We introduce risk aversion into the analysis of an optimal procurement contract for a basic service and an add-on whose costs are uncertain. Risk aversion has two impacts. First, the timing of payments for the basic service can be used to relax the incentive problem: An Income Effect which calls for lower output distortions of the basic service. Second, the firm must bear some risk associated with the add-on. This Risk Effect reduces the level of the add-on. The interaction between these effects have implications not only for contract renegotiation but also for the comparison between long-term and spot contracting.

KEYWORDS: Procurement, asymmetric information, uncertainty, change orders, risk aversion.

1. INTRODUCTION

Contracts for public utilities such as delegated management in water, sanitation and transportation are long-term contracts that last for up to several decades. For these kinds of long lasting relationships, ample evidence suggests that project managers and public authorities generally expect a certain amount of ex post adaptations, regardless of how well the project was planned and executed. For instance, the National Audit Office (NAO) acknowledges that over time UK Public Finance Initiative (PFI) deals need to be modified to meet inevitable but uncertain changes in the costs and demands for public services.\(^1\) In standard project management, modifications to an initial contract come in the form of change orders.\(^2\) These change orders may be related to modifications of

\(^1\)We thank participants at the Mannheim Workshop on Procurement and Contracts, the PSE Workshop on Financing Investments in Crisis Times, the MaCCI Competition and Regulation Day, the Fundação Getulio Vargas Workshop on Regulatory Environment and Institutions in Public Procurement, the 18\(^{th}\) SFB Meeting and the Padua Workshop on How Governance Complexity and Financial Constraints Affect Public-Private Contracts, seminar participants at the University of Mannheim, University of Tilburg, Humboldt-Universität zu Berlin, University of Vienna and NHH Norwegian School of Economics, Bernard Caillaud, Vinicius Carrasco, Elisabetta Iossa, Juan José Ganuza, Susanne Goldlücke, Georgia Kosmopoulou, Andras Niedermayer and Frank Rosar for extremely valuable comments. Financial support from the Deutsche Forschungsgemeinschaft (SFB/TR-15) and the program Investissements d’Avenir of the French government (ANR-10-LABX-93-01) is gratefully acknowledged. All errors are ours.

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\(^1\)NAO (2008).

\(^2\)Meredith and Mantel (2009).
the specifications of the basic good or service itself, but they may also correspond to additional work that was not clearly specified at the time of signing the contract because the relevant cost or demand characteristics were not known. In fact, the NAO points out that this second scenario is the most frequent one; the majority of changes to UK PFIs are additions rather than direct changes to the type or level of the service provided.\(^3\)

Although much has been reported on the role that uncertain add-ons play on contractual hazards by practitioners and legal scholars, the full consequences of such uncertainty and its timely resolution on long-term procurement contracts have not received much attention in the theoretical literature. This paper aims at filling this gap. Our overall goal is to evaluate how standard lessons from the procurement and incentive regulation literatures must be amended to take into account risk sharing between the contractor and the public Agency in charge when the procurement involves future rounds of contracting, or add-ons, that go beyond the basic requirement of the service. We are particularly interested in how the agency problems that may arise at these later stages, and that can be solved only by letting the contractor bear some risk, impact on first-period incentives, efficiency and participation. Our results have implications for the robustness of long-term contracts under the threat of renegotiation as additions to basic requirements become available, but also for the value of incomplete contracting and the amount of participation in tender procedures for long-term projects whose characteristics may change over time.

**Main elements of the model.** We consider a long-term procurement contract that covers two periods. In the first period, the contractor supplies a basic good or service at a cost which is its private information. This basic service is a long-term durable service, so that its level will remain fixed over the entire length of the relationship. In the second period, some additional work is needed due to changes in demand conditions. The cost of such an add-on is ex ante unknown to both the firm and the government Agency in charge of regulating the service and, for simplicity, remains independent of the first-period cost. Contracting for the add-on takes place under symmetric but incomplete information, although later on the firm will privately learn the realization of this cost. The firm is risk averse and cares about the share of the risk associated with the add-on that it has to bear. We are interested in the impact of such second-period background uncertainty on the design of the optimal long-term procurement contract. Of course, the level of this risk is endogenous and depends on the informational environment surrounding the contract.

**Overview of the results.** To fix ideas, suppose first that the cost of the add-on is

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\(^3\)See NAO (2008). Other prominent examples of procurement where additional work was required are the “Big Dig” highway project in Boston that led to changes in more than 150 contracts (Bajari et al. (2014) and Cleland and Ireland (2008)) and the Getty Center Art Museum in Los Angeles that had to be redesigned (without much change in the scope of the project) due to site conditions that were hard to anticipate ex ante (Bajari and Tadelis (2001) and Chakravarty and MacLeod (2009)).
The Agency should provide the firm with full insurance against these shocks. Because the firm is fully insured, it does not need to manipulate information on earlier procurement stages to reduce its own risk exposure. Yet, the concavity of the firm’s utility function is nevertheless important since the marginal gains from manipulating information on the cost of the basic service are, for the second period, evaluated at the marginal utility of income at this date. Backloading payments for the basic service is thus a way for the Agency to decrease this marginal utility and thereby relax the incentive constraints: An Income Effect. Turning to output distortions, the level of the basic service is closer to the first best than had the firm been risk neutral. Incentives are high-powered as a result of this first effect.

Consider now the more realistic scenario where the firm privately learns the cost of the add-on. Contracts now have to induce truthful revelation of such information. The risk averse firm must bear some endogenous risk to satisfy the second-period incentive compatibility constraint associated with this piece of private information. Even if the cost of the basic service were common knowledge, this Risk Effect would suffice to justify downward distortions of the level of the add-on and an additional increase of second-period payments to compensate for this risk. Reducing output decreases the risk borne by the firm and saves on the associated risk premium.

Clearly, this (endogenous) background risk may also have consequences on first-period incentives when the cost of the basic service is private information of the firm. The firm may be tempted to manipulate such information so as to limit its risk exposure. The timing and scale of payments for the basic service should thus respond to these novel incentives. Indeed, we show that solving the agency problem at one date exerts a contractual externality on the agency problem at the other date.

Because second-period incentive compatibility imposes that the firm bear some endogenous risk, its marginal utility of income increases, at least when its utility function satisfies the standard assumption of prudence. As a result, second-period risk makes it more valuable for the firm to manipulate first-period costs. The Income Effect is now exacerbated. Under these circumstances, the Agency first relaxes first-period incentive compatibility by backloading payments for the basic service even more. This provides the firm with enough liquidity to face the risk from the add-on. Second, the Agency is now more concerned with rent extraction and the level of the basic service is more distorted than in the absence of a second-period agency problem.

Under a quite natural assumption of Generalized Decreasing Absolute Risk Aversion, reducing second-period risk reduces the firm’s marginal utility of income in the second period. As a result, the first-period incentive constraint exacerbates the Risk Effect. The optimal contract also exhibits lower powered incentives in the second period and the

\footnote{Which reduces to the more standard Decreasing Absolute Risk Aversion property in the limit of small risks.}
Agency implements stronger distortions of the add-on than when there is no asymmetric information in the first period.

An important assumption in our environment is that the procurement Agency can fully commit to a long-term contract. Under very general conditions, the optimal long-term contract is *not renegotiation-proof*. Because of the contractual externality that we have just stressed, the Agency commits to strongly reducing the level of the add-on to make first-period incentives cheaper. However, once first-period costs have been revealed, these extra distortions are no longer needed. The Agency would want to renegotiate the contract so as to let the firm bear more risk for the sole purpose of inducing cheaper information revelation in the second period. When preferences exhibit Constant Absolute Risk Aversion (CARA), the optimal long term contract under full commitment remains renegotiation-proof. Furthermore, with CARA preferences, the optimal long-term contract can be implemented by delaying contracting on the add-on to the second-period. Beyond the CARA case, there is generally a value of writing a long-term contract for both the basic service and the add-on; a feature that echoes real-world practices.

In an extension, we consider a more competitive environment where the Agency allocates the long-term procurement contract through an auction. With projects in which both the basic service and add-ons are of fixed size, the Agency cannot play with the size of the provision to better trade off efficiency and rent extraction. A reserve price nevertheless helps to reduce rent by affecting participation. The contractual externality between the two stages of the projects no longer matters for the intensive margins but for the extensive one. The *Income* and *Risk Effects* are still at play in this environment. The *Income Effect* continues to make it attractive to backload payments to the risky second period since this relaxes first-period incentive compatibility. This effect raises participation. The *Risk Effect* instead requires to compensate the firm via a risk premium and this reduces participation.

**Literature review.** A key feature of our analysis is the explicit modeling of the firm’s risk attitude in a procurement context. Economic theory in general and, the literature on procurement and regulation more specifically, has mostly taken the shortcut of considering firms to be risk neutral. This common albeit questionable view is based on two implicit assumptions, one being related to the firm’s relationship with its outside financiers and the other being concerned with its internal organization. The first implicit assumption is that firms have perfect access to financial markets. Leland and Pyle (1977) nevertheless challenged this view. Because firms might have private information about their risky projects, outside investors who are solicited for financing new projects will then ask for credible signals of the venture’s quality. One such credible signal is the amount of risk kept by existing owners. Firms are then imperfectly diversified in order to convey information on project quality. More recently, Martimort, Pouyet and Sand-Zantman
(2014) study a screening environment and show that an optimal debt contract entails bankruptcy costs that may lead firms to behave as if they were risk averse.\textsuperscript{5} Arve (2014) explicitly takes into account differences in firms’ financial strength and characterizes the optimal procurement contract with risk neutral, but possibly financially constrained, firms.

The second implicit assumption is that firms do not suffer from any internal agency problems. By stressing the separation between ownership and control, the Theory of Contracts\textsuperscript{6} has pointed out the existence of an important trade-off between risk and incentives. Imperfect insurance is an incentive device that forces firms to remain imperfectly diversified. A similar trade-off arises when the firm itself subcontracts with independent firms and these relationships are plagued by agency problems. For instance, Kawasaki and McMillian (1987), Asanuma and Kikutani (1992) and Yun (1999) have applied a simple principal-agent framework to study subcontracting and risk-sharing in the relationship between manufacturers and contractors in Japan and Korea and empirically found that contractors are indeed risk averse. There is thus little doubt that risk matters for firms’ behavior and that this ingredient should be part of a more complete theory of procurement and regulation. This is especially true in the procurement contexts which are of prime interest for our study. Long-term relationships under changing conditions on demand and costs that require new developments and investments beyond existing services and assets, and which might also involve substantial subcontracting. This example features the two ingredients that justify introducing risk aversion as a modeling tool.\textsuperscript{7}

The existing literature on risk aversion in adverse selection settings is sparse.\textsuperscript{8} This scarcity is probably best explained by the technical complexity of the analysis when risk aversion and incentive constraints interact. Salanié (1990) illustrates this complexity in his study of an adverse selection problem where contracting takes place \textit{ex ante}, i.e., before the risk averse agent gets private information about his cost parameter. Laffont and Rochet (1998) instead focus on \textit{ex post} participation constraints. Risk aversion may induce greater output distortions, lower informational rents, and possibly some bunching in the limit of large degrees of risk aversion.\textsuperscript{9}

\textsuperscript{5}Asplund (2002) points out that there might be other reasons that make firms act \textit{as if} they were risk averse. These reasons include factors such as liquidity constraints, costly financial distress, and nonlinear tax system. For instance the project might be so important that any loss or gain related to it has a huge impact on the firm’s overall profits and survival.

\textsuperscript{6}Holmström (1979).

\textsuperscript{7}In this paper, we focus on risk averse firms and ignore the issue of risk aversion for the principals. Including risk averse principals, maybe as a short-cut for the budgetary pressures on public authorities, is an interesting extension that lies beyond the scope of this paper. We refer to Lewis and Sappington (1995) and Martimort and Sand-Zantman (2007) for examples of optimal regulatory designs by risk averse local governments and to Waehrer, Harstad and Rothkopf (1998) for an analysis of the preferences of a risk averse auctioneer over different “standard” auction formats.

\textsuperscript{8}Laffont (1994) and Armstrong and Sappington (2007).

\textsuperscript{9}In particular, Armstrong and Sappington (2007).
Long-term procurement contracts are mostly allocated through auction mechanisms. Auctions and, more generally bargaining procedures, are competitive environments in which risk aversion has been widely documented both in experimental works (see Kagel (1995) for a survey) and econometrically (Athey and Levin (2001)). These studies suggest that the assumption of risk neutrality is not always appropriate. Maskin and Riley (1984) and Matthews (1984) take this issue seriously and offer the first theoretical analysis of risk averse bidders. Eső and White (2004) show that bidders exhibiting decreasing absolute risk aversion may shade their bids for pure “precautionary bidding” purposes. In a dynamic bargaining context, White (2008) shows that a similar precautionary behavior makes bidders more patient. Our procurement model differs in many respects from these papers but we share the idea that incentive constraints and thus optimal contracts depend on how much risk a privately informed agent will bear. In this paper and in sharp contrast with Eső and White (2004) and White (2008), this risk is endogenously determined by incentive compatibility constraints.

From a more theoretical viewpoint, our paper also contributes to the dynamic mechanism design literature whose scope is certainly broader than the study of long-term relationships in the specific procurement context under scrutiny here (see Baron and Besanko (1984), Battaglini (2005), Battaglini and Lamba (2012), Eső and Szentes (2013), Pavan, Segal and Toikka (2014) among others). This literature stresses the value of history in long-term relationships, especially when types are serially correlated and/or when (for various reasons) current projects affect future technological frontiers. An important idea highlighted by this strand of literature is that commitment to future actions and payments help to screen current private information and reduce information rents. As a by-product, long-term contracts are generally not immune to renegotiation. When turning to applications, this literature mostly considers risk neutral agents and has thus overlooked the impact that their current reporting strategies might have on future marginal utility of income. When risk aversion is modelled it is often in contexts where the sole purpose of the contract is to provide insurance as in Thomas and Worall (1988) or to redistribute income as in Stantcheva (2014) among others. Yet, risk aversion is especially important in regulation and procurement when some production stages are long-lasting and firms may be reluctant to enter into these extra rounds of contracting. In these cases, the principal might find it attractive to distort future contracts to reduce the agent’s marginal utility.

10There is a related strand of the auction literature that studies bidding behavior under uncertainty. Calvares et al. (2004) and Burguet et al. (2012) study the effect of cost uncertainty on firms’ bidding behavior. Under limited liability, financially weak firms tend to bid more aggressively.

11Faure-Grimaud and Martimort (2003), Faure-Grimaud and Martimort (2007) and Strausz (2011) study a very specific risk borne by regulated firms, the political risk coming from fluctuations in the preferences of elected political principals in charge of designing regulatory policies.
of income and improve rent extraction. We show that this effect creates a new value of commitment that holds even when types are independently drawn over time. With risk neutrality, and more generally with CARA preferences, this value of commitment disappears. In this case, the optimal contract is renegotiation-proof and, surprisingly, can even be implemented through a sequence of spot contracts.

**Organization.** The model is presented in Section 2. The set of incentive-feasible allocations in our dynamic context is described in Section 3. Section 4 provides some important preliminary results. These results set the stage for the more complete analysis of optimal contracts that is undertaken in Section 5. Section 6 discusses whether the optimal contract is sequentially optimal and provides condition under which it can be implemented through a sequence of spot contracts, one for the basic and long-lasting service and another one for the add-on. Section 7 moves to a more competitive environment and shows how tender procedures must be modified to account for risk on subsequent add-ons. Section 8 concludes and discusses possible extensions. Proofs are relegated to the Appendix.

### 2. THE MODEL

We consider the following model of procurement: A government Agency (also referred to as the principal) contracts with a firm for the provision of a public service. The costs and benefits from this service accumulate over two periods. To be more precise, this service includes a basic good or service supplied in quantity $q$ in each period but also an additional service (add-on) which is provided in quantity $x$ but needs only to be delivered in the second period.\footnote{To make the model more realistic, we could allow for the add-on to only occur with some probability. This extra layer of uncertainty does not qualitatively change our results and is for simplicity ignored.} The idea is that the add-on represents long-term contractible variables associated with the service (a refined specification, some incremental services for new segments of demand, further stages of development of a prototype in defense procurement, etc.). The exact specifications required for the add-on are not completely known by contracting parties at the time of contracting. This uncertainty around the add-on puts the firm’s returns at risk.\footnote{We focus on the case where the basic service and the add-on are bundled tasks. There are several reasons for this. First of all, in many long-term contracts such as Public-Private Partnerships, the bidding consortia have a rather ephemeral life and in later stages of the contract only the winning consortium is still available for providing the add-on. Another reason could be that the competitors have too high costs because of project specific developments during the first stage or simply because it is not physically possible to find two different providers for the basic service and the add-on.} In the sequel, we will be particularly interested in the impact of this risk on contract design.

**Technology and Preferences.** The basic service generates a gross surplus $S(q)$ in each period. Motivated by the idea that this basic service is the choice of a capacity,
the fixed size of a network, or the basic version of a long-term durable good, we assume that the quantity \( q \) is chosen once and for all, and does not vary over time. The firm provides this service at a constant marginal cost \( \theta \). The function \( S(\cdot) \) is increasing and strictly concave \((S' > 0, S'' < 0)\) with \( S(0) = 0 \) and \( S'(0) = +\infty \).\(^{14}\) The gross surplus for consuming \( x \) units of the add-on is \( V(x) \) where \( V(\cdot) \) is increasing, strictly concave \((V' > 0, V'' < 0)\) with \( V(0) = 0 \) and \( V'(0) = +\infty \).

Payments for the basic service are denoted by \( t(\theta) \) in period 1 and \( t(\theta)+y(\theta) \) in period 2. \( t(\theta) \) can thus be viewed as a fixed per-period payment for the basic service \( q(\theta) \), while the payment \( y(\theta) \) represents an extra premium (or a penalty) for the second period. Below, we shall demonstrate how the payments for the basic service need not be stationary even though the quantity \( q(\theta) \) remains constant; the extra payment \( y(\theta) \) captures such non-stationarity. We also denote by \( p(\theta, \beta) \) the second-period payment for the \( x(\theta, \beta) \) units of the add-on which are provided in the second period.

Denoting by \( 1-\delta \) and \( \delta \) the relative lengths of the first and second period respectively, and normalizing intertemporal payoffs accordingly, the principal’s expected gains from dealing with a firm of type \( \theta \) can thus be written as:

\[
S(q(\theta)) - t(\theta) - \delta y(\theta) + \delta E_\beta \left( V(x(\theta, \beta)) - p(\theta, \beta) \right).
\]

Denoting by \( u(\theta) = t(\theta) - \theta q(\theta) \) the firm’s fixed per-period return on the basic service and by \( \mathcal{V}(\theta, \beta) = p(\theta, \beta) - \beta x(\theta, \beta) \) its second-period return on the add-on, the principal’s intertemporal payoff becomes:

\[
S(q(\theta)) - \theta q(\theta) - u(\theta) - \delta y(\theta) + \delta E_\beta \left( V(x(\theta, \beta)) - \beta x(\theta, \beta) - \mathcal{V}(\theta, \beta) \right).
\]

This expression already highlights the trade-off between efficiency and rent extraction that characterizes optimal contracting under informational asymmetries. The principal cares about the social value of the project but at the same time would like to minimize the share of this surplus that accrues to the firm.

We are interested in the consequences for contract design of introducing uncertainty on the cost of the add-on. Moving away from more standard models of procurement which postulate that the firm is risk neutral, we thus assume that the firm is risk averse and evaluates the second-period risky returns using a Bernoulli utility function \( v(\cdot) \) which is increasing and concave, \((v' > 0, v'' \leq 0)\) with the normalizations \( v(0) = 0 \) and \( v'(0) = 1 \) and an inverse function \( h = v^{-1} \). Risk aversion should be viewed as a proxy for financial constraints that may for instance limit the firm’s access to the capital market when it

\(^{14}\)These latter two conditions ensure that “shutting-down” production even with the least efficient service provider is never optimal. This simplifies our modeling without loss of economic insight. It allows us to concentrate on the impact of second-period risk on first-period incentives, an impact at the intrinsic margin. Section 7 relaxes this assumption in the specific case of a fixed-quantity service. There, our focus will be on the impact of second-period risk at the extrinsic margin.
wants to finance the investments needed to produce this add-on. The normalization $v'(0) = 1$ implies that the marginal utility at zero income is the same in both periods. This implies that, under complete information, the principal should not transfer payments across periods. Any non-stationarity in payments thus follows from asymmetric information.

With these remarks in mind and our previous notations in hand, the firm’s intertemporal payoff becomes:

$$(1 - \delta)u(\theta) + \delta E_\beta \left( v(u(\theta) + y(\theta) + V(\theta, \beta)) \right).$$

**Information.** At the time of contracting the firm has private information on the cost parameter $\theta$. This variable is drawn from a (common knowledge) cumulative distribution $F(\cdot)$ with an atomless and everywhere positive density $f(\theta) = F'(\theta)$ whose support is $\Theta = [\theta, \overline{\theta}]$. Following a standard assumption in the screening literature, the monotone hazard rate property holds:

**Assumption 1** Monotone hazard rate property:

$$\frac{d}{d\theta} \left( \frac{F(\theta)}{f(\theta)} \right) \geq 0 \text{ for all } \theta \in \Theta.$$

The social value of the add-on is uncertain. Ex ante, there is symmetric but incomplete information on the cost parameter $\beta$. However, before producing the add-on, the firm learns the value of $\beta$. To maintain a tractable analysis, we consider the case where $\beta$ is drawn from a common knowledge distribution on the discrete support $\mathcal{B} = \{\beta, \overline{\beta}\}$ (where $\Delta\beta = \overline{\beta} - \beta > 0$) with respective probabilities $\nu$ and $1 - \nu$, where $\nu \in (0, 1)$.

First- and second-period cost parameters are independently drawn and, more generally, there is no technological linkage between periods. When deriving the optimal dynamic contract in this environment, any departure from the optimal spot contracts that would be optimal with a risk neutral firm comes from the fact that the firm is risk averse. Furthermore, our analysis unveils conditions under which a contractual externality between the first and the second period arises.

**Incentive mechanisms.** The principal commits to a long-term contract that regu-

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15See Martimort, Pouyet and Sand-Zantman (2014) for a model along these lines. In general, the overall environment in which a firm operates might affect its attitude towards risk. Here we abstract from these factors and assume that risk attitudes are only affected by the return and risk of the given period. We also assume that the firm is risk neutral with respect to first-period payoffs. One possible justification is that all background uncertainty that could affect first-period payoffs has already been costlessly diversified (maybe by calling outside investors who can be attracted by the stability of long-term returns that comes with the long-term part of the project).

16Bagnoli and Bergstrom (2005).
lates the basic service over both periods and the add-on in the second period. There are several justifications for this assumption on commitment. First, in some contexts to which our model applies, for instance PPP contracts, public officials commit over periods up to thirty years but include adaptation clauses to react to changes in the environment. Changes that are outside these clauses might also be limited by law. Second, the desire to build a reputation forces the principal to stick to her initial commitment even if renegotiation might become attractive as time unfolds. Third, focusing on the full commitment scenario characterizes an upper bound on what long-term contracting can achieve and allows us to unveil intertemporal links across periods that arise under asymmetric information. The issue of renegotiation is nevertheless discussed in Section 6.

From the (dynamic version of the) Revelation Principle (Baron and Besanko (1984), Myerson (1986)), there is no loss of generality in restricting the analysis to incentive-compatible direct revelation mechanisms. These mechanisms stipulate payments and outputs in each period as a function of the firm’s report of its current type and, possibly, the past history of reports. Mechanisms are thus of the form:

\[ C = \{ t(\hat{\theta}), y(\hat{\theta}), q(\hat{\theta}), p(\hat{\theta}, \hat{\beta}), x(\hat{\theta}, \hat{\beta}) \} \quad \hat{\theta} \in \Theta, \hat{\beta} \in \mathcal{B}, \]

where \( \hat{\theta} \in \Theta \) and \( \hat{\beta} \in \mathcal{B} \) are the firm’s announcements of its cost parameters in the first and the second period respectively.\(^{18}\)

TIMING. The contracting game unfolds as follows:

1. The firm privately learns its cost parameter \( \theta \) for the basic service.
2. The principal offers a long-term contract \( C \). If the firm refuses, parties get their reservation payoffs which are, without loss of generality, normalized to zero.
3. If the firm accepts, it reports its first-period cost. This report \( \hat{\theta} \) determines both the level of the basic service \( q(\hat{\theta}) \) and the corresponding payments \( t(\hat{\theta}) \) and \( t(\hat{\theta}) + y(\hat{\theta}) \).
4. The firm learns the value of the second-period cost \( \beta \). The firm then reports \( \hat{\beta} \) and provides the corresponding level of the add-on, \( x(\hat{\theta}, \hat{\beta}) \), at a price \( p(\hat{\theta}, \hat{\beta}) \).

MORE ASSUMPTIONS ON PREFERENCES. In the remainder of the paper, detailed (albeit standard) properties of \( v(\cdot) \) play a key role in the characterization of the optimal contract. Beyond the familiar assumption of a Bernoulli utility function, we also assume that \( v(\cdot) \) satisfies:

\(^{17}\)In the European Union, add-ons that are outside the scope of the initial contract might be seen as a violation of Art. 101 of the Treaty on the Functioning of the European Union.

\(^{18}\)For technical reasons, we will assume that all feasible \( q \) and \( x \) are respectively bounded above by some levels \( Q \) and \( X \) (both large enough).
Assumption 2

∀z, \ v''(z) \geq 0 \text{ and } v^{(4)}(z) \leq 0.

The first condition implies that the firm exhibits some prudent behavior while the second means that its temperance is non-negative.\textsuperscript{19} Although supported by experimental evidence, these requirements are less common and we therefore point out whenever they are required for our results to hold.\textsuperscript{20}

To express the firm’s intertemporal payoff in a more compact form, a first useful step is to evaluate general properties of payoff functions once one adds some (zero-mean) background risk to a fixed-return project that yields a profit z. This background risk yields \((1 - \nu)\varepsilon\) with probability \(\nu\) and \(-\nu\varepsilon\) with probability \(1 - \nu\). Consider thus a utility function \(w(z, \varepsilon)\) defined over wealth and risk levels \((z, \varepsilon)\) as:

\[
w(z, \varepsilon) \equiv \nu v(z + (1 - \nu)\varepsilon) + (1 - \nu)v(z - \nu\varepsilon).
\]

The function \(w(\cdot)\) inherits some important properties from \(v(\cdot)\). First, \(w(\cdot)\) is increasing and concave in \(z\) while it remains decreasing in \(\varepsilon\).\textsuperscript{21} The firm’s marginal utility of income is positive for a given level of second-period risk but decreases in \(z\). Following Assumption 2, the firm exhibits a prudent behavior and its second-period marginal utility of income increases in more uncertain environments:

\[
w_{z\varepsilon}(z, \varepsilon) = \nu(1 - \nu)(v''(z + (1 - \nu)\varepsilon) - v''(z - \nu\varepsilon)) \geq 0.
\]

Intuitively, the firm is better able to cope with background risk when the initial fixed-return project yields a higher return. This captures the idea that agency costs from accessing financial markets may be lower for firms that can pledge more income from the basic service.\textsuperscript{22}

For further references, let \(\varphi(\zeta, \varepsilon)\) be the wealth level that guarantees \(\zeta\) utils to the firm when the risk level is \(\varepsilon\):

\[
\zeta = w(\varphi(\zeta, \varepsilon), \varepsilon).
\]

\textsuperscript{19}The concept of prudence goes back to Leland (1968) and Sandmo (1970). The notion of temperance (or outer risk aversion) was developed in Kimball (1992) and Menezes and Wang (2005).

\textsuperscript{20}Experimental evidence (Deck and Schlesinger (2010), Maier and Rüger (2011), Deck and Schlesinger (2014), Ebert and Wiesen (2014) and Noussair et al (2014)) is mostly in line with these assumptions. They all provide results for prudence (and so do Tarazona-Gomez (2004) and Ebert and Wiesen (2011)). All but one confirm our assumption of temperance.

\textsuperscript{21}Indeed, we have: \(w_z(z, \varepsilon) = \nu v'(z + (1 - \nu)\varepsilon) + (1 - \nu)v'(z - \nu\varepsilon) > 0\), \(w_{zz}(z, \varepsilon) = \nu v''(z + (1 - \nu)\varepsilon) + (1 - \nu)v''(z - \nu\varepsilon) \leq 0\) and \(w_{z\varepsilon}(z, \varepsilon) = \nu(1 - \nu)(v'(z + (1 - \nu)\varepsilon) - v'(z - \nu\varepsilon)) \leq 0\) (since \(v'' \leq 0\)).

\textsuperscript{22}Similarly, \(v^{(4)} \leq 0\) also implies: \(w_{z\varepsilon\varepsilon}(z, \varepsilon) = \nu(1 - \nu)(v'''(z + (1 - \nu)\varepsilon) - v'''(z - \nu\varepsilon)) \leq 0\).
The function $\varphi(\cdot)$ is increasing in $\zeta$ and $\varepsilon$ with $\varphi(\zeta,0) = h(\zeta)$. Let us denote by $\zeta^*(\varepsilon)$ the solution to:

$$w_z(\varphi(\zeta^*(\varepsilon),\varepsilon),\varepsilon) = 1.$$ 

By definition, $\zeta^*(\varepsilon)$ is the utility level such that the marginal utility of income remains equal to one (its value in the first period). By differentiating this definition, we get:

$$(2.1) \dot{\zeta}^*(\varepsilon) = -\frac{w_z(\varphi(\zeta^*(\varepsilon),\varepsilon),\varepsilon)}{w_{zz}(\varphi(\zeta^*(\varepsilon),\varepsilon),\varepsilon)} H(\varphi(\zeta^*(\varepsilon),\varepsilon),\varepsilon),$$

where the function $H(\cdot)$ is defined, for all pairs $(z,\varepsilon)$, as:

$$H(z,\varepsilon) = w_{z\varepsilon}(z,\varepsilon) - \frac{w_{zz}(z,\varepsilon)w_z(\varepsilon)}{w_z(z,\varepsilon)}.$$ 

We restrict attention to preferences such that $\dot{\zeta}^*(\varepsilon) \geq 0$. If second-period returns become riskier, the second-period utility that yields a marginal utility of income equal to one should increase to keep the marginal utility of income constant. To ensure that this property is satisfied, we impose the following assumption.

**Assumption 3** Generalized decreasing absolute risk aversion (GDARA):

$$H(z,\varepsilon) \geq 0 \text{ for all } (z,\varepsilon).$$

Some intuition for Assumption 3 can be given by observing that, in the limit of small risks, the following approximation holds:

$$H(z,\varepsilon) \approx \nu(1 - \nu)\varepsilon \left( v''(z) - \frac{(v''(z))^2}{v'(z)} \right) = -\nu(1 - \nu)\varepsilon v'(z) \frac{d}{dz} \left( -\frac{v''(z)}{v'(z)} \right).$$

In the limit of small risks, $\zeta^*(\varepsilon)$ being non-decreasing simply follows from decreasing absolute risk aversion, a standard assumption in the risk literature. More generally, $H(\cdot)$ is the sum of two terms with opposite signs. The first one, $w_{z\varepsilon}(z,\varepsilon)$, is non-negative when the firm exhibits prudence. It simply means that more background risk increases the marginal utility of income, making it more valuable to transfer revenues towards the second period: an *Income Effect*. The second term, $-\frac{w_{zz}(z,\varepsilon)w_z(\varepsilon)}{w_z(z,\varepsilon)}$, is negative. It captures the idea that more background risk decreases second-period utility, making such transfers less attractive: a *Risk Effect*. Assumption 3 ensures that the *Income Effect* 23Indeed, we have: $\varphi(\zeta,\varepsilon) = \frac{1}{w_z(\varphi(\zeta,\varepsilon),\varepsilon)} \geq 0$, and $\varphi(\zeta,\varepsilon) = -\frac{w_z(\varphi(\zeta,\varepsilon),\varepsilon)}{w_z(\varphi(\zeta,\varepsilon),\varepsilon)} \geq 0$. 24The literature on risk aversion provides evidence for *decreasing absolute risk aversion* (Holt and Laury (2002)). When interpreting risk aversion as a proxy for a costly access to financial markets, it seems reasonable that wealthy firms face less of these constraints simply because they can pledge more income against future borrowing. Small or specialized firms instead face tighter financial constraints.
dominates. In the sequel, we point out specifically when results rely on Assumption 3.

Example (CARA preferences). Suppose that \( v(\cdot) \) is a CARA utility function. Given our normalizations this means that \( v(z) = 1 - \exp(-rz) \). We can write \( w(z, \varepsilon) = 1 - \exp(-r(1 - \nu)\varepsilon) + (1 - \nu)\exp(r\nu\varepsilon) \). Finally, we have \( H(z, \varepsilon) \equiv 0 \) for all \((z, \varepsilon)\) so that the Income Effect and the Risk Effect cancel out.

3. INCENTIVE FEASIBLE ALLOCATIONS

We now describe the set of incentive feasible allocations. Since the Revelation Principle applies in this dynamic context with full commitment, we can define the firm’s intertemporal payoff as:

\[
U(\theta) = \max_{\hat{\theta} \in \Theta, \hat{\beta} \in B} (1 - \delta)(t(\hat{\theta}) - \theta q(\hat{\theta}))) + \delta E_{\beta} \left( v \left( t(\hat{\theta}) - \theta q(\hat{\theta}) + y(\hat{\theta}) + p(\hat{\theta}, \hat{\beta}) - \beta x(\hat{\theta}, \hat{\beta}) \right) \right),
\]

where the maximum above is achieved for truthful strategies.

Second-period incentive compatibility. The requirement of incentive compatibility can be applied recursively.\(^{25}\) To get a compact characterization of incentive compatibility, define the firm’s second-period profit \( V(\theta, \beta) \) as:

\[
V(\theta, \beta) = \max_{\hat{\beta} \in B} p(\theta, \hat{\beta}) - \beta x(\theta, \hat{\beta}), \quad \forall \theta \in \Theta.
\]

Second-period incentive compatibility requires that a firm facing a low cost of producing the add-on prefers the requested option:

\[
V(\theta, \beta) \geq V(\theta, \beta) + \Delta \beta x(\theta, \beta), \quad \forall \theta \in \Theta.\(^{26}\)
\]

This condition tells us that, when the firm has \textit{ex post} private information on the cost of producing the add-on, the distribution of second-period profits must be risky in order to satisfy incentive compatibility.\(^{27}\) This source of \textit{endogenous risk} plays a significant role in the sequel.

\(^{25}\)See Baron and Besanko (1984), Battaglini (2005) and Pavan, Segal and Toikka (2014).

\(^{26}\)In this two-type model, it is routine to check that the second-period incentive constraint of a firm facing a high cost of producing the add-on, namely \( u(\hat{\theta}, \beta) \geq u(\hat{\theta}, \beta) - \Delta \beta x(\hat{\theta}, \beta) \), is automatically satisfied when (3.3) is binding and \( x(\theta, \beta) \geq x(\hat{\theta}, \beta) \) as required by the standard monotonicity condition. This monotonicity condition holds for the optimal contract that will be derived below. Therefore, we simplify the presentation by focusing only on the low-cost incentive constraint (3.3).

\(^{27}\)This is true as long as the add-on is produced in positive quantities, \( x(\theta, \beta) > 0 \). This non-negativity requirement is satisfied by the optimal levels when the Inada condition holds.
Furthermore, there is no loss of generality in assuming that the firm makes zero expected profits on the add-on since any non-zero expected payment could be incorporated into the payment for the basic service itself:

\[ E_{\beta} (\mathcal{V}(\theta, \beta)) = 0. \]

From this remark, second-period profits can be expressed as:

\[ \mathcal{V}(\theta, \beta) = (1 - \nu)\varepsilon(\theta) \text{ and } \mathcal{V}(\theta, \beta) = -\nu\varepsilon(\theta), \]

for some function \( \varepsilon(\theta) \). It is thus equivalent to view a direct mechanism \( \mathcal{C} \) as a menu \( \{ t(\hat{\theta}), y(\hat{\theta}), q(\hat{\theta}), \varepsilon(\hat{\theta}), x(\hat{\theta}, \beta) \}_{\theta \in \Theta} \) where \( \varepsilon(\hat{\theta}) \) is the amount of risk borne by the firm in the second period. Because the mechanism is truthful, the amount of risk \( \varepsilon(\theta) \) is endogenously determined by second-period incentive compatibility. The corresponding constraint (3.3) amounts to:

\[ \varepsilon(\theta) \geq \Delta\beta x(\theta, \beta), \quad \forall \theta \in \Theta. \]

In the remainder of the paper, we are particularly interested in the consequences of such second-period endogenous risk on first-period incentives.

**First-period incentive compatibility.** It is of course equivalent to characterize incentive compatibility conditions by means of a direct and truthful mechanism \( \mathcal{C} \) or through the allocation \( (U(\theta), q(\theta), u(\theta), y(\theta), \varepsilon(\theta), x(\theta, \beta)) \) induced by that mechanism, we adopt this dual approach and rewrite the firm’s intertemporal payoff as:

\[ U(\theta) = \max_{\hat{\theta} \in \Theta} (1 - \delta)(t(\hat{\theta}) - \theta q(\hat{\theta})) + \delta w(t(\hat{\theta}) - \theta q(\hat{\theta}) + y(\hat{\theta}), \varepsilon(\hat{\theta})). \]

Recalling that \( u(\theta) = t(\theta) - \theta q(\theta) \), the following lemma provides necessary and sufficient conditions satisfied by any incentive compatible allocation.

**Lemma 1.** Necessary condition. \( U(\theta) \) is absolutely continuous in \( \theta \) and thus almost everywhere differentiable with at any point of differentiability:

\[ \hat{U}(\theta) = -q(\theta) (1 - \delta + \delta w_{z}(u(\theta) + y(\theta), \varepsilon(\theta))). \]

\(^{28}\)Suppose on the contrary that the second-period profit on the add-on has a non-zero mean \( E_{\beta} (p(\theta, \beta) - \beta x(\theta, \beta)) \neq 0 \) for some payments \( (t(\theta), y(\theta), p(\theta, \beta)) \). Keeping all outputs unchanged, we can redefine a set of new payments \( (\tilde{t}(\theta), \tilde{y}(\theta), \tilde{p}(\theta, \beta)) \) such that \( \tilde{p}(\theta, \beta) = p(\theta, \beta) - E_{\beta} (p(\theta, \beta) - \beta x(\theta, \beta)) \), \( \tilde{t}(\theta) = t(\theta) + E_{\beta} (p(\theta, \beta) - \beta x(\theta, \beta)) \) and \( \tilde{y}(\theta) = y(\theta, \beta) - E_{\beta} (p(\theta, \beta) - \beta x(\theta, \beta)) \). The per-period total payment remains unchanged since \( \tilde{t}(\theta) + \tilde{y}(\theta) = t(\theta, \beta) + y(\theta) \) and \( t(\theta) + \tilde{p}(\theta, \beta) = t(\theta) + p(\theta, \beta) \) while, by construction, second-period profits have zero mean: \( E_{\beta} (\tilde{p}(\theta, \beta) - \beta x(\theta, \beta)) = 0 \).
Sufficient condition. A rent profile \( U(\theta) \) which is absolutely continuous in \( \theta \) and satisfies (3.8) corresponds to an incentive compatible allocation if it is convex.

A firm which has private information on its cost parameter \( \theta \) must receive an information rent to reveal its type. By reporting a higher type, \( \theta + d\theta \), this firm can produce the same amount as the slightly less efficient type \( \theta + d\theta \) but does so at a lower marginal cost, thereby saving \( q(\theta + d\theta) d\theta \approx q(\theta) d\theta \) in each period. Of course, these gains are weighted by the marginal utility of income in each period. This explains the term \( w_z(u(\theta) + y(\theta), \varepsilon(\theta)) \) that appears on the right-hand side of (3.8). To prevent such mimicking behavior, the type \( \theta \) firm must receive an extra informational rent worth \( U(\theta) - U(\theta + d\theta) \approx -\dot{U}(\theta) d\theta \approx q(\theta) (1 - \delta + \delta w_z(u(\theta) + y(\theta), \varepsilon(\theta))) d\theta. \)

It is clear from (3.8) that the convexity of \( U(\cdot) \) is guaranteed when the future does not matter much (i.e., \( \delta \) small enough) and \( q(\theta) \) itself is decreasing. As we will see in the characterization of the optimal outputs under various scenarios, these conditions are always satisfied when Assumption 1 holds. This proviso on \( \delta \) validates our approach which consists of solving the relaxed problem obtained by omitting the convexity condition. To simplify the exposition and rule out the uninteresting technicalities that would arise with bunching, we will make this proviso implicit in all our statements below.

Observe also that (3.8) implies that \( U(\theta) \) is non-increasing and that the participation constraint always holds if it holds for the highest cost parameter, \( \bar{\theta} \):

\[
(3.9) \quad U(\bar{\theta}) \geq 0.
\]

For future reference, we express the second-period profit \( u(\theta) + y(\theta) \) in terms of \( U(\theta) \) as:

\[
(3.10) \quad u(\theta) + y(\theta) = \varphi \left( \frac{U(\theta) - (1 - \delta)u(\theta)}{\delta}, \varepsilon(\theta) \right).
\]

This allows us to get the more compact expression of incentive compatibility:

\[
(3.11) \quad \dot{U}(\theta) = -q(\theta) \left[ 1 - \delta + \delta w_z \left( \varphi \left( \frac{U(\theta) - (1 - \delta)u(\theta)}{\delta}, \varepsilon(\theta) \right), \varepsilon(\theta) \right) \right].
\]

This expression shows that, when the firm’s output is reduced, the incentive-compatibility constraint is relaxed and less rent is left to the firm. This is a familiar distortion in screening environments. However, here the firm’s rent is also reduced if the second-period marginal utility of income decreases. This is a specific feature of our environment which comes from the concavity of the firm’s utility function in the second period. Playing both on the second-period profit \( u(\theta) + y(\theta) \) and on the risk \( \varepsilon(\theta) \) helps the principal extract more of the firm’s rent.
4. PRELIMINARY RESULTS

This section stresses different effects that are at play when a risk averse firm faces second-period risky returns. To build intuition and before undertaking the full-fledged analysis, we decompose the analysis into several elementary steps that can be studied independently of each other.

First, we analyze the benchmark case where all cost realizations are common knowledge. Second, we turn to the case of a risk neutral firm and demonstrate that making the firm residual claimant for the second-period social value of the add-on solves the agency problem. Finally, we move to the case of risk aversion and distinguish between two elementary effects stemming from the concavity of the firm’s utility function. The first one, called the Income Effect, arises even under symmetric information on the cost of the add-on. The second effect, the Risk Effect, follows from the second-period agency problem and occurs even under complete information on the cost of the basic service.

4.1. Complete Information

Suppose that $\theta$ and $\beta$ are both common knowledge, but recall that at the time of contracting the second-period cost $\beta$ is not yet realized. The solution to the contracting problem is obvious. First, it entails perfect insurance against second-period cost realizations. Second, the firm must have the same marginal utility of income in both periods so as to smooth the marginal cost of public subsidies over time. Lastly, production should be efficient in both periods.

To prepare for the rest of the analysis, we will nevertheless be more explicit about the nature of the principal’s problem in this simple informational environment. First, equipped with our more compact notations, we rewrite the firm’s intertemporal payoff and participation constraint as:

$$U(\theta) = (1 - \delta)u(\theta) + \delta w(u(\theta) + y(\theta), \varepsilon(\theta)) \geq 0, \quad \forall \theta \in \Theta. \tag{4.1}$$

Taking into account that second-period expected profits from the add-on are zero, the principal’s expected payoff becomes:

$$S(q(\theta)) - \theta q(\theta) - (1 - \delta)u(\theta) - \delta \varphi \left( \frac{U(\theta) - (1 - \delta)u(\theta)}{\delta}, \varepsilon(\theta) \right) + \delta E_{\beta} \left( V(x(\theta, \beta)) - \beta x(\theta, \beta) \right).$$

Maximizing this expression subject to the participation constraint (4.1) unveils several important facts. First, the principal reduces payments so that the firm breaks even (i.e., (4.1) is always binding which follows from $\varphi \geq 0$):

$$U^*(\theta) = 0, \quad \forall \theta \in \Theta. \tag{4.2}$$
Second, because it is costly for the firm to bear risk ($\varphi_\varepsilon \geq 0$), it should be fully insured against fluctuations in second-period costs:

$$
\varepsilon^*(\theta) = 0, \quad \forall \theta \in \Theta.
$$

Third, moving to the intertemporal distribution of profits and differentiating the principal’s objective with respect to $u(\theta)$ yield the following first-order condition:

$$
\varphi (\delta u^*(\theta), 0) = 1 \iff v' \left( h \left( \frac{1-\delta}{\delta} u^*(\theta) \right) \right) = 1, \quad \forall \theta \in \Theta.
$$

In other words, the firm’s marginal utility of income remains constant over time. Because $v'(0) = 1$ and $h(0) = 0$, this condition simply means that the firm should make zero profits in each period:

$$
u^*(\theta) = y^*(\theta) = 0, \quad \forall \theta \in \Theta.
$$

As a result, the optimal contract under complete information entails a stationary payment for the basic service. Second-period payments only provide insurance against fluctuations in the cost of the add-on. Finally, the basic service and the add-on are both produced at their respective first-best levels $q^*(\theta)$ and $x^*(\beta)$:

$$
S'(q^*(\theta)) = \theta, \quad \forall \theta \in \Theta, \quad \text{and} \quad V'(x^*(\beta)) = \beta, \quad \forall \beta \in \mathcal{B}.
$$

### 4.2. The Simple Case of Risk Neutrality

Let us now move to the case of asymmetric information, but assume that the firm is risk neutral. This case best illustrates the situation where the firm is a well-diversified venture that has perfect access to financial markets. Under these circumstances, the principal can easily structure incentives to induce efficient production of the add-on and extract all profits from this activity. The only contracting friction then comes from the firm’s private information on the cost of the basic service.

To see this more clearly, consider the following simple mechanism (actually a nonlinear price) which is designed to regulate the production of the add-on by making the firm residual claimant for its social value:

$$
p(x) = V(x) - V^*, \quad \forall x,
$$

where $V^* = E_\beta(V(x^*(\beta)) - \beta x^*(\beta))$ is the expected (first-best) surplus generated by the add-on. The payment schedule $p(x)$ is independent of the first-period announcement of $\theta$. Facing this nonlinear contract, the firm chooses to provide the add-on at the first-best level and the principal reaps the entire expected surplus from the add-on.
With this second-period continuation, the characterization of first-period incentive compatibility in Lemma 1 applies. The profile $U(\theta)$ is obtained as a maximum of linear functions of $\theta$, and, as such, is convex, absolutely continuous and thus almost everywhere differentiable with at any point of differentiability:

\begin{equation}
\dot{U}(\theta) = -q(\theta).
\end{equation}

The requirement of convexity then amounts to the familiar monotonicity condition:

\begin{equation}
q(\theta) \text{ non-increasing.}
\end{equation}

We summarize the main features of the optimal contract in the next proposition.\footnote{The proof is omitted. Under risk neutrality, the analysis reduces to the standard framework à la Baron and Myerson (1982).}

**Proposition 1** Assume that the firm is risk neutral.

- The add-on is produced at the first-best level $x^*(\beta)$ for all $\beta \in B$.
- The level of the basic service is distorted below the first-best level, $q_{bm}(\theta) \leq q^*(\theta)$ for all $\theta \in \Theta$. This output is given by the standard Baron and Myerson (1982) formula:

\begin{equation}
S'(q_{bm}(\theta)) = \theta + \frac{F(\theta)}{f(\theta)}, \quad \forall \theta \in \Theta.
\end{equation}

When Assumption 1 holds, $q_{bm}(\theta)$ is non-increasing in $\theta$.
- The firm’s information rent is always non-negative:

\begin{equation}
U_{bm}(\theta) = \int_{\Theta} q_{bm}(x) \, dx, \quad \forall \theta \in \Theta.
\end{equation}

The pricing scheme in (4.6) allows the principal to make the risk neutral firm residual claimant for the social value of the add-on. An efficient production decision for the add-on follows directly from this. Provided that \textit{ex post} the firm bears all the risk, the solution to the first-period screening problem reduces to the standard Baron and Myerson (1982) allocation. Because at the time of contracting the firm is privately informed about $\theta$, it must receive an information rent to reveal this parameter. The optimal contract exhibits the usual trade-off between information rent and efficiency. Output is distorted downward below the first-best level for all but the most efficient type. Second-period risk has no impact on first-period incentives.
4.3. Income Effect

Suppose now that the principal and the firm share *ex post* knowledge about \( \beta \). Contracting still takes place *ex ante*, i.e. before the realization of this parameter, and the firm remains privately informed about \( \theta \) at the time of contracting. Second-period incentive constraints do not matter in this environment and letting the firm bear some risk is costly for two reasons. First, the firm must receive a risk premium to be willing to participate. Second, when Assumption 2 holds, the second-period marginal utility of income increases with risk (\( w_{2\varepsilon} \geq 0 \)). Increasing second-period risk would thus harden first-period incentive compatibility. Together, these two reasons lead the principal to fully insure the firm against second-period uncertainty:

\[
\varepsilon^i(\theta) = 0, \quad \forall \theta \in \Theta.
\]

**Profit levels.** Because the firm’s utility function is concave, the principal wants to transfer payments between periods to reduce the firm’s second-period marginal utility of income and reduce the social cost of its information rent: an *Income Effect*.

**Proposition 2** Assume that \( \beta \) is common knowledge in the second period.

- The firm’s marginal utility of income decreases over time.\(^{30}\)

\[
(4.10) \quad v'(u^i(\theta)+y^i(\theta)) = 1 + q^i(\theta) \frac{F(\theta)}{f(\theta)} v''(u^i(\theta)+y^i(\theta)) \leq 1, \quad \forall \theta \in \Theta \quad (\text{with equality for } \theta \text{ only}).
\]

- The firm’s information rent is always non-negative:

\[
(4.11) \quad U^i(\theta) \geq U^i(\overline{\theta}) = 0, \quad \forall \theta \in \Theta.
\]

Starting from the full information scenario and increasing second-period profits from the basic service above zero, i.e. setting \( u^i(\theta) + y^i(\theta) \geq 0 \) as implied by (4.10), decreases the firm’s second-period marginal utility of income. The firm finds it less attractive to lie about its first-period cost because the second-period benefits of doing so are lower.

Payments for the basic service are no longer stationary under asymmetric information. To see why, let us first put together (4.10) and (4.11) to obtain that, for \( \theta \) close enough to \( \overline{\theta} \), the following string of inequalities holds:

\[
u^i(\theta) \leq 0 \leq u^i(\theta) + y^i(\theta).
\]

\(^{30}\)The optimal contract is deterministic. This follows from Strausz (2006) who shows that, when the optimal deterministic mechanism does not involve bunching, stochastic mechanisms cannot help. For the same reasons, the optimal mechanism is also deterministic in the more general scenario of Section 5.
To deter the most efficient types from exaggerating their costs, the first-period payoff is negative if a firm claims a high first-period cost. Since first-period payoffs are evaluated at a higher marginal utility of income than second-period payoffs, for a firm having a low first-period cost $\theta$ exaggerating its cost parameter is costly. Payments for firms with sufficiently high cost are backloaded towards the future to relax incentive compatibility. Furthermore, the stronger the risk aversion, the more attractive backloading is.

Turning to the lower tail of the cost distribution, observe that (4.10) also implies that $u^i(\theta) + y^i(\theta) = 0$ (with $u^i(\theta) + y^i(\theta) \geq 0$ in the right-neighborhood of $\theta$). Since a $\theta$ firm also gets a positive rent (i.e., $\mathcal{U}^i(\theta) > 0$), payments remain frontloaded for types close enough to $\theta$:

$$u^i(\theta) > u^i(\theta) + y^i(\theta) \geq 0.$$  

Again, the principal may shift payoffs of an efficient firm towards the first period to make it more attractive for the firm to reveal a low cost parameter.

**Outputs.** Output distortions reflect the incentives to manipulate costs and thus depend on the concavity of the firm’s utility function.

**Proposition 3** Assume that $\beta$ is common knowledge in the second period.

- The optimal production of the basic service is distorted downwards below the first-best level, $q^i(\theta) \leq q^*(\theta)$ (with equality at $\underline{\theta}$ only):

$$S'(q^i(\theta)) = \theta + \frac{F(\theta)}{f(\theta)}(1 - \delta + \delta v'(u^i(\theta) + y^i(\theta))), \quad \forall \theta \in \Theta.

- The add-on is always produced at the first-best level:

$$x^i(\theta, \beta) = x^*(\theta), \quad \forall (\theta, \beta) \in \Theta \times \mathcal{B}.$$  

From (4.10), we know that the firm’s benefits from exaggerating its cost are evaluated at a lower marginal utility of income in the second period. As a result, the principal does not need to distort production as much. Output distortions are lower than in the risk neutral case and contracts for the basic service are tilted towards high-powered incentives:

$$q^i(\theta) \geq q^{bm}(\theta), \quad \forall \theta \in \Theta.$$  

Furthermore, these effects are more pronounced as $\delta$ increases. Intuitively, as the second period matters more, it becomes easier to relax first-period incentive compatibility by

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31Measured by the importance of $v''(\cdot)$.
paying more for the basic service in the second period. This of course makes output distortions less useful to induce information revelation.

Example (CARA preferences - continued). Using (4.10) and (4.12), we obtain the following closed-form expressions for output, per-period profits and information rent:

\[ S'(q^i(\theta)) = \theta + \frac{F(\theta)}{f(\theta)} \left(1 - \delta + \frac{\delta}{1 + rq^i(\theta)\frac{F(\theta)}{f(\theta)}}\right), \quad \forall \theta \in \Theta, \]

\[ u^i(\theta) + y^i(\theta) = \frac{1}{r} \ln \left(1 + rq^i(\theta)\frac{F(\theta)}{f(\theta)}\right), \quad \forall \theta \in \Theta, \]

\[ u^i(\theta) = \frac{1}{1 - \delta} \left(U^i(\theta) - \frac{\delta q^i(\theta)\frac{F(\theta)}{f(\theta)}}{1 + rq^i(\theta)\frac{F(\theta)}{f(\theta)}}\right), \quad \forall \theta \in \Theta, \]

\[ U^i(\theta) = \int_\theta^\theta q^i(s) \left(1 - \delta + \frac{\delta}{1 + rq^i(s)\frac{F(s)}{f(s)}}\right) ds, \quad \forall \theta \in \Theta. \]

4.4. Risk Effect

Suppose now that \( \beta \) is privately learned by the firm while \( \theta \) remains common knowledge so that the first-period incentive constraint (3.8) disappears from the principal’s optimization problem. The principal is now torn between the conflicting objectives of providing insurance against uncertain second-period costs and inducing revelation of these costs.

Profit levels. The firm’s risk behavior now affects the second-period incentive problem.

**Proposition 4** Assume that \( \beta \) is privately observed by the firm, but that \( \theta \) remains common knowledge.

- The firm always bears some risk which remains independent of the first-period cost: \( \varepsilon^r(\theta) = \varepsilon^r > 0 \) for all \( \theta \in \Theta \).
- The firm’s marginal utility of income remains constant over time and second-period profits are independent of the first-period cost: \( u^r(\theta) + y^r(\theta) = u^r + y^r \) for all \( \theta \in \Theta \), with:

\[ w_z(u^r + y^r, \varepsilon^r) = 1. \]

- There is always full extraction of the firm’s rent:

\[ U^r(\theta) = 0, \quad \forall \theta \in \Theta. \]
There is no need to condition second-period payments and outputs on $\theta$ when this parameter is common knowledge. Indeed, payments for the add-on are used to provide insurance and incentives in the second period but also to ensure that the firm’s marginal utility of income remains constant over time. These objectives are independent of the cost of the basic service.

From (3.6), the firm must bear some risk to satisfy second-period incentive compatibility. This implies that the firm’s participation becomes more costly for the principal since a risk premium must be paid. Yet, the principal’s marginal cost of public funds must remain the same over time which requires adjusting the second-period subsidies to ensure that the firm’s marginal utility of income remains constant over time. Because $w_{ze} \geq 0$, (4.13) then implies that $w_z(u_r + y^r, 0) = v'(u_r + y^r) < 1$ and second-period profits are necessarily positive:

$$u_r + y^r = \varphi \left( \frac{-(1 - \delta)u_r}{\delta}, \varepsilon^r \right) > 0,$$

where the first equality follows from the definition of $\varphi(\cdot)$ and the fact that $U_r(\theta) = 0$.

More precise results on the intertemporal profile of payments can be obtained by using the more detailed properties of $v(\cdot)$.

**Corollary 1** Assume that $\beta$ is privately observed by the firm, that $\theta$ remains common knowledge, and that Assumptions 2 and 3 both hold. Then payments are always backloaded:

$$y^r \geq u_r + y^r > 0 \geq u^r, \quad \forall \theta \in \Theta.$$

There are two effects at play in determining profit levels in each period. First, moving payments to the second period and choosing $y^r \geq 0$ acts as precautionary savings against second-period uncertainty; an optimal response when the firm exhibits prudent behavior ($v'' \geq 0$). Yet, risk aversion also calls for giving the firm a risk premium which requires that the overall intertemporal profit $u_r + \delta y^r$ remains positive. Taken in tandem, Assumptions 2 and 3 ensure that the second effect is strong enough even when first-period profits are negative.

**Outputs.** Because $\theta$ is common knowledge, there is no reason to distort production of the basic service. A downward distortion of output only arises for the add-on. It helps relax the second-period incentive-compatibility constraint (3.6).

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32 Kimball and Weil (2009) investigate the interaction between risk aversion and precautionary savings.
Proposition 5 Assume that $\beta$ is privately observed by the firm, but that $\theta$ remains common knowledge.

- The basic service is always produced at the first-best level: $q^r(\theta) = q^*(\theta)$ for all $\theta \in \Theta$.
- The add-on is always produced at the first-best level when second-period costs are low and below the first-best level otherwise. This level $x^r$ is independent of the first-period cost: $x^r(\theta, \beta) = x^*(\beta)$ and $0 < x^r(\theta, \beta) = x^r = \frac{\epsilon^r}{\Delta \beta} \leq x^*(\beta)$ where:

\[
(4.15) \quad (1 - \nu)(V'(x^r) - \beta) = \Delta \beta \varphi_r(w(u^r + y^r, \epsilon^r), \epsilon^r).
\]

When $\theta$ is common knowledge, the optimal contract is the same regardless of the importance of the second period (as measured by $\delta$). Because the first-period incentive problem can be ignored, there is no reason for the principal to play with the firm’s marginal utility of income. Furthermore, all distortions related to the Risk Effect arise in the second period and therefore are unaffected by the value of $\delta$.

Example (CARA preferences - continued). We now use (4.13), (4.14) and (4.15) to obtain closed-form expressions for per-period profits and second-period output as follows:

\[
(4.16) \quad (1 - \nu)\left( V'(x^r) - \beta \right) = \Delta \beta \frac{\eta_r(r, \epsilon^r)}{r \eta_r(r, \epsilon^r)} > 0,
\]

\[
(4.17) \quad y^r = \frac{1}{r} \ln \left( \eta(r, \epsilon^r) \right) > 0 = u^r.
\]

5. OPTIMAL CONTRACT: THE GENERAL CASE

Suppose now that both $\theta$ and $\beta$ are privately observed by the firm. The analysis now merges the two specific contexts analyzed in Sections 4.3 and 4.4 and highlights how the Income and Risk Effects interact. The optimal contract exhibits a contractual externality of one agency problem on the other even though there is no technological linkage across periods. Output distortions on the basic service are more pronounced than when there is no agency problem in the second period, and the firm must bear less risk in the second period compared to when there is no incentive problem in the first period. In light of our preliminary results, risk aversion leads to lower powered incentives in both periods.

Profit levels. A first obvious finding is that, at the optimal contract, there is full rent extraction only for the least efficient firm:

\[
(5.1) \quad U^a_*(\theta) \geq U^a_*(\bar{\theta}) = 0, \quad \forall \theta \in \Theta.
\]

\[\text{Formally, this can be viewed in (4.13) and (4.15) which are independent of } \delta.\]
More specific to the present context are changes in the firm’s marginal utility of income both over time and by comparison to the case where there is no private information in the second period. Presenting these changes is the purpose of the next proposition. To get a clear characterization of the solution to the optimal contracting problem, we shall assume that the corresponding Hamiltonian is always concave so that the necessary conditions for optimality given by Pontryagyn Principle are also sufficient (see Seierstad and Sydsaeter (1987, p. 105, Theorem 4)). \(^{34}\)

**Proposition 6**  Assume that both \(\theta\) and \(\beta\) are private information.

- The firm’s marginal utility of income is lower in the second period than in the first period:

\[
(5.2) \quad w_z(u^{as}(\theta) + y^{as}(\theta), \varepsilon^{as}(\theta)) = 1 + q^{as}(\theta) \frac{F(\theta)}{f(\theta)} w_z(u^{as}(\theta) + y^{as}(\theta), \varepsilon^{as}(\theta)) \leq 1, \quad \forall \theta \in \Theta \quad \text{(with equality for } \theta \text{ only}).
\]

- When Assumption 2 holds, we have:

\[
(5.3) \quad v'(u^{as}(\theta) + y^{as}(\theta)) \leq 1 + q^{as}(\theta) \frac{F(\theta)}{f(\theta)} v''(u^{as}(\theta) + y^{as}(\theta)), \quad \forall \theta \in \Theta.
\]

Again, the *Income Effect* implies that payments are backloaded so as to reduce the cost of first-period incentive compatibility. But with an incentive problem in the second period, the firm must also bear some risk and \(\varepsilon^{as}(\theta) > 0\). Assumption 2 guarantees that the firm’s marginal utility of income is higher when second-period returns are also riskier. To compensate for this effect of the second-period agency problem, the cost of first-period incentive compatibility is reduced by increasing even further second-period profits. Payments are even more backloaded as can be seen by comparing (4.10) and (5.3). \(^{35}\)

*Outputs.* The magnitude of output distortions now depends on how much risk is borne by the firm.

**Proposition 7**  Assume that both \(\theta\) and \(\beta\) are private information.

- The production of the basic service is distorted downward below the first-best level but remains above the Baron and Myerson (1982) outcome, \(q^{bm}(\theta) \leq q^{as}(\theta) \leq q^*(\theta)\)

\[^{34}\text{As explained in the Appendix, this concavity is ensured when } V(\cdot) \text{ is sufficiently concave and } \delta \text{ small enough so that overall the model remains “close” to the case without the add-on.}\]

\[^{35}\text{The attentive reader will of course have noticed that the outputs } q'(\theta) \text{ and } q''(\theta) \text{ change across the two scenarios. Yet our reasoning remains indicative of the direction in which payments move.}\]
(with equality at \( \theta \) only):

\[
S'(q^{as}(\theta)) = \theta + \frac{F(\theta)}{f(\theta)} (1 - \delta + \delta w_z(u^{as}(\theta) + y^{as}(\theta), \varepsilon^{as}(\theta))), \quad \forall \theta \in \Theta.
\]

- When Assumption 2 holds, we have:

\[
S'(q^{as}(\theta)) \geq \theta + \frac{F(\theta)}{f(\theta)} (1 - \delta + \delta v'(u^{as}(\theta) + y^{as}(\theta))), \quad \forall \theta \in \Theta.
\]

Since \( w_z \geq 0 \), the fact that the firm bears some risk in the second period implies a higher marginal utility of income at this date. The firm therefore has stronger incentives to lie about its costs because the corresponding cost savings are evaluated at a higher rate of return for the second-period. This suggests that the principal should be more concerned with rent extraction than in the absence of such risk (condition (5.5)) and output distortions are stronger. Condition (5.2) further ensures that distortions remain less pronounced than if the firm was risk neutral in the second period.

The interaction between the agency problems in each period also goes the other way. The rent left to the firm for the production of the add-on has an impact on the amount of risk it has to bear. This is unveiled in the next proposition.

**Proposition 8** Assume that both \( \theta \) and \( \beta \) are private information.

- The firm always bears some risk in the second period: \( \varepsilon^{as}(\theta) = \Delta \beta x^{as}(\theta, \beta) > 0 \).
- The add-on is always produced at the first-best level when second-period costs are low but below the first-best level otherwise: \( x^{as}(\theta, \beta) = x^*(\beta) \) and \( 0 < x^{as}(\theta, \beta) \leq x^*(\beta) \) where

\[
(1 - \nu)(V'(x^{as}(\theta, \beta)) - \beta) = \Delta \beta \left( \varphi(\frac{w(u^{as}(\theta) + y^{as}(\theta), \varepsilon^{as}(\theta)), \varepsilon^{as}(\theta)) + q^{as}(\theta) \frac{F(\theta)}{f(\theta)} H(u^{as}(\theta) + y^{as}(\theta), \varepsilon^{as}(\theta)) \right), \quad \forall \theta \in \Theta.
\]

- When Assumption 3 holds, we have:

\[
(1 - \nu)(V'(x^{as}(\theta, \beta)) - \beta) \geq \Delta \beta \varphi(\frac{w(u^{as}(\theta) + y^{as}(\theta), \varepsilon^{as}(\theta)), \varepsilon^{as}(\theta)), \forall \theta \in \Theta.
\]

The findings in Proposition 8 are closely related to those highlighted in Section 4.4 where only second-period incentive compatibility was a concern. The first difference is illustrated in condition (5.6). When choosing how much risk should be borne by the firm, the principal anticipates the consequences of this risk on the firm’s marginal utility of
income. There are two effects at play here. First, keeping second-period profits fixed, more second-period risk increases the marginal utility of income which makes first-period incentive compatibility more costly. This first effect calls for reducing the level of risk in the second period. However, leaving the firm with more risk in the second period also requires an increase in second-period profits so as to keep the firm’s intertemporal utility constant. This second effect reduces the marginal utility of income in the second period, making first-period incentives less costly. Overall, Assumption 3 guarantees that the first effect dominates. The principal thus implements lower powered incentives and more insurance against second-period uncertainty. As illustrated by (5.7), the optimal contract exhibits lower powered incentives in the second period and a smaller distortion of the add-on compared to when there is no asymmetric information in the first period.

Example (CARA preferences - continued). Straightforward computations lead to:

\[ \varepsilon^a(\theta) = \varepsilon^r \quad \text{and} \quad q^a(\theta) = q^r(\theta), \quad \forall \theta \in \Theta. \]

Output distortions are the same as when the incentive problems at each date are taken in isolation. The second-period endogenous risk is independent of whether there is asymmetric information on first-period costs or not. Reciprocally, distortions of the basic service are independent of whether there is symmetric or asymmetric information on second-period costs.

However, second-period uncertainty requires payments to be backloaded and a second-period risk premium to be paid for the endogenous risk that the privately informed firm must bear. Indeed, with CARA preferences, the second-period risk premium paid to the firm, namely \( \frac{1}{r} \ln (\eta(r, \varepsilon^r)) \), is independent of the profits from the basic service. Therefore, there is no reason to modify the firm’s information rent and first-period output in view of reducing second-period agency costs.

An immediate consequence of these findings is that the firm’s information rent is also the same as when only the Income Effect is at play:

\[ \mathcal{U}^a(\theta) = \mathcal{U}^i(\theta), \quad \forall \theta \in \Theta. \]

Second-period profits are simply the sum of the values obtained when considering the Income and Risk Effects separately:

\[ u^a(\theta) + y^a(\theta) = \frac{1}{r} \ln (\eta(r, \varepsilon^r)) + \frac{1}{r} \ln \left( 1 + r q^i(\theta) \frac{F(\theta)}{f(\theta)} \right), \quad \forall \theta \in \Theta. \]

As a result, the marginal utility of income is unchanged by the addition of some endoge-
ous risk for the second period; namely:

\[ w_z(u^a(\theta) + y^a(\theta), \varepsilon^r) \equiv v'(u^i(\theta) + y^i(\theta)), \quad \forall \theta \in \Theta. \]

This equality has an important implication: With CARA preferences, the second-period agency problem has no impact on first-period incentives.

Overall, and similarly to the case of a risk neutral firm analyzed in Section 4.2, there is now a dichotomy between solving the incentive problems in each period. The case of risk neutrality is simply obtained by taking the limit when \( r \) goes to 0 and, doing so, we recover the findings in Proposition 1. When \( r \) goes to infinity, the firm becomes infinitely risk averse and we obtain a screening distortion which is the same as in the two-type screening models with \textit{ex post} participation constraints: 36

\[
\lim_{r \to +\infty} V'(x^r) = V'(x^r_\infty) = \bar{\beta} + \frac{\nu}{1 - \nu} \Delta \beta.
\]

Indeed, an infinitively risk averse firm only cares about the worst possible returns for the second period. This requirement hardens the participation constraint and calls for strong distortions in the second period. When considering information manipulations in the first period, the firm now anticipates that cost savings provide no utility gains in the second period. This reduces how much rent the firm can make from the basic service. As a result, output distortions are also weaker and information rents are lower:

\[
\lim_{r \to +\infty} S'(q^{as}(\theta)) = S'(q^{as}_\infty(\theta)) = \theta + (1 - \delta) \frac{F(\theta)}{f(\theta)}, \quad \forall \theta \in \Theta,
\]

\[
\lim_{r \to +\infty} \mathcal{U}^{as}(\theta) = (1 - \delta) \int_{\theta}^{\bar{\theta}} q^{as}_\infty(s) ds, \quad \forall \theta \in \Theta.
\]

6. RENEGOTIATION

In this section, we argue that, under very general conditions, the dynamics of the optimal contract opens the door to renegotiation. Our aim here is not to characterize the optimal renegotiation-proof contract, but to give some insights into when renegotiation may or may not hinder the performance of long-term agreements. We highlight a difficult trade-off between flexibility and rigidity. In general, a long-term contract needs to be flexible enough to include the add-on, but rigid enough to avoid (costly) renegotiation.

Under full commitment, the principal can fully anticipate the impact of the firm’s private information in the second period on first-period incentive compatibility. Therefore, the principal might find it attractive to play on the second-period risk borne by the firm to ease information revelation earlier on. This important feature of long-term agreements

36See for instance Chapter 2, Laffont and Martimort (2002).
contracting is apparent from comparing the second-period distortions with and without informational asymmetry in the first period, i.e. (4.15) and (5.6). It is immediate that, as soon as \( H(z, \varepsilon) > 0 \), these expressions differ. Whenever these two expressions differ, history matters: First-period costs have an impact on second-period risk-sharing. More precisely, the principal cares about reducing the second-period risk to save on the firm’s information rent and this concern is more pronounced the greater the level of the basic service.\(^{37}\) Yet, once in the second period, these extra output distortions are no longer needed. In other words, the optimal long-term contract is not renegotiation-proof in a sense that we now make more precise.

Let us consider a long-term contract \( C = \{ t(\hat{\theta}), y(\hat{\theta}), q(\hat{\theta}), \varepsilon(\hat{\theta}) \}_{\hat{\theta} \in \Theta} \). In the second period, this contract gives an expected utility \( w(t(\hat{\theta}) + y(\hat{\theta}) - \theta q(\hat{\theta}), \varepsilon(\hat{\theta})) \) to the firm of type \( \theta \) when it chooses to report a first-period cost \( \hat{\theta} \). With limited commitment, the Revelation Principle may fail and there is no reason to expect that the firm reports truthfully in the first place. In fact, by misreporting its type early on, the firm might secure a more attractive renegotiation later on. We denote by \( M(\hat{\theta} | \theta) \) the distribution of first-period reports that the type \( \theta \) may choose.

When the second period comes, but before second-period costs are observed by the firm, the principal might already want to propose a new contract for the provision of the add-on. This new offer may let the firm bear more risk than the initial contract. For the firm to accept this new offer, it must stipulate an average extra profit level \( \tilde{y}(\hat{\theta}) \), production levels of the add-on \( \tilde{x}(\hat{\theta}, \beta) \) and an amount of risk \( \tilde{\varepsilon}(\hat{\theta}) \)\(^{39}\) that are more profitable than the initial contract. For any type \( \theta \) such that the report \( \hat{\theta} \) is in the support of \( M(\cdot | \theta) \), we must therefore have:

\[
(6.1) \quad w(t(\hat{\theta}) + y(\hat{\theta}) + \tilde{y}(\hat{\theta}) - \theta q(\hat{\theta}), \tilde{\varepsilon}(\hat{\theta})) \geq w(t(\hat{\theta}) + y(\hat{\theta}) - \theta q(\hat{\theta}), \varepsilon(\hat{\theta})).
\]

The new contract must also remain incentive compatible:

\[
(6.2) \quad \tilde{\varepsilon}(\hat{\theta}) \geq \Delta \beta \tilde{x}(\hat{\theta}, \beta).
\]

Among all contracts that are acceptable at the renegotiation stage, there is always the null contract that consists of offering no extra payment to the firm, \( \tilde{y}(\hat{\theta}) \equiv 0 \), and leaving

---

\(^{37}\)Formally, the term \( \Delta \beta q^{ws}(\theta) \frac{\partial \tilde{x}(\theta)}{\partial \beta} H(w^{ws}(\theta) + y^{ws}(\theta), \varepsilon^{ws}(\theta)) \) depends on how much of the basic service is provided.

\(^{38}\)For the sake of simplifying notations and the presentation, we first take the short-cut of considering that the second-period contract is fully determined by the condition of zero expected profits in (3.4) and by an amount of risk in the second-period that satisfies (3.6). Second, we restrict attention to the case of direct mechanisms. The analysis of Bester and Strausz (2001) shows that, in the case of a finite set of types, this restriction is without loss of generality even in an environment with limited commitment if one is ready to entertain the possibility that reports are no longer truthful.

\(^{39}\)In other words the new prices for each levels of the add-ons are now \( \tilde{p}(\hat{\theta}, \beta) - \theta \tilde{x}(\hat{\theta}, \beta) = \tilde{y}(\hat{\theta}) + (1 - \nu)\tilde{\varepsilon}(\hat{\theta}) \) and \( \tilde{p}(\hat{\theta}, \beta) - \theta \tilde{x}(\hat{\theta}, \beta) = \tilde{y}(\hat{\theta}) - \nu \tilde{\varepsilon}(\hat{\theta}) \).
the levels of the add-on and the overall risk unchanged, so that $\tilde{\varepsilon}(\hat{\theta}) \equiv \varepsilon(\hat{\theta})$.

A long-term contract $C$ (with an associated first-period reporting strategy $M(\cdot|\theta)$) is renegotiation-proof if, given the principal’s posterior beliefs about the firm’s type $\theta$, the principal finds it optimal to offer the null contract ($\tilde{y}(\hat{\theta}) = 0$ and $\tilde{\varepsilon}(\hat{\theta}) = \varepsilon(\hat{\theta})$) at the renegotiation stage.\textsuperscript{40}

We are now ready to check whether the optimal contract under full commitment $C^{as}$ is actually robust to renegotiation.

**Proposition 9**  
\begin{itemize}
  \item Suppose that there exists $\theta \in \Theta$ such that
  \begin{equation}
  H(u^{as}(\theta) + y^{as}(\theta), \varepsilon^{as}(\theta)) > 0.
  \end{equation}
  Then the optimal contract under full commitment $C^{as}$ is not renegotiation-proof.
  \item The optimal contract under full commitment $C^{as}$ is always renegotiation-proof if the firm has CARA preferences.
\end{itemize}

Section 5 showed how the principal wants to reduce the second-period risk borne by the firm to decrease the second-period marginal utility of income and facilitate information revelation earlier on. When information has already been revealed, this extra insurance is no longer needed. Renegotiation may push the principal to increase risk and, by second-period incentive compatibility, raise the level of the add-on.

However, with CARA preferences, the amount of risk borne by the firm in the second period is, even in the full commitment optimal contract, independent of how much profit has been promised for the delivery of the basic service. Incentive problems in each period are no longer linked and the optimal contract under full commitment remains renegotiation-proof.

A particularly interesting case arises when the firm is risk neutral. To the extent that risk neutrality captures the idea that the firm has a perfect access to financial markets, our model predicts that long-term contracts with firms having perfect access to financial markets are stable and robust to further rounds of negotiations. Instead, costly access to financial markets might destabilize long-term contracts when add-ons become necessary.

**SPOT CONTRACTING.** Suppose now that, at the time when the long-term contract is signed, the contract is highly incomplete and doesn’t even specify payments and output requirements for the add-on. With CARA preferences, it can easily be checked that even if parties are restricted to trade through a sequence of contracts, the same allocation as in the optimal long-term contract $C^{as}$ can be implemented. Indeed, consider a long-term agreement $\{t^{as}(\hat{\theta}), y^{as}(\hat{\theta}) - y^r, q^{as}(\hat{\theta})\}_{\theta \in \Theta}$ which is signed \textit{ex ante} and regulates the

\textsuperscript{40}Even if the informational contexts differ slightly, we use the definition of renegotiation-proofness given in Dewatripont (1988).
basic service over the whole relationship. This contract does not include any specification regarding the add-on. However, in the second period, parties agree on a spot contract to regulate the add-on. This spot contract specifies an extra payment $y^r$, the levels of the add-on in the different states of nature, $x^{as}(\cdot, \beta)$, and the associated risk $\varepsilon^r$ borne by the firm. The firm accepts this spot contract because for any report $\hat{\theta}$ it has made earlier on, it increases its profit by doing so:

$$(6.4) \quad w(t^{as}(\hat{\theta}) + y^{as}(\hat{\theta}) - \theta q^{as}(\hat{\theta}), \varepsilon^r) = w(t^{as}(\hat{\theta}) + y^{as}(\hat{\theta}) - y^r - \theta q^{as}(\hat{\theta}), 0) \iff w(y^r, \varepsilon^r) = 0$$

where $y^r$ follows from (4.10) and $w(\cdot)$ takes the specific form of CARA preferences.

From (5.8) the following equality holds:

$$\max_{\hat{\theta} \in \Theta} (1 - \delta)((t^{as}(\hat{\theta}) - \theta q^{as}(\hat{\theta})) + \delta w(t^{as}(\hat{\theta}) - \theta q^{as}(\hat{\theta}) + y^{as}(\hat{\theta}) - y^r + y^r, \varepsilon^r) = \max_{\hat{\theta} \in \Theta} (1 - \delta)((t^{as}(\hat{\theta}) - \theta q^{as}(\hat{\theta})) + \delta w(t^{as}(\hat{\theta}) - \theta q^{as}(\hat{\theta}) + y^{as}(\hat{\theta}), \varepsilon^{as}(\theta)) = U^{as}(\theta).$$

Therefore, anticipating acceptance of the second-period spot contract regulating the add-on, the firm also chooses to report $\theta$ truthfully. We summarize this important finding in the next proposition.

**Proposition 10** Suppose that the firm has CARA preferences. There is no loss of generality in delaying contracting on the add-on to the beginning of the second period.

With CARA preferences, there is no loss of using an incomplete long-term contract. Contracting on the add-on can be delayed until this option arises. This proposition bears some strong resemblance with Fudenberg, Holmström and Milgrom (1990) who also discuss conditions under which an optimal long-term contract can be implemented through short-term contracts. Our paper and their more general inquiry differ in several ways. First, those authors allow the principal and the firm to borrow from the financial markets on equal terms. While this assumption is certainly relevant for some employment relationships, it is less relevant in our procurement context where a public principal might have access to borrowing on more favourable terms. Indeed our modeling of second-period risk aversion for the firm is precisely meant to capture such frictions. Second, while we insist on (repeated) adverse selection as a fundamental friction in the contract, Fudenberg, Holmström and Milgrom (1990) study a repeated contracting model under the assumption that technology is common knowledge (in the sense that future contract outcomes is completely determined by current history). This is clearly not the case in our environment. Furthermore, they focus on moral hazard environments. These conditions (plus other more technical assumptions) are then shown to be sufficient to obtain the irrelevance of long-term contracting. Our model illustrates that such sequential optimality can be found in other more informationally constrained environments as well.
7. COMPETING FIRMS

In most infrastructure sectors (water, transportation, waste disposal, energy, . . . ), service provision is allocated through competitive bidding. Yet, at the time of tender, bidders as well as the principal organizing the auction may not be aware of specific needs and costs for future stages of the project. Actual bids may reflect how bidders perceive this future risk. In response, the principal may have to modify reserve prices to foster participation. In this section, we investigate how the principal can do so. This focus on participation allows us to stress a new impact of the second-period endogenous risk that now bites at the extrinsic margin.

Since our focus is on competition and how the Income and Risk Effects influence bidding behavior, we simplify the modeling on the demand side and assume that the public authority wants to procure only one unit of the basic good or service whose social value is denoted by $S$. In the second period, there is also only one unit of the add-on to procure. Because of switching costs, or complementarities (on the extensive margin) between production stages, the winning firm in the first stage of the tender is also in charge of supplying the add-on later on.\footnote{To motivate this setting, remember that tenders for long-lived projects like PPPs involve consortia that are short-term ventures created for the sole purpose of bidding for this particular project. Most of the time, these consortia do not even exist later on, leaving the winning firm and the principal in a situation of bilateral lock-in.} We will assume that its value $V$ is large enough to ensure that this add-on is always valuable even under asymmetric information.\footnote{Typically $V > \bar{\beta} + \frac{\Delta \beta}{1 - \nu}$ gives a sufficient condition.} The firm is thus paid a fixed amount for the provision of the add-on that exactly covers the highest possible cost $\bar{\beta}$. No quantity screening can be used to help rent extraction in the second period. Thus the only tools that are available to the principal to improve the dynamic rent/efficiency trade-off are the amount of participation induced by the choice of a reserve price and the timing of payments. In this setting, we show that backloading payments remains a key ingredient in improving incentives and that the Income and Risk Effects impact on the choice of the reserve price. From our more general analysis in Section 5, asymmetric information on the second-period cost requires the firm to bear all risk associated with second-period returns. Formally, we have $\varepsilon(\theta) = \Delta \beta$ for all $\theta \in \Theta$.

In the first period, the principal runs a first-price auction with a reserve price to allocate the service provision. We assume that there are $n + 1$ competing bidders whose first-period costs are independently drawn from the same distribution $F(\cdot)$. Firms are also symmetric in terms of the distribution of their second-period costs. We thus look for a symmetric equilibrium bidding strategy $b(\theta)$ that determines a price for the basic service as a function of the firm’s announcement of its costs. As before, we denote by $y(\theta)$ a second-period bonus paid to the firm whenever the principal opts for non-stationary payments. From the principal’s viewpoint, the reserve price indirectly defines a cutoff for the winning bidder’s first-period cost above which it is preferable not to engage in the
long-term project. We denote by $\tilde{\theta}$ this cutoff.\footnote{In Section 5, we assumed that surplus functions satisfy the Inada conditions. These conditions imply that it is always optimal to procure the service even when the firm had the highest possible cost $\bar{F}$. With 0-1 projects (which technically amounts to assuming linear surplus cum a capacity constraint), shutting down production for the least efficient firms by setting a reserve price may improve the rent/efficiency trade-off.}

Assuming that the bidding strategy $b(\cdot)$ is increasing, the probability that a bidder who reports a first-period cost $\hat{\theta}$ wins the auction is $(1 - F(\hat{\theta}))^n$. This allows us to rewrite the requirement of incentive compatibility for a bidder with type $\theta$ as:

$$U(\theta) = \max_{\theta \in \Theta} (1 - F(\hat{\theta}))^n \left( (1 - \delta)(b(\hat{\theta}) - \theta) + \delta w(b(\hat{\theta}) - \theta + y(\hat{\theta}), \Delta \beta) \right).$$

Following our earlier notations, we denote the first-period profit conditionally on winning by $u(\theta) = b(\theta) - \theta$. Using the Envelope Theorem, we obtain the following necessary condition for incentive compatibility:\footnote{In the sequel, we will content ourselves with the analysis of a relaxed optimization problem for the principal where we neglect global conditions for incentive compatibility. Following Lemma 1, we could show that this condition now amounts to $w_z (u(\theta) + y(\theta), \Delta \beta)$ being non-increasing in $\theta$.}

$$\dot{U}(\theta) = -(1 - F(\theta))^n (1 - \delta + \delta w_z (u(\theta) + y(\theta), \Delta \beta)).$$

Expressing second-period profits $u(\theta) + y(\theta)$ in terms of the rent $U(\theta)$, we rewrite this envelope condition as:

$$\dot{U}(\theta) = -(1 - F(\theta))^n \left( 1 - \delta + \delta w_z \left( \varphi \left( \frac{U(\theta)}{(1 - F(\theta))^n} - \frac{(1 - \delta)u(\theta)}{\delta}, \Delta \beta \right), \Delta \beta \right) \right).$$

Once a bidding strategy $b(\theta)$ (or equivalently a first-period profit $u(\theta)$ that is chosen by the principal so as to indirectly control that bidding strategy) is given, the rent profile $U(\theta)$ is fully determined by the differential equation (7.2) for types $\theta \leq \tilde{\theta}$ and the boundary condition that is implicitly defined by the reserve price:

$$U(\tilde{\theta}) = 0.$$

Due to the symmetry of bidders, everything happens as if the principal was actually dealing with a single firm (referred to as the “winning firm” below) but this firm would have a first-period cost drawn from the distribution of the minimum of $n + 1$ independent variables. The corresponding distribution function is thus $G(\theta) = 1 - (1 - F(\theta))^{n+1}$ (with density $g(\theta) = (n + 1)f(\theta)(1 - F(\theta))^n$). This remark facilitates the derivation of the optimal contract under which the winning firm operates.

**Proposition 11** The second-period profit $u^{as}(\theta) + y^{as}(\theta)$ of the winning firm satisfies:
When Assumptions 2 holds, this profit is greater than when \( \beta \) is common knowledge.

Proposition 11 captures the distortions needed to ease information revelation in the first period. Formally, (7.4) looks very similar to (5.2) with the only proviso that projects are now 0-1 in both periods. In order to reduce the firm’s marginal utility of income in the second period, the principal still pays an extra premium for the basic service in this period. Again, inducing information revelation of the first-period cost becomes easier. The Income Effect is stronger when there is also asymmetric information in the second period. Such asymmetry makes it more valuable to backload payments for precautionary purposes.

Observe that (7.4) is independent of the number of competing firms. Competition plays no role in determining second-period profits which are identical to those achieved when there is a single provider. The amount of payments that is backloaded to the second-period is independent of how competitive the environment is.

Of course, bids depend on the magnitude of competition. Using (7.1), the equilibrium bidding strategy can easily be obtained in terms of the second-period profit \( u^a_s(\theta) + y^a_s(\theta) \):

\[
b^a_s(\theta) = b_0(\theta) + \\
\frac{\delta}{1-\delta} \left( \int_{\theta}^{\tilde{\theta}^a} \frac{(1 - F(\tilde{\theta}))^n}{(1 - F(\theta))^n} w_z(u^a(\tilde{\theta}) + y^a(\tilde{\theta}), \Delta \beta) d\tilde{\theta} - w(u^a(\theta) + y^a(\theta), \Delta \beta) \right),
\]

where \( b_0(\cdot) \) is the optimal bidding strategy of a risk-neutral firm in the first-price auction with reserve price \( \tilde{\theta}^a \). The expression of this latter strategy is well known and easily derived as:

\[
b_0(\theta) = \theta + \frac{1}{(1 - F(\theta))^n} \int_{\theta}^{\tilde{\theta}^a} (1 - F(\tilde{\theta}))^n d\tilde{\theta}.
\]

Let us now turn to the optimal reserve price or, more precisely, its consequences on participation. For the sake of the comparison, it is useful to recall the value of the cutoff \( \tilde{\theta}^r_n \) that would be achieved had the firm been risk neutral. When fixing this reserve price, the principal trades off the overall value of the project (including the expected benefits from the add-on) and its cost, taking into account information rents left to the winning firm. It is routine to verify that the cutoff \( \tilde{\theta}^r_n \) solves:

\[
S + \delta V = \tilde{\theta}^r_n + \frac{F(\tilde{\theta}^r_n)}{f(\tilde{\theta}^r_n)} + \delta E_\beta(\beta).
\]
This condition simply means that, for the cutoff $\tilde{\theta}^{rn}$, the overall value of the projects (the left-hand side of (7.5)) is equal to its virtual costs (the right-hand side). To ensure an interior solution $\tilde{\theta}^{rn}$, we will from now on assume:

**Assumption 4**

$$\theta < S + \delta(V - E_\beta(\beta)) < \bar{\theta} + \frac{1}{f(\bar{\theta})}.$$ 

We also define $\tilde{\theta}^{i}$ as the cutoff when $\beta$ is common knowledge. We now turn to the characterization of these cutoffs.

**Proposition 12** Assume that $\beta$ is common knowledge, the Income Effect increases participation:

(7.6) $\tilde{\theta}^{i} \geq \tilde{\theta}^{rn}$.

When $\beta$ is private information and Assumptions 2 holds, the Risk Effect decreases participation:

(7.7) $\tilde{\theta}^{as} \leq \tilde{\theta}^{i}$.

Following the intuition built in Section 4.3, the Income Effect makes it less attractive to exaggerate first-period costs since payments are now backloaded. The principal can raise the optimal reserve price beyond its value had firms been risk neutral and thereby foster more participation. However, the impact of second-period uncertainty on that reserve price goes in the other direction. First, the Risk Effect requires an extra risk premium to be paid to ensure firms’ participation. This calls for a lower reserve price and reduces participation. Second, risk on the add-on increases the marginal utility of income (since $w_{ze} \geq 0$) and makes first-period incentive compatibility more costly. This also pushes towards a lower reserve price and further reduces participation.

*Example (CARA preferences - continued).* This example allows us to quantify the relative impact of both effects and show that whether more risk on the add-on hardens or exacerbates participation is ambiguous. First, observe that (7.4) now gives us the following closed-form expression of second-period profits:

$$u^{as}(\theta) + y^{as}(\theta) = \frac{1}{r} \ln (\eta(r, \Delta \beta)) + \frac{1}{r} \ln \left(1 + r \frac{E(\theta)}{f(\theta)}\right), \quad \forall \theta \in \Theta.$$
Inserting this expression into (7.2) and taking into account (7.3), we obtain:

\[ U^{as}(\theta) = \int_{\tilde{\theta}^{as}} (1 - F(s))^n \left( 1 - \delta + \frac{\delta}{1 + r \frac{F(s)}{f(s)}} \right) ds, \quad \forall \theta \in \Theta, \]

where \( \tilde{\theta}^{as} \) solves:

\[ S + \delta (V - E_\beta(\beta)) = \tilde{\theta}^{as} + \frac{F(\tilde{\theta}^{as})}{f(\tilde{\theta}^{as})} + \delta \left( \frac{1}{r} \ln(\eta(r, \Delta \beta)) + \frac{1}{r} \ln \left( 1 + r \frac{F(\tilde{\theta}^{as})}{f(\tilde{\theta}^{as})} \right) - \frac{F(\tilde{\theta}^{as})}{f(\tilde{\theta}^{as})} \right). \]

When \( \Delta \beta \) is sufficiently small, the risk-premium \( \frac{1}{r} \ln(\eta(r, \Delta \beta)) \) that is required to induce the firm’s participation is also small and the bracket on the right-hand side remains negative. The Income Effect drives the direction of the distortion and second-period risk increases participation.

8. CONCLUSION AND DISCUSSION

This paper has studied optimal dynamic procurement contracts in the context characterized by long-lasting basic services which are later augmented by add-ons whose costs are ex ante unknown. The firm’s risk aversion has complex impacts on both second- and first-period incentives even when the costs of these two products are independently drawn and there are no intertemporal technological linkages between these two stages.

Backloading payments for the basic service reduces the firm’s marginal utility of income and eases first-period incentive compatibility: An Income Effect which calls for lower output distortions of the basic service, higher powered incentives, and more participation in tender procedures. Second, the firm must keep part of the risk associated with the add-on to satisfy second-period incentive compatibility. This Risk Effect reduces the level of the add-on. Under the assumption of Generalized Decreasing Absolute Risk Aversion, backloading payments turns out to be more attractive the more of the risk associated with the add-on the firm must bear. At the same time, transferring payments towards the future also reduces risk aversion and eases second-period incentive compatibility. We show that this contractual externality between two otherwise independent provision stages leads to lower powered incentives and higher output distortions on both the basic service and the add-on. This contractual externality also has strong implications for the robustness of long-term contracts to renegotiation, their implementability through spot contracts and the participation of bidders in tender procedures in risky environments.

We now discuss the robustness of some of our simplifying assumptions.

First of all, we have focused on a basic service which level does not change over time. Although we believe that this is a natural assumption when we consider durable goods and services, there might also be cases where the level of the basic service can be adjusted
over time, perhaps as a consequence of uncertain demand being revealed. In this case, the final (second-period) level of the basic service would be less distorted as a result of the effects described in this paper while the initial level of the basic service would still be given by the standard Baron and Myerson (1982) formula.

We have also made the simplifying assumption that the cost of the uncertain add-on comes from a binary distribution. Although this assumption allows us to consider the consequences of an endogenous background risk on earlier incentives in a stripped down manner, a more symmetric treatment of costs would call for an analysis of the case where the cost of the add-on takes a continuum of values. Both Salanié (1990) and Laffont and Rochet (1998) show that, even in simpler static settings, bunching might arise under this scenario. Focusing on a discrete model has allowed us to depart from such technical complexity and to focus solely on the new economic forces at play. Since in that case second-period incentive compatibility also imposes an endogenous risk on the firm, we conjecture that our results will to a large extent be robust to the introduction of a continuum of types. Yet, the general tractability of the approach might be a concern.

Furthermore, we have assumed that there is symmetric but incomplete information on the cost of the add-on at the time of contracting. In some environments, the firm may already have some private signals about the cost of the add-on or the future demand conditions even at the *ex ante* stage. Existing models of sequential screening, for instance Courty and Li (2000) and Krähmer and Strausz (2011), suggest that such an extra layer of informational asymmetries might be a source of further information rents for the firm. However, as long as this early signal remains imperfect and future uncertainty puts the firm’s profits at risk, the effects described in this paper are likely to persist. Similar robustness is also expected in models with more complex technological linkages across stages, for instance by introducing serial correlation across types, or learning-by-doing effects directly embedded into the expression of second-period costs.

We have focused our analysis on service provision. However, our model can straightforwardly be reinterpreted in terms of the provision of a physical good (for instance a building or some other kind of infrastructure) and our results still hold. In that case, the discount factor should be reinterpreted as the probability that an add-on is required. Furthermore what was previously interpreted as backloaded payments should now be reinterpreted as payments being shifted to states in which the add-on is required. Alternatively the model presented in Section 2 can be rewritten to exclude surplus and production costs in the first period, i.e. both the production of the basic good and production of the add-on take place in the second period but the payment for the basic service can be made both upfront before the production phase starts and in the production phase.

We have considered a complete contracting framework where additional work can be included in the contract at the *ex ante* stage. In reality, firms are often faced with unforeseen contingencies that could not be anticipated and written into the initial contract. Our
analysis of renegotiation shows that the optimal full commitment contract is in general not renegotiation-proof. This also implies that whether additional work can be included in the initial contract or not matters. Only in the CARA case do we obtain the same level of services and overall expected payments regardless of whether add-ons are ex ante contractible or not. In practice, most contracts involve clauses that describe how parties should react when unforeseen risks materialize. So to some extent, even unforeseen events are included and governed by the initial contract, which provides a justification for our complete contracting approach. How to cope with additional works is also an issue of tantamount importance in the context of relational contracting, as illustrated by many examples of subcontracting in highway procurement. Indeed, and as expected, legal disputes surroundings change orders matter a great deal when contracts are incomplete. Those issues as well as of the conditions under which a change can already be included into an initial ("more complete") contract in a public procurement setting are discussed in Hartlev and Liljenbøl (2013).

Bundling of the basic service and the add-on into one contract with a sole provider is, by assumption, the sole possibility in our model. This might be the most appropriate assumption for many add-ons that build on the expertise and the firm’s investments in the basic service. When these add-ons are less specific, provision might be adequately undertaken by other firms. In such contexts, whether the two tasks should be bundled or not becomes an important issue. The extant literature on bundling versus unbundling in PPP contexts has focused on investment and cost externalities (Hart (2003), Bennett and Iossa (2006), Martimort and Pouyet (2008) and Iossa and Martimort (2015)). In our model there is no technological linkage across projects; only agency costs of one project feed back into the other. Of course, the CARA case offers an interesting illustration of a situation where bundling and unbundling reach similar welfare levels because agency costs at each stage are not linked. Beyond this specific case, contracting for the add-on with a separate firm reduces the risk borne by the provider of the basic service and decreases agency cost on that task. At the same time, the (unbundled) contract for the add-on becomes more costly because, under unbundling, the second stage project becomes more risky. The Income Effect no longer requires to transfer payments for the basic service and this increases the effect of risk aversion. Overall whether bundling or unbundling dominates remains an open question.

We hope to investigate some of those issues in future research.

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45 In the U.K standardized PFI contracts, these mechanisms are known as change mechanism clauses.
46 See Gil and Marion (2013).
47 A noticeable exception from the aforementioned literature is Schmitz (2013) who studies the effect of the government’s budget constraint on the bundling decision.
REFERENCES


NAO, National Audit Office (2008), Making Changes in Operational PFI Projects, National Audit Office, HC 205.


**APPENDIX A: PROOFS**

**PROOF OF LEMMA 1: NECESSITY.** From Theorem 2 and Corollary 1 in Milgrom and Segal (2002) and the fact that $x$ and $q$ are positive and bounded above, it immediately follows that $U(\theta)$ is absolutely continuous and thus almost everywhere differentiable with (3.8) holding at any point of differentiability.

**SUFFICIENCY.** $\forall(\theta, \hat{\theta})$, we rewrite (3.7) as:

\[
U(\theta) \geq U(\hat{\theta}) + (1-\delta)(\hat{\theta} - \theta)q(\hat{\theta}) + \delta \left( w(u(\hat{\theta}) + y(\hat{\theta}) + (\hat{\theta} - \theta)q(\hat{\theta}), \varepsilon(\hat{\theta})) - w(u(\theta) + y(\theta), \varepsilon(\theta)) \right).
\]

Using (3.8) and absolute continuity, the rent profile $U(\theta)$ satisfies:

\[
U(\theta) - U(\hat{\theta}) = \int_{\theta}^{\hat{\theta}} q(\theta') \left( 1 - \delta + \delta w_z(u(\theta') + y(\theta'), \varepsilon(\theta')) \right) d\theta', \quad \forall(\theta, \hat{\theta}) \in \Theta^2.
\]

Condition (A.1) thus holds when:

\[
\int_{\theta}^{\hat{\theta}} q(\theta') \left( 1 - \delta + \delta w_z(u(\theta') + y(\theta'), \varepsilon(\theta')) \right) d\theta' \geq (1-\delta)(\hat{\theta} - \theta)q(\hat{\theta}) + \delta \left( w(u(\hat{\theta}) + y(\hat{\theta}) + (\hat{\theta} - \theta)q(\hat{\theta}), \varepsilon(\hat{\theta})) - w(u(\theta) + y(\theta), \varepsilon(\theta)) \right), \quad \forall(\theta, \hat{\theta}).
\]

Because $w(\cdot)$ is concave in its first argument, we have:

\[
w(u(\hat{\theta}) + y(\hat{\theta}) + (\hat{\theta} - \theta)q(\hat{\theta}), \varepsilon(\hat{\theta})) - w(u(\theta) + y(\theta), \varepsilon(\theta)) \leq (\hat{\theta} - \theta)q(\hat{\theta})w_z(u(\hat{\theta}) + y(\hat{\theta}), \varepsilon(\hat{\theta})).
\]

A sufficient condition for (A.1) to hold is thus:

\[
\int_{\theta}^{\hat{\theta}} q(\theta') \left( 1 - \delta + \delta w_z(u(\theta') + y(\theta'), \varepsilon(\theta')) \right) d\theta' \geq (\hat{\theta} - \theta)q(\hat{\theta})(1 - \delta + \delta w_z(u(\hat{\theta}) + y(\hat{\theta}), \varepsilon(\hat{\theta})),
\]

or equivalently:

\[
(A.2) \quad \int_{\theta}^{\hat{\theta}} \left( (1-\delta)(q(\theta') - q(\hat{\theta})) + \delta(q(\theta')w_z(u(\theta') + y(\theta'), \varepsilon(\theta')) - q(\theta)w_z(u(\hat{\theta}) + y(\hat{\theta}), \varepsilon(\hat{\theta}))) \right) d\theta' \geq 0.
\]

Observe now that $q(\theta)(1 - \delta + \delta w_z(u(\theta) + y(\theta), \varepsilon(\theta)))$ weakly decreasing implies:

\[
\int_{\theta}^{\hat{\theta}} q(\theta')(1 - \delta + \delta w_z(u(\theta) + y(\theta), \varepsilon(\theta)))d\theta' \geq (\hat{\theta} - \theta)q(\hat{\theta})(1 - \delta + \delta w_z(u(\hat{\theta}) + y(\hat{\theta}), \varepsilon(\hat{\theta})).
\]
Hence, a sufficient condition to get (A.2) and thus (A.1) is given by:

\[ q(\theta)(1 - \delta + \delta w_z(u(\theta) + y(\theta), \varepsilon(\theta))) \text{ weakly decreasing.} \]

Inserting into (3.8), this condition amounts to having \( \mathcal{U}(\cdot) \) convex. \( Q.E.D. \)

Proofs of Propositions 2 and 3: Taking into account that \( \varepsilon(\theta) \equiv 0 \) and simplifying the principal’s and the firm’s objectives accordingly (and in particular taking into account that \( w(z, 0) = v(z) \)), the principal’s (relaxed) problem can be stated as follows:

\[
\left( \mathcal{P}^i \right): \max_{(q(\theta), x(\theta, \beta), u(\theta), \varepsilon(\theta), \mathcal{U}(\theta))} \int_{\theta}^{\theta} E_\beta \left( \mathcal{W}(q(\theta), x(\theta, \beta), u(\theta), 0, \mathcal{U}(\theta)) \right) f(\theta) d\theta \text{ subject to (3.9)-(3.11)}. \]

First, observe that at the optimum of \( \left( \mathcal{P}^i \right) \), \( x(\theta, \beta) \) is set at the first-best level: \( x^i(\theta, \beta) = x^*(\beta) \), for all \( (\theta, \beta) \). Accordingly, we may simplify the expression of the principal’s second-period payoff to:

\[
E_\beta \left( V(x^*(\beta)) - \beta x^*(\beta) \right) - \varphi \left( \frac{\mathcal{U}(\theta) - (1 - \delta)u(\theta)}{\delta}, 0 \right). \]

We now write the Hamiltonian for this problem as:

\[
\mathcal{H}_0(q, u, \mathcal{U}, \lambda, \theta) = f(\theta) \left( S(q) - \theta q - (1 - \delta)u - \delta \varphi \left( \frac{\mathcal{U} - (1 - \delta)u}{\delta}, 0 \right) + \delta E_\beta \left( V(x^*(\beta)) - \beta x^*(\beta) \right) \right) \]

\[
- \lambda q \left( 1 - \delta + \delta u^\prime \left( \varphi \left( \frac{\mathcal{U} - (1 - \delta)u}{\delta}, 0 \right) \right) \right). \]

The optimization then follows the same steps (except for the optimization with respect to \( \varepsilon(\theta) \) since \( \varepsilon(\theta) \equiv 0 \)) as in the Proof of Propositions 6, 7 and 8. Details are thus omitted. \( Q.E.D. \)

Proofs of Propositions 4 and 5: The principal’s (relaxed) problem can be written as:

\[
\left( \mathcal{P}^r \right): \max_{(q(\theta), x(\theta, \beta), u(\theta), \varepsilon(\theta), \mathcal{U}(\theta))} E_\beta \left( \mathcal{W}(q(\theta), x(\theta, \beta), u(\theta), \varepsilon(\theta), \mathcal{U}(\theta)) \right) \text{ subject to (4.1) and (3.6)}. \]

Because \( \varphi_\zeta \geq 0 \), (4.1) is necessarily binding. Inserting \( \mathcal{U}(\theta) \equiv 0 \) into the maximand, this maximand is decreasing in \( \varepsilon \) and thus obtained when (3.6) is also binding. From there, optimizing yields the final results. In particular, it is trivial to check that second-period distortions and the fixed per-period profits are independent of the first-period cost. Moreover, the Inada condition \( V'(0) = +\infty \) ensures that \( x^r \) and thus \( \varepsilon^r \) are both positive. \( Q.E.D. \)

Proof of Corollary 1: From (4.13), (4.14) and the definition of \( \zeta^*(\cdot) \), we first get:

\[ -\frac{1 - \delta}{\delta} u^r = \zeta^*(\varepsilon^r) \geq 0, \quad \forall \theta \in \Theta, \]
where the inequality follows from $\varepsilon^r > 0$, $\zeta^*(0) = 0$ and the fact that Assumption 3 holds so that $\zeta^*(\cdot)$ is non-decreasing.

Second, observe that (4.14) implies:

$$y^r = \varphi \left( -\frac{1-\delta}{\delta} u^r, \varepsilon^r \right) - u^r.$$  

Using (4.13), we may rewrite this as:

$$y^r = \varphi (\zeta^*(\varepsilon^r), \varepsilon^r) + \frac{\delta}{1-\delta} \zeta^*(\varepsilon^r).$$

Let us now define $\psi(\varepsilon)$ as:

$$\psi(\varepsilon) = \varphi (\zeta^*(\varepsilon), \varepsilon) + \frac{\delta}{1-\delta} \zeta^*(\varepsilon).$$

Differentiating this yields:

$$\psi'(\varepsilon) = \left( \varphi \zeta (\zeta^*(\varepsilon), \varepsilon) + \frac{\delta}{1-\delta} \right) \dot{\zeta}^*(\varepsilon) + \varphi \varepsilon (\zeta^*(\varepsilon), \varepsilon).$$

Taking into account (2.1), we obtain:

$$\psi'(\varepsilon) = -\frac{w_{x\varepsilon}(\varphi(\zeta^*(\varepsilon), \varepsilon), \varepsilon)}{w_{zz}(\varphi(\zeta^*(\varepsilon), \varepsilon), \varepsilon)} + \frac{\delta}{1-\delta} \dot{\zeta}^*(\varepsilon).$$

From Assumption 2, the first term is positive while Assumption 3 ensures that the second term is also positive. Because $\psi(0) = 0$ and $\varepsilon^r > 0$, we deduce that:

$$y^r = \psi(\varepsilon^r) > 0.$$

\[Q.E.D.\]

**Proofs of Propositions 6, 7 and 8:** We start by proving Proposition 6 before turning our attention to Propositions 7 and 8.

When both $\theta$ and $\beta$ are private information, the principal’s (relaxed) problem becomes:

$$(\mathcal{P}^{as}) : \max_{(q(\theta), x(\theta, \beta), u(\theta), \varepsilon(\theta), U(\theta))} \int_{\theta} E_\beta (W(q(\theta), x(\theta, \beta), u(\theta), \varepsilon(\theta), U(\theta))) f(\theta)d\theta$$

subject to (3.6)-(3.9)-(3.11).

First, observe that the second-period incentive constraint (3.6) is necessarily binding at the optimum of $(\mathcal{P}^{as})$ so that $\varepsilon(\theta) = \Delta \beta x(\theta, \beta)$. Moreover, it should also be clear that optimizing with respect to $x(\theta, \beta)$ immediately gives $x^{as}(\theta, \beta) = x^*(\beta)$ for all $\theta$. We may simplify the expression of the principal’s second-period payoff accordingly:

$$\nu(V(x^*(\beta)) - \beta x^*(\beta)) + (1-\nu) \left( V \left( \frac{\varepsilon(\theta)}{\Delta \beta} \right) - \beta \varepsilon(\theta) \right) - \varphi \left( \frac{U(\theta) - (1-\delta)u(\theta)}{\delta}, \varepsilon(\theta) \right).$$
Equipped with this expression, and denoting by $\lambda$ the costate variable for (3.11) we now write the Hamiltonian for the problem $\mathcal{P}^{as}$ as:

$$
\mathcal{H}(q, u, \varepsilon, \mathcal{U}, \lambda, \theta) =
\begin{multline}
\begin{aligned}
&f(\theta)
\left(S(q) - \theta q - (1 - \delta)u - \delta \varphi \left(\frac{U - (1 - \delta)u}{\delta}, \varepsilon\right) + \nu \delta \left(V(x^*(\overline{\beta})) - \beta x^*(\overline{\beta})\right) + (1 - \nu)\delta \left(V\left(\frac{\varepsilon}{\Delta \beta}\right) - \overline{\beta} \frac{\varepsilon}{\Delta \beta}\right)\right) \\
- &\lambda q \left(1 - \delta + \delta w_z \left(\varphi \left(\frac{U - (1 - \delta)u}{\delta}, \varepsilon\right), \varepsilon\right)\right).
\end{aligned}
\end{multline}
$$

We shall assume that $\mathcal{H}(q, u, \varepsilon, \mathcal{U}, \lambda, \theta)$ is concave in $(q, u, \varepsilon, \mathcal{U})$ and use the Pontryagin Principle to get necessary and sufficient conditions for the optimum (see Chapter 2, Theorems 2 and 4 in Seierstad and Sydsaeter (1987)). These necessary and sufficient conditions are listed below. It can be checked that concavity is ensured provided that $\delta$ is small enough and $V(\cdot)$ is sufficiently concave. (Proof available upon request.)

- **Costate variable.** The costate variable $\lambda(\theta)$ is continuous, differentiable and such that:

$$
\dot{\lambda}(\theta) = -\frac{\partial \mathcal{H}}{\partial \mathcal{U}}(q(\theta), u(\theta), \varepsilon(\theta), \mathcal{U}(\theta), \theta) \iff 
\begin{multline}
\dot{\lambda}(\theta) = 
\left(f(\theta) + \lambda(\theta)q(\theta)w_{zz} \left(\varphi \left(\frac{U(\theta) - (1 - \delta)u(\theta)}{\delta}, \varepsilon(\theta)\right), \varepsilon(\theta)\right)\right) \varphi \left(\frac{U(\theta) - (1 - \delta)u(\theta)}{\delta}, \varepsilon(\theta)\right).
\end{multline}
$$

- **Transversality condition.** Because (3.9) is necessarily binding at the optimum (otherwise, the principal’s payoff could be improved by reducing $\mathcal{U}(\overline{\theta})$ by a small amount while still respecting the participation constraint for all types), the transversality condition writes as:

$$
\lambda(\theta) = 0.
$$

- **Optimality condition with respect to $u$.** The first-order condition with respect to $u$ writes as:

$$
\begin{multline}
\begin{aligned}
f(\theta) = \varphi \left(\frac{U(\theta) - (1 - \delta)u(\theta)}{\delta}, \varepsilon(\theta)\right) \left(f(\theta) + \lambda(\theta)q(\theta)w_{zz} \left(\varphi \left(\frac{U(\theta) - (1 - \delta)u(\theta)}{\delta}, \varepsilon(\theta)\right), \varepsilon(\theta)\right)\right) \varphi \left(\frac{U(\theta) - (1 - \delta)u(\theta)}{\delta}, \varepsilon(\theta)\right).
\end{aligned}
\end{multline}
$$

- **Optimality condition with respect to $q$.** The first-order condition with respect to $q$ writes as:

$$
S'(q(\theta)) = \theta + \frac{\lambda(\theta)}{f(\theta)} \left(1 - \delta + \delta w_z \left(\varphi \left(\frac{U(\theta) - (1 - \delta)u(\theta)}{\delta}, \varepsilon(\theta)\right), \varepsilon(\theta)\right)\right).
$$
• Optimality condition with respect to $\varepsilon$. The first-order condition with respect to $\varepsilon$ writes as:

\[(A.7) \quad \frac{1 - \nu}{\Delta \beta} \left( V' \left( \frac{\varepsilon(\theta)}{\Delta \beta} \right) - \beta \right) = \varphi_{\varepsilon} \left( \frac{U(\theta) - (1 - \delta)u(\theta)}{\delta}, \varepsilon(\theta) \right) \]

\[\quad + q(\theta) \lambda(\theta) \left( w_{zz} \left( \varphi \left( \frac{U(\theta) - (1 - \delta)u(\theta)}{\delta}, \varepsilon(\theta) \right), \varepsilon(\theta) \right) \varphi_{\varepsilon} \left( \frac{U(\theta) - (1 - \delta)u(\theta)}{\delta}, \varepsilon(\theta) \right) \right) \]

\[\quad + w_{\varepsilon} \left( \varphi \left( \frac{U(\theta) - (1 - \delta)u(\theta)}{\delta}, \varepsilon(\theta) \right), \varepsilon(\theta) \right) \right). \]

We now use these optimality conditions to derive more specific results.

• Proposition 6. Inserting (A.5) into (A.3) and simplifying yields:

\[\dot{\lambda}(\theta) = f(\theta),\]

which together with (A.4) yields:

\[(A.8) \quad \lambda(\theta) = F(\theta).\]

Inserting this expression into (A.5), taking into account that $\varphi_{\varepsilon}(z, \varepsilon) = \frac{1}{w_{z}(\varphi(z, \varepsilon), \varepsilon)}$, the definition

\[(A.9) \quad u^{as}(\theta) + y^{as}(\theta) = \varphi \left( \frac{U^{as}(\theta) - (1 - \delta)u^{as}(\theta)}{\delta}, \varepsilon(\theta) \right) \]

and simplifying yields (5.2). The inequality (5.3) then immediately follows from Assumption 2.

• Proposition 7. Inserting (A.8) into (A.5) and simplifying using (A.9) gives us (5.4). The inequality (5.5) immediately follows from Assumption 2.

From (5.2), we know that $w_{z}(u^{as}(\theta) + y^{as}(\theta), \varepsilon^{as}(\theta)) \leq 1$. Therefore, (5.4) implies that

\[S'(q^{as}(\theta)) \leq \theta + \frac{F(\theta)}{f(\theta)}, \quad \forall \theta \in \Theta,\]

and thus $q^{as}(\theta) \geq q^{bm}(\theta)$.

• Proposition 8. Inserting (A.8) into (A.7) and simplifying using (A.9) gives us (5.6). The inequality (5.7) immediately follows from Assumption 3.

Q.E.D.

Proof of Proposition 9: To check whether a long-term contract $C$ (and in particular the optimal contract under full commitment, $C^{as}$) is renegotiation-proof when the firm truthfully reveals its type in the first period (i.e., the first-period strategy $M(\cdot | \theta)$ puts unit mass on $\hat{\theta} = \theta$) we first look for the optimal continuation for the second period.
The first step is to observe that (6.1) becomes:

\[(A.10) \quad w(t(\theta) + y(\theta) + \tilde{y}(\theta) - \theta q(\tilde{\theta}), \tilde{\varepsilon}(\theta)) \geq w(t(\theta) + y(\theta) - \theta q(\theta), \varepsilon(\theta)), \quad \forall \theta \in \Theta.\]

The principal’s problem at the renegotiation stage can thus be written as follows:

\[(P^r_e) : \max_{(\tilde{x}(\theta, \beta), \tilde{\varepsilon}(\theta), \tilde{y}(\theta))} E_\beta \left( V(\tilde{x}(\theta, \beta)) - \beta \tilde{x}(\theta, \beta) \right) - \tilde{y}(\theta) \text{ subject to (3.6)-(A.10)}.\]

Observe that the participation constraint (A.10) is necessarily binding. Denoting again the fixed per-period profit in the long-term contract as 

\[u(\theta) = t(\theta) - \theta q(\theta)\]

and the firm’s reservation payoff for the second period as 

\[w_0(\theta) = w(u(\theta) + y(\theta), \varepsilon(\theta)),\]

we can thus write:

\[(A.11) \quad u(\theta) + y(\theta) + \tilde{y}(\theta) = \phi(w_0(\theta), \tilde{\varepsilon}(\theta)), \quad \forall \theta \in \Theta.\]

Inserting this expression of \(\tilde{y}(\theta)\) into the maximand of \((P^r_e)\) and optimizing with respect to \(\tilde{\varepsilon}(\theta)\), we get that (3.6) is necessarily binding. The last step of the optimization gives us 

\[x^r_e(\theta, \beta) = \tilde{x}(\theta, \beta) \text{ for all } \theta \in \Theta.\]

When the realized second-period cost is \(\beta\), the distortion of the add-on is given by:

\[(A.12) \quad (1 - \nu)(V'(x^r_e(\theta, \beta)) - \beta) = \Delta \beta \varphi_\varepsilon(w_0(\theta), \varepsilon^r(\theta)), \quad \forall \theta \in \Theta.\]

If \(C^{as}\) is renegotiation-proof, we should have:

\[w_0(\theta) = w(u^{as}(\theta) + y^{as}(\theta), \varepsilon^{as}(\theta)), \quad x^r_e(\theta, \beta) = x^{as}(\theta, \beta) \text{ and } \tilde{y}(\theta) = 0, \quad \forall \theta \in \Theta.\]

Inserting this into (A.12) yields:

\[(A.13) \quad (1 - \nu)(V'(x^{as}(\theta, \beta)) - \beta) = \Delta \beta \varphi_\varepsilon(w(u^{as}(\theta) + y^{as}(\theta), \varepsilon^{as}(\theta)), \varepsilon^{as}(\theta)), \quad \forall \theta \in \Theta.\]

Comparing this to (5.6), we can conclude that whenever (6.3) holds \(C^{as}\) is not renegotiation-proof.

On the other hand, for CARA preferences, \(H(z, \varepsilon) \equiv 0\) and thus the long-term contract \(C^{as}\) is renegotiation-proof.

\[Q.E.D.\]

**Proof of Proposition 10:** The proof follows directly from Section 6. First Proposition 9 shows that spot contracting under CARA yields \(x^{as}(\theta, \beta)\) as the effective level of the add-on. In the main text we also show that the first-period incentive compatibility constraint when anticipating the continuation contract for the add-on is the same incentive constraint as in the full commitment long-term contract and (6.4) tells us that no additional rent is left to the firm.

\[Q.E.D.\]
Proof of Propositions 11 and 12: Given the type distribution \( G(\cdot) \) of the winning firm’s bid, the principal’s intertemporal payoff when dealing with this firm can be written as:

\[
W(u(\theta), U(\theta)) = S - \theta + \delta(V - E(\beta)) - (1-\delta)u(\theta) - \delta \varphi \left( \frac{U(\theta)}{(1-F(\theta))^n} - \frac{(1-\delta)u(\theta)}{\delta}, \Delta \beta \right).
\]

The (relaxed) problem with this representative firm can now be written as follows:

\[
(P^{as}) : \max_{(u(\theta),U(\theta))} \int_0^\beta W(u(\theta),U(\theta))g(\theta)d\theta \text{ subject to (7.2)-(7.3)}.
\]

Equipped with this expression, and denoting by \( \lambda \) the costate variable for (7.2) we can now write the Hamiltonian for the problem \((P^{as})\) as:

\[
H(u,U,\lambda,\theta) = g(\theta)W(u(\theta),U(\theta)) - \lambda(1-F(\theta))^n (1-\delta + \delta w_z) \left( \varphi \left( \frac{U(\theta)}{(1-F(\theta))^n} - \frac{(1-\delta)u(\theta)}{\delta}, \Delta \beta \right), \Delta \beta \right).
\]

Since \( H(u,U,\lambda,\theta) \) is concave in \((u,U)\), we can use the Pontryagyn Principle to get necessary and sufficient conditions for the optimum These necessary and sufficient conditions are listed below.

- **Costate variable.** There exists a continuous and differentiable \( \lambda \) such that:

\[
(\text{A.14}) \quad \dot{\lambda}(\theta) = (n+1)f(\theta) + \lambda(\theta)w_{zz} \left( \varphi \left( \frac{U(\theta)}{(1-F(\theta))^n} - \frac{(1-\delta)u(\theta)}{\delta}, \Delta \beta \right), \Delta \beta \right) - \varphi \left( \frac{U(\theta)}{(1-F(\theta))^n} - \frac{(1-\delta)u(\theta)}{\delta}, \Delta \beta \right).
\]

- **Transversality condition.** Because (3.9) is necessarily binding at the optimum, the transversality condition is still given by (A.4).

- **Optimality condition with respect to \( u \).** Using the first-order condition, we find:

\[
(\text{A.15}) \quad 1 = \varphi \left( \frac{U(\theta)}{(1-F(\theta))^n} - \frac{(1-\delta)u(\theta)}{\delta}, \Delta \beta \right) \left( 1 + \frac{\lambda(\theta)}{(n+1)f(\theta)}w_{zz} \left( \varphi \left( \frac{U(\theta)}{(1-F(\theta))^n} - \frac{(1-\delta)u(\theta)}{\delta}, \Delta \beta \right), \Delta \beta \right) \right).
\]

We now use these optimality conditions to derive more specific results.

\[\text{It is straightforward to show that } H(u,U,\theta) \text{ is concave in } (u,U) \text{ if and only if it is concave in } U. \]

The condition for concavity in \( U \) simplifies to

\[
1 \geq \frac{\lambda(\theta)}{(n+1)f(\theta)}w_{zz} \left( \varphi \left( \frac{U(\theta)}{(1-F(\theta))^n} - \frac{(1-\delta)u(\theta)}{\delta}, \Delta \beta \right), \Delta \beta \right), \text{ which always holds given the value of } \lambda(\theta) \text{ found below and given that } w_{zz} \leq 0.
\]
• Proposition 11. Inserting (A.15) into (A.14) yields \( \dot{\lambda}(\theta) = (n + 1)f(\theta) \). Taking into account (A.4) yields \( \lambda(\theta) = (n + 1)F(\theta) \). Inserting this expression into (A.15), and simplifying yields (7.4).

• Proposition 12. We follow Seierstad and Sydsaeter (1985, p.145) to get the following optimality condition with respect to \( \tilde{\theta} \):

(A.16) \( \mathcal{H}(u(\tilde{\theta}), U(\tilde{\theta}), \lambda(\tilde{\theta}), \tilde{\theta}) = 0. \)

Taking into account (7.3), this optimality condition can be written as:

\[
S + \delta(V - E_{\beta}(\beta)) = \tilde{\theta} + \frac{F(\tilde{\theta})}{f(\tilde{\theta})} + \delta \left(u(\tilde{\theta}) + y(\tilde{\theta}) - w(u(\tilde{\theta}) + y(\tilde{\theta}), \Delta \beta) + \frac{F(\tilde{\theta})}{f(\tilde{\theta})}(w(u(\tilde{\theta}) + y(\tilde{\theta}), \Delta \beta) - 1)\right).
\]

Mutatis mutandis, we can also use these conditions when \( \beta \) is common knowledge. It is enough to replace \( \Delta \beta \) by 0 in (A.17) to get the following expression for \( \theta^i \):

(A.18) \( S + \delta(V - E_{\beta}(\beta)) = \theta^i + \frac{F(\theta^i)}{f(\theta^i)} + \delta \left(u(\theta^i) + y(\theta^i) - v(u(\theta^i) + y(\theta^i)) + \frac{F(\theta^i)}{f(\theta^i)}(v'(u(\theta^i) + y(\theta^i)) - 1)\right). \)

Define \( \mu(\theta) \equiv u^i(\theta) + y^i(\theta) \). From (7.4), we know that \( \mu(\theta) \) solves:

(A.19) \( v'(\mu(\theta)) = 1 + \frac{F(\theta)}{f(\theta)}v''(\mu(\theta)). \)

We will use the function \( \mu(\theta) \) to define \( J(\theta) \) as:

(A.20) \( J(\theta) = \mu(\theta) - v(\mu(\theta)) + \frac{F(\theta)}{f(\theta)}(v'(\mu(\theta)) - 1). \)

First observe that from (7.4) and the normalizations of \( v(\cdot) \), \( J(\theta) = 0 \). Second, differentiating and taking into account (A.19) yields:

(A.21) \( \dot{J}(\theta) = \frac{d}{d\theta} \left( \frac{F(\theta)}{f(\theta)} \right) \frac{F(\theta)}{f(\theta)}v''(\mu(\theta)) \leq 0, \)
where the last inequality follows from Assumption 1. From this, it follows that $J(\tilde{\theta}^i) < 0$ when $\tilde{\theta}^i > \theta$. Inserting into (A.17), we deduce that:

$$S + \delta(V - E_\beta(\beta)) < \tilde{\theta}^i + \frac{F(\tilde{\theta}^i)}{f(\tilde{\theta}^i)},$$

which gives us (7.6).

From Assumption 2, we have:

$$-w(u(\theta) + y(\theta), \Delta \beta) + \frac{F(\theta)}{f(\theta)} w_z(u(\theta) + y(\theta), \Delta \beta) \geq -v(u(\theta) + y(\theta)) + \frac{F(\theta)}{f(\theta)} v'(u(\theta) + y(\theta)).$$

Using this for $\theta = \tilde{\theta}^{\text{pos}}$ and inserting it into (A.17) yields:

(A.22)

$$S + \delta(V - E_\beta(\beta)) \geq \tilde{\theta}^1 + \frac{F(\tilde{\theta})}{f(\tilde{\theta})} + \delta \left( u(\tilde{\theta}) + y(\tilde{\theta}) - v(u(\tilde{\theta}) + y(\tilde{\theta})) + \frac{F(\tilde{\theta})}{f(\tilde{\theta})} (v'(u(\tilde{\theta}) + y(\tilde{\theta})) - 1) \right),$$

which implies (7.7). Q.E.D.