$

Equation 14 can be rewritten as follows:

$$|S_p^q|_\infty^{(1/q)} \geq (u_{pq-1})^{p-1/q}$$

For any norm, Gelfand's Theorem shows that $\lim_{q\to\infty}||X^q||_\infty^{(1/q)} = \rho(X)$. Thus, when q tends to infinity, 14 leads to:

$$\lim_{k\to\infty} u_k^p \leq \rho(S_p)$$

Thus, as $p > 1$,

$$\lim_{k\to\infty} u_k \leq \rho(S_p)^{1/p} \tag{16}$$

Combining Equations (14) and (16), we find the following upper and lower bounds:

$$\lim_{k\to\infty} u_k \leq \rho(S_p)^{1/p} \leq u_{p-1}^{1-1/p}$$

and thus, $\left(\rho(S_p)^{1/p}\right)$ is convergent and has the same limit as (u_p).

D Proof of Proposition 2

Proposition 2 follows directly from Equation (16).

To prove Proposition 2, we notice that

$$u_k^k = \sum_{(i_1,\cdots,i_k,i_{k+1})\in\{1,\cdots,N\}^k} p_{i_1 i_2}\cdots p_{i_{k-1}i_k} p_{i_k i_{k+1}} \|\Gamma_{i_1}^{-1}\cdots\Gamma_{i_k}^{-1}\|$$

Then by considering the multiples of p ($k = np$) and by only keeping the diverging trajectory (the hypothesis of the Lemma), we can rewrite the above equation as follows:

$$u_{np}^{np} \geq [p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0}\|\Gamma_{i_0}^{-1}\cdots\Gamma_{i_{p_0}}^{-1}\|]^n$$

and hence,

$$u_{np} \geq [p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0}\|\Gamma_{i_0}^{-1}\cdots\Gamma_{i_{p_0}}^{-1}\|]^{1/p}$$

Besides,

$$[p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0}\|\Gamma_{i_0}^{-1}\cdots\Gamma_{i_{p_0}}^{-1}\|]^{1/p} \geq \rho(p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0}\Gamma_{i_0}^{-1}\cdots\Gamma_{i_{p_0}}^{-1})$$

Thus,

$$\lim_{n\to\infty} u_{np} \geq \rho(p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0}\Gamma_{i_0}^{-1}\cdots\Gamma_{i_{p_0}}^{-1})$$

The right-hand-side of the inequality is larger than one by hypothesis which implies that $\nu > 1$.

E Different classes of equilibria

E.1 Markovian equilibria

In this section, we give the assumptions under which Proposition 1 of Davig and Leeper (2007) is valid, and thus explain the debate between Davig and Leeper (2007) and Farmer et al. (2010a). The authors restrict the solution space to bounded Markovian solutions.

Definition 3. *A solution, z_t, is said to be Markovian of order p, if it depends on the p last regimes, $\{s_t, s_{t-1}, \cdots, s_{t-p}\}$, and all past shocks, ε^t, i.e. there exists a measurable function, ϕ, mapping $\{1, \cdots, N\}^{p+1} \times (\mathbb{R}^p)^\infty$ into $\mathbb{R}^n$ such that $z_t = \phi(\{s_t, s_{t-1}, \cdots, s_{t-p}\}, \varepsilon^t)$.*

This definition is a generalization of what Branch et al. (2007) call a "regime dependent equilibrium". In this paper, the authors define a regime dependent equilibrium as a solution depending on the current regime only. In our terminology, such an equilibrium is a 0-order Markovian solution. Davig and Leeper (2007) considers 0-order Markovian solution, whereas the counter-example provided by Farmer et al. (2010a) does not belong to this solutions space.

We introduce a matrix, $\mathbf{M}$, as a combination between transition probabilities matrix, P, and a $(n.N \times n.N)$ real matrix, diagonal by blocks, $\text{diag}(\Gamma_1, \cdots, \Gamma_N)$: $\mathbf{M} = (P \otimes \mathbb{1}_n) \times \text{diag}(\Gamma_1, \cdots, \Gamma_N)$. The mathematical symbol $\otimes$ denotes the standard Kronecker product. Proposition 2 refines and complements the main proposition by Davig and Leeper (2007).

Proposition 4. *1. Model (1) admits a unique Markovian bounded solution, z_t^0, if and only if the spectral radius of $\mathbf{M}$, i.e. the largest eigenvalue in absolute value, $\rho(\mathbf{M})$, is strictly less than one.*

2. Otherwise, all the bounded Markovian solutions can be put into the following form:

$$z_t = z_t^0 + V_{s_t} w_t, \quad \text{where}, \quad w_t = J_w w_{t-1} + \xi_t,$$

with, ξ_t being any bounded martingale ($\mathbb{E}_t \xi_{t+1} = 0$) independent from current and past regimes. The martingale can be either a sunspot shock as defined in Cass and Shell (1983) or a fundamental disturbance. Matrices J_w and V_{s_t} are defined in equations (19) and (20).

The proof of the first point is undertaken in two steps:

- If $\phi \in \mathcal{M}$ is solution of Equation (1), then $\phi \in \mathcal{M}_0$
- Furthermore if $\phi \in \mathcal{M}$, then defining $\mathbf{\Phi}$ by:

$$\mathbf{\Phi}(\varepsilon^t) = \begin{bmatrix} \phi(1s^{t-1}, \varepsilon^t) \\ \vdots \\ \phi(Ns^{t-1}, \varepsilon^t) \end{bmatrix}$$

  $\mathbf{\Phi}$ is solution of a linear rational expectations model with regime-independent parameters. We thus can apply Blanchard and Kahn (1980).

Assume that there exists a p-order Markovian solution of (1), ϕ, we define $\mathcal{P}(q)$ the statement that the solution only depends on the last q regimes:

$$\mathcal{P}(q): \qquad \phi(is_1 \cdots s_q w, \varepsilon^t) = \phi(is_1 \cdots s_q w', \varepsilon^t)$$

$$\forall (s_1,\cdots,s_q)\in\{1,\cdots,N\}^q,\qquad \forall w\in\{1,\cdots,N\}^\infty,\qquad \forall w'\in\{1,\cdots,N\}^\infty,\qquad \forall \varepsilon^t\in V^\infty$$

$\mathcal{P}(p)$ is satisfied by assumption. Let us assume that $\mathcal{P}(q)$ is satisfied for $q\in\{1,\cdots,p\}$. Since ϕ is a solution of (1), for any w, we compute:

$$\phi(s_t s_1\cdots s_{q-1}w,\varepsilon^t) = -\Gamma_{s_t}^{-1}(\sum_i p_{s_t i}\int\phi(is_t s_1\cdots s_{q-1}w,\varepsilon\varepsilon^t)d\varepsilon - \Gamma_{s_t}^{-1}C_{s_t}\varepsilon_t$$

Due to $\mathcal{P}(q)$, we know that:

$$-\Gamma_{s_t}^{-1}(\sum_i p_{s_t i}\int\phi(is_t s_1\cdots s_{q-1}w,\varepsilon\varepsilon^t)d\varepsilon = -\Gamma_{s_t}^{-1}(\sum_i p_{s_t i}\int\phi(is_t s_1\cdots s_{q-1}w',\varepsilon\varepsilon^t)d\varepsilon$$

for any w', and hence, ϕ does not depend on w. $\mathcal{P}(q-1)$ is thus satisfied. By decreasing induction we eventually show that ϕ is Markovian of order 0.

More generally if the solution is Markovian, its order is the same than ψ_0. Here, ψ_0 is Markovian of order 0, thus, ϕ is also Markovian of order 0.

If $\phi\in\mathcal{M}_0$ is a solution of 1, ϕ is a solution of:

$$\forall i\in\{1,\cdots,N\}\qquad \phi(i,\varepsilon^t)+\Gamma_i^{-1}\left(p_{i1}\int\phi(1,\varepsilon\varepsilon^t)d\varepsilon + p_{i2}\int\phi(2,\varepsilon\varepsilon^t)d\varepsilon\right) = -\Gamma_i^{-1}C_i\varepsilon_t$$

We consider

$$\mathbf{\Phi}(\varepsilon^t) = \begin{bmatrix}\phi(1,\varepsilon^t)\\ \phi(2,\varepsilon^t)\\ \vdots\\ \phi(N,\varepsilon^t)\end{bmatrix} \tag{17}$$

Thus, by introducing $\mathbf{D} = -\begin{bmatrix}\Gamma_1^{-1}C_1\\ \vdots\\ \Gamma_N^{-1}C_N\end{bmatrix}$, the system is rewritten as :

$$\mathbf{\Phi}(\varepsilon^t) + \mathbf{M}\int\mathbf{\Phi}(\varepsilon\varepsilon^t)d\varepsilon = D\varepsilon_t \tag{18}$$

where $\mathbf{\Phi}$ is defined in Equation (17). Model (18) is a standard linear rational expectations model with constant parameters. We hence easily prove Proposition 2 by applying Blanchard and Kahn (1980).

We denote by $\mathcal{B}_0$ the set of bounded functions on V^∞, and by $\mathcal{F}$ the bounded operator acting in $\mathcal{B}_0$:

$$\mathcal{F}:\mathbf{\Phi}\mapsto\left((\varepsilon^t)\mapsto\int\mathbf{\Phi}(\varepsilon\varepsilon^t)d\varepsilon\right)$$

We rewrite equation (18) as: $[\mathbb{1}+\mathbf{M}\mathcal{F}]\mathbf{\Phi} = \mathbf{\Psi_0}$, where $\mathbf{\Psi_0}(\varepsilon_t) = \mathbf{D}\varepsilon_t$. The solution $\mathbf{\Phi}$ is then: $\mathbf{\Phi} = \sum_{k=0}^\infty\left[-\mathbf{M}\mathcal{F}\right]^k\mathbf{\Psi_0}$. Knowing that:

$$(\mathcal{F}^k\mathbf{\Psi_0})(\varepsilon_t) = \mathbf{D}\mathbb{E}_t\varepsilon_{t+k}$$

The solution is then given by:

$$\mathbf{\Phi}(\varepsilon^t) = -\sum_{k=0}^{\infty} \mathbf{M}^k \mathbf{D} \mathbb{E}_t \varepsilon_{t+k}$$

thus,

$$\phi(s_t, \varepsilon^t) = U_{s_t} \sum_{k=0}^{\infty} \mathbf{M}^k \mathbf{D} \mathbb{E}_t \varepsilon_{t+k}$$

where

$$\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_N \end{bmatrix} = -\mathbb{1}$$

Defining $z_t^0 = U_{s_t} \sum_{k=0}^{\infty} \mathbf{M}^k \mathbf{D} \mathbb{E}_t \varepsilon_{t+k}$, we notice that the solution z_t^0 only depends on s_t and ε_t. In existing literature, z_t^0 is called fundamental or Minimum State Variable solution.

In the case where $\rho(\mathbf{M}) > 1$, we apply the strategy of solving linear rational expectations models (see Blanchard and Kahn (1980) or Lubik and Schorfheide (2004)). There exists an invertible matrix Q such that

$$\mathbf{M} = \mathbf{Q} \times \begin{bmatrix} \Delta_u & R_u \\ 0 & \Delta_s \end{bmatrix} \mathbf{Q}^{-1}$$

with $\rho(\Delta_u) > 1$ and $\rho(\Delta_s) < 1$. Writing $Z_t = Q \begin{bmatrix} Z_t^u \\ Z_t^s \end{bmatrix}$, Z^s and Z^u are such that:

$$Z_t^s = \sum_{k=0}^{\infty} (-\Delta_s)^k [\ 0 \quad \mathbb{1}\ ] QC \mathbb{E}_t \varepsilon_{t+k}$$

$$\Delta_u \mathbb{E}_t Z_{t+1}^u + R_u \mathbb{E}_t Z_{t+1}^s + Z_t^u = [\ \mathbb{1} \quad 0\ ] QC \varepsilon_t$$

A general solution of the previous equation is then

$$Z_t^u = Z_t^{u,0} + \sum_{k=0} (-\Delta_u)^{-k} \xi_{t-k}$$

where ξ_t is any martingale and the solution $Z_t^{u,0}$ is such that:

$$Z_t^{u,0} = \sum_{k=0}^{\infty} (-\Delta_u)^{-k} [\ \mathbb{1} \quad 0\ ] QC \varepsilon_{t-k-1} - \sum_{k=0}^{\infty} (-\Delta_u)^{-k} Z_{t-k}^s$$

Then, the solutions are given by

$$Z_t = Z_t^0 - Q \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} (-\Delta_u)^{-k} \xi_{t-k}$$

And finally,

$$z_t = z_t^0 + V_{s_t} w_t$$

with

$$V_1 = -[\ \mathbb{1}\quad 0\ ]Q \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} \text{ and } V_2 = -[\ 0\quad \mathbb{1}\ ]Q \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} \tag{19}$$

w satisfies:

$$w_t = (-\Delta_u)^{-1} w_{t-1} + \xi_t$$

Defining J_w by

$$J_w = (-\Delta_u)^{-1} \tag{20}$$

leads to the result.

E.2 Non-Markovian equilibria : a proof of Proposition 3

For $\alpha = [\alpha(i_0, i_1, \cdots, i_{q-1}, i_q)]_{(i_0,i_1,\cdots,i_{q-1},i_q)\in\{1,\cdots,N\}^{q+1}}$, we denote by $K(\alpha)$ the matrix introduced in Proposition 3 as follows:

$$K(\alpha) = \left[\sum_{(i_1,\cdots,i_{q-1})\in\{1,N\}^{q-1}} p_{ii_1} p_{i_1i_2} \cdots p_{i_{q-1}j} \Gamma_i^{-1} \cdots \Gamma_{i_{q-1}}^{-1} \alpha(i, i_1, \cdots, i_{q-1}, j) \right]_{(i,j)} . \tag{21}$$

Assumptions of Proposition 3 imply that, for a certain integer q, there exist N^{q+1} real numbers in the (interior of the) unit disk, $\alpha(i_0, \cdots, i_q)$, and a $(nN \times 1)$ column vector, U, satisfying:

$$K(\alpha)U = U, \tag{22}$$

Then, we introduce, for $k \in \{0, \cdots, q\}$ the $(k+1)^N$ vectors $V(s_{t-k}, \cdots, s_t)$. For $k = q$, $V(s_{t-q}, \cdots, s_t)$ satisfies:

$$V(s_{t-q}, \cdots, s_t) = \alpha(s_{t-q}, \cdots, s_t) U_{s_t} \tag{23}$$

and, for k from $q-1$ to 0, $V(s_{t-k}, \cdots, s_t)$ is defined by backward induction by

$$\Gamma_{s_t} V(s_{t-k}, \cdots, s_t) = p_{s_t1} V(s_{t-k}, \cdots, s_t, 1) + p_{s_t2} V(s_{t-k}, \cdots, s_t, 2) \tag{24}$$

By construction, we see that

$$V(s_t) = U_{s_t}$$

Now, we construct some specific solutions.

Lemma 4. *Under assumptions of Proposition 3, then a unique Markovian stable equilibrium co-exists with multiple stable cyclical equilibria. An example of such bounded equilibria is, for any given t_0.*

$$\begin{cases} z_t = z_t^0, \ \textit{for } t < t_0 \\ z_t = z_t^0 + w_t, \ \textit{for } t \geq t_0 \\ w_t = \frac{V'(s_{t-1-(t-1-t_0)[q]},\cdots,s_{t-1})w_{t-1}}{V'(s_{t-1-(t-1-t_0)[q]},\cdots,s_{t-1})V(s_{t-1-(t-1-t_0)[q]},\cdots,s_{t-1})} V(s_{t-(t-t_0)[q]},\cdots,s_t) + V(s_{t-(t-t_0)[q]},\cdots,s_t)\xi_t, \end{cases}$$

where $(t-t_0)[q]$ represents the rest of the division of $(t-t_0)$ by q. The vectors V are defined in equations (23), ξ_t is any bounded real-valued martingale independent of s^t.

Lemma 4 gives the explicit form of sunspots and thus implies Proposition 3. Moreover, we notice that for any $\lambda \in]1, \frac{1}{\max(\alpha)}[$, the matrix $K(\lambda\alpha)$ has an eigenvalue larger than 1, thus, according to (23), there exists a continuum of solutions w.

To prove Lemma 4, we have to check that w is solution.

We first notice that by construction, w_t is collinear to $V(s_{t-(t-t_0)[q]},\cdots,s_t)$, for any $t \geq t_0$. Moreover, we notice that $(t-t_0)[q]$ belongs to $\{0,\cdots,q-1\}$, thus:

$$\mathbb{E}_t(w_{t+1}) = (\frac{V'(s_{t-(t-t_0)[q]},\cdots,s_t)w_t}{V'(s_{t-(t-t_0)[q]},\cdots,s_t)V(s_{t-(t-t_0)[q]},\cdots,s_t)} + \mathbb{E}_t\xi_{t+1})\mathbb{E}_t V(s_{t+1-(t+1-t_0)[q]},\cdots,s_{t+1})$$

We know that $\mathbb{E}_t(\xi_{t+1}) = 0$, and according to equation (24), that

$$\mathbb{E}_t V(s_{t+1-(t+1-t_0)[q]},\cdots,s_{t+1}) = \Gamma_{s_t} V(s_{t+1-(t+1-t_0)[q]},\cdots,s_t)$$

And finally:

$$\mathbb{E}_t(w_{t+1}) = \frac{V'(s_{t-(t-t_0)[q]},\cdots,s_t)w_t}{V'(s_{t-(t-t_0)[q]},\cdots,s_t)V(s_{t-(t-t_0)[q]},\cdots,s_t)} \Gamma_{s_t} V(s_{t+1-(t+1-t_0)[q]},\cdots,s_t) = \Gamma_{s_t} w_t$$

We represent in figure 7, the different determinacy regions, for $q=1$ and $q=6$ for the calibration chosen in Davig and Leeper (2007).

E.3 The solutions of Farmer et al. (2009b)

Farmer et al. (2009b) focus on equilibria that can be put under the following form:

$$z_t = z_t^0 + w_t,$$

$$w_t = \Lambda_{s_{t-1}s_t} w_{t-1} + V_{s_t} V'_{s_t} \xi_t$$

This class of equilibria is a subclass of our solutions in Lemma 4 and Proposition 3, for $q=1$.

The result of Farmer et al. (2009b) can be extended in the following Proposition.

Proposition 5. *If $\rho(\mathbf{M}) \neq 1$, any solution of model (1) can then be put under the form*

$$z_t = z_t^0 + V_{s_t} V'_{s_t} w_t,$$

$$w_t = V_{s_t} \Phi_{s_t,\cdots,s_{t-p}} V'_{s_{t-p}} w_{t-p} + \eta_t,$$

where η_t *is a martingale when* $p = 1$ *and satisfies the following property otherwise:*

$$\mathbb{E}_{t-p} \prod_{i=t-p}^{t-1} \Gamma_{s_i}^{-1} \eta_t = 0.$$

$\Phi_{s_t,\cdots,s_{t-p}}$ *and* V_{s_t} *satisfy:*

$$\Gamma_{s_t} V_{s_t} = \sum_{s_{t+1}\dots s_{t+p}} p_{s_t s_{t+1}} \dots p_{s_{t+p-1} s_{t+p}} \Gamma_{s_{t+1}}^{-1} \dots \Gamma_{s_{t+p}}^{-1} V_{s_{t+p}} \Phi_{s_t \dots s_{t+p}} \tag{25}$$

The stability of processes described in Proposition 5 is given by the maximum eigenvalue of all the possible products involving $V_{s_t}\Phi_{s_t,\cdots,s_{t-p}}V'_{s_{t-p}}$ as already mentioned in **?**. This maximum is known in mathematics as the joint spectral radius. The joint spectral radius of a set of matrices $\mathcal{M} = \{M_1, \cdots M_N\}$ is defined by:

$$\overline{\rho}(\mathcal{M}) = \lim_{t\to+\infty} \max\{\|M_{i_1}\cdots M_{i_t}\|, \quad ,(M_{i_1},\cdots,M_{i_t}) \in \mathcal{M}^t\} \tag{26}$$

We have thus the following corollary of Proposition 5:

Corollary 1. *If there exists* $p > 0$ *such that matrices* $\Phi_{\{.\}}$ *and* $V_{\{.\}}$ *satisfy equations 25 and*

$$\overline{\rho}\left(\{V_{i_1}\Phi_{i_1,\cdots,i_{p+1}}V'_{i_{p+1}},(i_1,\cdots,i_{p+1}) \in \{1,\cdots,N\}^{p+1}\}\right) < 1 \tag{27}$$

Then, there exists at least one non-Markovian bounded solution to model (1).

In Farmer et al. (2009b), the authors give determinacy conditions for first-order Markov switching Vectorial autoregressive process, i.e. when for $p = 1$ the solution is stable (mean square stable in their paper, but that is not key) and η_t only depends on ε^t but is independent from current and past regimes.

However, there exists region of parameters for which there is no such a stable solution but indeterminacy. That may be the case when for $p = 1$, η_t depends on past regimes and is unbounded so as the autoregressive process. Indeed, the sum of two unbounded processes can be unbounded. This proves that determinacy region provided by the application of Farmer et al. (2009b) is too large and incomplete.

Proposition 5 combined with Corollary 1 theoretically provides a way to refine the region of indeterminacy. However, it requires finding all the possible matrices $\Phi_{\{.\}}$ and $V_{\{.\}}$ satisfying equations 25 which is computationally expensive.

The proof of Proposition 5 consists in extending the strategy of Farmer et al. (2009b). Let us consider a solution to model 1, z_t. We define w_t as the difference between this solution and the solution z_t^0. w_t is thus solution of the homogenous model:

$$\Gamma_{s_t} w_t = \mathbb{E}_t w_{t+1}, \tag{28}$$

We denote by V_{s_t} a $(n \times k_{s_t})$ matrix whose column vectors are a base of the linear space spanned by w_t in regime s_t.

Let us consider an element, w_k of V_i $(k \leq k_{s_t})$ solution of the homogenous model (28).

$$\begin{aligned}
\Gamma_i w_k &= \mathbb{E}(\Gamma_{s_t} w_t | s_t = i, w_t = w_k) && (29)\\
&= \mathbb{E}(\mathbb{E}_t(w_{t+1}) | s_t = i, w_t = w_k) && (30)\\
&= \mathbb{E}(\sum_{s_{t+1}} p_{s_t s_{t+1}} \mathbb{E}_t(w_{t+1}|s_{t+1})|s_t = i, w_t = w_k) && (31)\\
&= \sum_{s_{t+1}} p_{s_t s_{t+1}} \mathbb{E}(w_{t+1}|s_t = i, w_t = w_k, s_{t+1}) && (32)\\
&= \sum_{s_{t+1}} p_{s_t s_{t+1}} E(\Gamma_{s_{t+1}}^{-1} \mathbb{E}_{t+1} w_{t+2} | s_t = i, w_t = w_k, s_{t+1}) && (33)\\
&= \sum_{s_{t+1}} p_{s_t s_{t+1}} \Gamma_{s_{t+1}}^{-1} \mathbb{E}(\mathbb{E}_{t+1} w_{t+2} | s_t = i, w_t = w_k, s_{t+1}) && (34)\\
&= \sum_{s_{t+1}, s_{t+2}} p_{s_t s_{t+1}} p_{s_{t+1} s_{t+2}} \Gamma_{s_{t+1}}^{-1} \mathbb{E}(w_{t+2} | s_t = i, w_t = w_k, s_{t+1}, s_{t+2}) && (35)\\
& && (36)
\end{aligned}$$

And by recurrence,

$$\Gamma_i w_k = \sum_{s_{t+1}, \ldots s_{t+p}} p_{s_t s_{t+1}} \ldots p_{s_{t+p-1} s_{t+p}} \Gamma_{s_{t+1}}^{-1} \ldots \Gamma_{s_{t+p}}^{-1} \mathbb{E}(w_{t+p} | s_t = i, w_t = w_k, s_{t+1}, s_{t+2}, s_{t+p})$$

As $(w_{t+p} | s_t = i, w_t = w_k, s_{t+1}, s_{t+2}, s_{t+p})$ is in $V_{s_{t+p}}$, there exists a matrix such that:

$$\mathbb{E}(w_{t+p} | s_t = k, w_t = w_i, s_{t+1}, s_{t+2}, s_{t+p}) = V_{s_{t+p}} \phi_{s_t \ldots s_{t+p}}^{\{k\}}$$

By doing so for any column vector of V_{s_t} we obtain:

$$\Gamma_{s_t} V_{s_t} = \sum_{s_{t+1}, \ldots s_{t+p}} p_{s_t s_{t+1}} \ldots p_{s_{t+p-1} s_{t+p}} \Gamma_{s_{t+1}}^{-1} \ldots \Gamma_{s_{t+p}}^{-1} V_{s_{t+p}} \Phi_{s_t \ldots s_{t+p}}$$

where $\Phi_{s_t \ldots s_{t+p}} = [\phi_{s_t \ldots s_{t+p}}^{\{1\}} \ldots \phi_{s_t \ldots s_{t+p}}^{\{k_{s_t}\}}]$.

Now, we define η_t, the difference between w_t and the autoregressive term: $V_{s_t} \Phi_{s_{t-p} \ldots s_t} V'_{s_{t-p}} w_{t-p}$.

We notice that this difference satisfies:

$$\mathbb{E}_{t-p} \prod_{i=t-p}^{t-1} \Gamma_{s_i}^{-1} \eta_t = 0.$$

F Different determinacy conditions

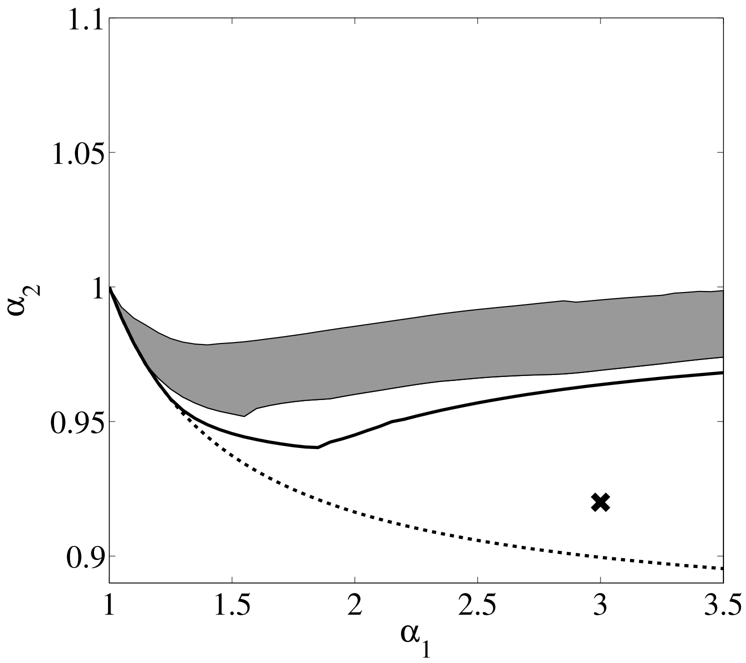

Figure 7: Indeterminacy regions and classes of equilibria: new Keynesian model with Markov-switching monetary policy

Note: We represent the regions corresponding to the different classes of equilibria, the dashed line is the limit of indeterminacy for Markovian solutions, the thick black line is the limit of the determinacy region for the solutions of Farmer et al. (2009b), the grey area is the region in which we cannot conclude. Below the grey area, sunspots equilibria depending on last six regimes at most exist. The cross corresponds to the counterexample of Farmer et al. (2010a). Probabilities are set to $p_{11} = 0.8$ *and* $p_{22} = 0.95$.

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