Bounds on Price Setting

Narayana R. Kocherlakota*

July 2018

University of Rochester and NBER

Abstract

In macroeconomic models, price-setting firms are free to choose any positive price and so have non-compact action sets. In the first part of this paper, I first show that this non-compactness induces a powerful, but unrealistic, equilibrium selection mechanism that underlies core conclusions in macroeconomic theory. I then study a class of infinite horizon models in which all firms can costlessly choose any price within a common compact interval that is indexed to last period's price level. These models have a host of stochastic equilibria. I describe how the central bank can or cannot select among these equilibria using interest rate rules. In particular, I show that, regardless of how loose the price-setting bounds are, secular stagnation is an unavoidable risk: for *any* interest rate rule, there is a set of equilibria in which the output gap is negative for a long - possibly infinitely long - period of time.

*This is a much revised version of a paper that previously circulated under the titles, "Monetary Economies with Bounded Competition," and "The L-Shaped Phillips Curve: Theoretical Justification and Empirical Implications" (NBER WP 24086). I thank Marco Bassetto, Mariacristina deNardi, Martin Eichenbaum, Greg Kaplan, Chen Kan, Matteo Maggiori, Stephen Morris, Ivan Werning, and participants in seminars at Chicago-Booth, Cornell University, the European Central Bank, the Federal Reserve Bank of San Francisco, the London School of Economics, Northwestern University, University College-London, and Yale University for valuable comments.

1 Introduction

A key presumption in macroeconomics is that in the long run, the behavior of real variables like output and consumption is largely independent of the behavior of nominal variables like inflation (Friedman (1968)). In this paper, I argue that the usual theoretical justification for this presumption is flawed because it is based on models in which the key actors - price-setting firms - have non-compact action sets. In these models, long-run outcomes are discarded as equilibria for an uncompelling reason: they induce a race among firms to lower or raise prices as much as possible and the models - with their non-compact action sets - are uninformative about how these price races get resolved.[1] To fix this problem, I instead consider a class of models in which all firms are able to adjust prices costlessly at each date within a closed and bounded interval. Regardless of how loose these price-setting bounds are, the resultant models imply that the long-run behavior of real economic variables is shaped in important ways by the combined effects of monetary policy and self-fulfilling beliefs about inflation.

The details are as follows. The paper has two parts. In the first part of the paper, I analyze a class of two-period monetary models with monopolistically competitive firms. The firms can choose any positive price, so that they are playing a game in which they all have non-compact action sets. I show that this non-compactness gives rise to a powerful, but unrealistic, form of equilibrium selection.

The relevant problem can be illustrated via the following two-period game. In period 2, two players C and D simultaneously choose actions c and d from the set $\{0, 1\}$. They both get the payoff cd if their choices are the same and get the payoff 0 otherwise. In period 1, two players A and B simultaneously choose their actions a and b from the non-negative reals. At the end of period 2, A and B receive the payoff $(a+1)(b+1)c + ab(c-1)$.

What are the equilibria to this game? Suppose that, in period 1, players A and B believe that player C will choose 1 in period 2. Then A and B are playing a game in which neither

[1] Asen Kochov pointed out to me that this argument is reminiscent of Jackson's (1992) critique of the use of unbounded mechanisms in implementation theory. Bassetto and Phelan (2015) also criticize the use of non-compact action sets in macroeconomic models.

player's optimization problem has a solution. There is no equilibrium to this game. So, the unique equilibrium to this two-period game is that all players choose 0.

But the *predictive relevance* of this unique equilibrium is, at best, highly questionable. *Why would players C and D ever co-ordinate in period 2 on playing (the weakly dominated strategy of) 0?* The more compelling prediction is that players C and D would co-ordinate on 1, and that players A and B would respond in period 1 by choosing a very large number. But this logic reveals the model's key weakness: because of the non-compact nature of the action sets of A and B, the model provides no guidance about what their choices might be.

The point of the first part of this paper is that models without price floors or ceilings have this same defect. The real interest rate in a given period depends on future policy choices and equilibrium actions. If the real interest rate is "too high" , then the price-setting firms try to raise demand by lowering prices as much as possible. If the real interest rate in a given period is "too low", then the price-setting firms try to lower demand by raising prices as much as possible. The non-compact action sets mean that "as much as possible" is not well-defined. It is typical for modelers to respond to this situation by specifying that the future outcomes associated with overly high or low real interest rates cannot be part of an equilibrium. But the right conclusion is that, just as in the above example game, the model provides insufficient guidance about what will happen in these eventualities if they were to occur.

With that conclusion in mind, in the second part of the paper, I study the implications of a class of infinite-horizon models in which all firms can costlessly adjust their prices, relative to last period's price level, within a compact interval. The model is non-stochastic and stationary, so that there is a constant *natural* real interest rate r^* that would characterize equilibria if the model were moneyless. I treat fiscal policy as Ricardian as in Woodford (1995) (or passive as in Leeper (1991)), so that it plays no role in price level determination. Even though the fundamentals in the economy are non-stochastic, there are many stochastic equilibria. I focus on equilibria in which output is uniformly bounded from above (which ensures that firm profits don't become arbitrarily negative over time). I define the output gap, at any date and state, to be the negative difference between logged marginal utility in that date and state and its counterpart in the constant moneyless equilibrium.[2]

I obtain three sets of results. First, in contrast to Cochrane (2011), **the central bank can ensure that inflation does not get too high**. In particular, suppose it uses a nominal interest rate rule that satisfies the Taylor Principle (so that the nominal interest rate responds more than one for one to inflation) when the inflation rate is above some target. I show that, in any equilibrium, inflation is bounded from above by the target and the output gap is non-positive in all dates and states.

Second, assuming that the central bank uses a rule of the above form, **the Phillips Curve has the same L-shape both within and across equilibria**. If the output gap is negative, then the curve is flat in the sense that inflation is at its lower bound. If the output gap is zero, then the curve is vertical, in the sense that inflation can take on any value between its upper and lower bounds.

Finally, **the central bank cannot keep the output gap from being persistently negative.** I show that for any monetary policy rule and for any horizon, there is a set of equilibria in which the output gap is negative over that horizon. Even more strongly, suppose the monetary policy rule implies that, when inflation is permanently at its lower bound, the resulting real interest rate is *no larger than* the constant natural real interest rate r^*. Then, given any $B < 0$, there is a set of equilibria in which the output gap is always less than B.

How can the output gap be negative for so long? In this class of economies, households hold money in order to pay their taxes. If expectations about future inflation and/or output are low, then households demand relatively little consumption. Faced with this low level of demand, firms always find it optimal to cut prices. This situation wouldn't be an equilibrium in a conventional model with non-compact price sets. In this model, the incentive to keep cutting prices instead serves to drive inflation down to its lowest possible level. The low

[2]The moneyless equilibrium features inefficiently low production because of the monopolistic distortion. In this sense, a positive output gap is actually desirable within the model.

inflation-low output outcome is self-fulfilling, because it feeds back to create low consumption demand in earlier periods.

This is a theory paper. But the results about negative output gaps do suggest a possible way to think about a broad swath of macroeconomic data. Over the past decade, around much of the developed world, inflation has been running below the central banks' targets. At the same time, inflation has also been surprisingly insensitive to usual measures of the output gap. Real output is currently (2018) much lower than was anticipated a dozen years ago, suggesting that the output gap may have fallen permanently.

These observations seem puzzling when viewed through the lens of standard monetary models.[3] But they become completely natural once one realizes that such models are incompletely specified in the absence of pricing bounds. If firms face a lower bound to price-setting, then the central bank has no ability to rule out outcomes in which the inflation rate is low and stable, the output gap is negative, firms' profits shares are high, and the real interest rate is low over long periods of time.

I close the introduction with a final comment about the role of money in the models studied in this paper. Money has no transaction role. (It does serve as a unit of account, because firms denominate their prices in units of money.) In each period, agents' consumption spending equals their wage income and their share of firms profits. They hold money only to pay their taxes. Accordingly, the Friedman Rule is always satisfied: the risk-adjusted real rate of return on money is the same as that on any other asset. The point of this paper is that, even though the Friedman Rule is always satisfied, money can be highly distortionary in this economy. When its anticipated real return is too high, the agents in the economy demand relatively little consumption. Firms bid down their prices and inflation to its lowest possible level, so that the beliefs about money's high real return become self-fulfilling.

[3] See Cochrane (2017).

2 Why Compactness is Necessary

In this section, I illustrate through a two-period example why it is necessary for price-setting firms in monetary models to have compact choice sets. I first consider a (standard) flexible price model in which firms can choose any positive price. In this model, the government imposes a lump-sum tax in period 2 equal to the average amount of money outstanding. As a result, any rate of inflation is an equilibrium in period 2. However, the anticipation of many (possibly almost all) of these period 2 equilibria implies that the price-setting firms want to cut or raise prices as much as possible in period 1. Since "as much as possible" is not well-defined given the firms' non-compact action sets, we have to impose an artificial restriction on the set of period 2 equilibrium outcomes to ensure existence in the overall dynamic economy.

My baseline analysis is for a model with flexible prices. However, in Appendix A, I show that the unrealistic equilibrium selection mechanism carries over to settings in which almost all prices are fixed. I also show that it generalizes to settings in which markets for risk sharing are incomplete, fiscal policy is being used for price level determination, some money is held only for its liquidity services, and agents are able to store resources.

The artificial equilibrium selection is being driven by the non-compactness of the firms' action sets. I eliminate the problem by restricting the firms' choices of prices to lie in a compact interval. Given this restriction, I demonstrate that there is a dynamic equilibrium for any period 2 equilibrium.

2.1 Two Period Example Setup

There are two periods and a unit measure of agents who all live for two periods. There is also a unit measure of goods in period 1 and a single good in period 2. The agents maximize the expectation of a cardinal utility function of the form:

$$u((\int c_1(j)^{1-1/\eta}dj)^{\frac{\eta}{\eta-1}}) - v(N_1) + u(C_2)$$

Here, $c_1(j)$ is consumption of good j in period 1, C_2 is consumption of the single good in period 2, and N_1 is labor in period 1. The utility function u satisfies typical restrictions:

$$u', -u'', v', v'' > 0$$

$$lim_{c\to 0}u'(c) = \infty$$

$$lim_{c\to\infty}u'(c) = 0$$

As is conventional, I restrict $\eta > 1$ (in order to ensure that monopolistically competitive firms have finite solutions to their maximization problems).

In period 2, the agents are each endowed with Y units of consumption.

In period 1, each good is produced by a monopolistically competitive firm that treats the wage as exogenous. Each firm has identical constant returns to scale technologies that transform a measure n units of time in period 1, $n \geq 0$, into n units of consumption goods. The agents have equal ownership of all firms. New firms are not allowed to enter and (perhaps less intuitively) existing firms are not allowed to exit.

I treat money as an interest-bearing asset (akin to the interest-bearing reserves that banks hold with the Federal Reserve). Each person is endowed with M dollars of money in period 1. The government commits to an interest rate rule: in period 2, money pays a gross nominal interest rate $R(P_1)$, where R is a continuous function of the period 1 price level P_1. In terms of fiscal policy, all agents are required to pay a lump-sum tax of $MR(P_1)$ dollars in period 2.

2.2 Equilibrium and Equilibrium Selection

In this subsection, I discuss and characterize two distinct notions of equilibrium. In the first, I consider the set of equilibria in a static economy defined by period 2. In the second, I consider equilibria in the full dynamic (two-period) economy. I show that the latter equilibrium concept is a refinement or selection, in the sense that not all period 2 equilibria are associated with a dynamic equilibrium. I explain why this selection should be viewed as highly artificial.

2.2.1 Equilibrium in Period 2

Suppose that the economy were to start in period 2. Households begin the period with $MR(P_1^*)$ units of money, trade money and consumption in a competitive market, and then pay taxes. Given a period 2 price level P_2, the generic household's problem is:

$$max_{c_2, M_2} u(C_2)$$

$$s.t. P_2C_2 + M_2 \leq P_2Y + MR(P_1^*)$$

$$M_2 \geq MR(P_1^*)$$

where P_1^* is the period 1 price level. The last constraint is necessary to ensure that the household has enough money at the end of the period to pay its taxes. An equilibrium in period 2 is a specification of (P_2^*, C_2^*, M_2^*) such that (C_2^*, M_2^*) solve the household's problem and markets clear ($M_2^* = MR(P_1^*)$ and $C_2^* = Y$).

It is straightforward to show that, for any P_2, it's optimal for households to set $M_2 = MR(P_1^*)$ and to set $C_2 = Y$. Given these choices, markets clear. It follows that any positive real P_2 is an equilibrium in period 2 and that any gross inflation rate $\Pi = P_2/P_1^*$ can occur in a period 2 equilibrium.

2.2.2 Dynamic Equilibrium

Now, we move back in time to period 1 and consider equilibrium in the overall dynamic two-period economy.

It is straightforward to show that, in period 1, all firms face the same problem, given a nominal wage W^*:

$$max_{P_1 \in (0,\infty)}(P_1^{1-\eta} - W^* P_1^{-\eta})$$

They earn equilibrium nominal profits:

$$\Phi^* = (P_1^* - W^*)N_1^*$$

Note that the firms are allowed to choose any positive price.

Households in period 1 correctly expect that the equilibrium period 2 price level P_2^* will equal the equilibrium period 1 price level P_1^* multiplied by (an endogenously determined) gross inflation rate Π^*. Given these expectations, they trade money, consumption, and labor in period 1. The households' money-consumption-labor problem in period 1 is then:

$$\begin{aligned} & max_{(c_1,c_2,n_1,M_1')} u(C_1) - v(N_1) + u(C_2) \\ s.t. & P_1^* C_1 + M_1' = W^* n_1 + M + \Phi^* \\ & \Pi^* P_1^* C_2 = M_1' R(P_1^*) - M R(P_1^*) + \Pi^* P_1^* Y \\ & C_1, C_2, M_1' \geq 0 \end{aligned}$$

where P_1^* is the period 1 price level, W^* is the period 1 wage (in terms of dollars), and Φ^* are the firms' monopoly profits in dollars.

We can then define a dynamic equilibrium in this economy to be a specification of $(P_1^*, \Pi^*, W^*, \Phi^*, C_1^*, N_1^*, C_2^*, M_1^*)$ such that:

- $(C_1^*, C_2^*, N_1^*, M_1^*)$ solves the household's problem, given $(P_1^*, W_1^*, \Phi^*, \Pi^*)$.

- P_1^* solves the firm's problem given W^*, so that $P_1^* = (1 - 1/\eta)^{-1} W^*$ and $\Phi^* = (P_1^* - W^*) N_1^*$.
- Markets clear, so that $C_1^* = N_1^*$; $M_1^* = M$; and $C_2^* = Y$.

We have seen that, if the economy only consisted of period 2, any gross inflation rate is part of an equilibrium. The next proposition provides conditions under which a gross inflation rate Π^* is part of a dynamic equilibrium. I define C^{mono} to be the solution to:

$$(1 - 1/\eta) = v'(C^{mono})/u'(C^{mono})$$

Proposition 1. *Given a monetary policy rule R and a period 2 gross inflation rate Π^*, there exists a dynamic equilibrium with that period 2 inflation rate if and only if there exists some P_1^* such that:*

$$u'(C^{mono}) = \frac{R(P_1^*)}{\Pi^*} u'(Y)$$

Proof. In any equilibrium, the households satisfy their intratemporal consumption-labor first order conditions:

$$u'(C_1^*) W^* = v'(C_1^*) P_1^*$$

where we've exploited market-clearing to substitute out for consumption. We can then use the firms' first order conditions to conclude that:

$$C_1^* = N_1^* = C^{mono}$$

where C^{mono} is defined as in the Proposition. The Proposition then follows from the households' consumption-money Euler equation. □

Proposition 1 shows that the real interest rate is pinned down in a monetary equilibrium by the households' marginal willingness to hold money in a monetary equilibrium. That real interest rate in turn determines the price level in period 1.

Proposition 1 implies that the set of dynamic equilibrium outcomes is a selection from the set of period 2 equilibrium outcomes described in the prior subsection. For example, suppose $R_{min} \leq R(P_1) \leq R_{max}$ for all P_1. Consider any Π^* such that:

$$u'(C^{mono})\Pi^*/u'(Y) < R_{min}$$

or such that:

$$u'(C^{mono})\Pi^*/u'(Y) > R_{max} \tag{1}$$

Then, given this interest rate rule R, Π^* is not part of a dynamic equilibrium.

2.2.3 Artificial Equilibrium Selection

In this subsection, I explain why we should view the refinement associated with dynamic equilibrium as being unrealistic. As before, I define C^{mono} to be the level of output/labor/consumption in a purely real version of the economy:

$$u'(C^{mono})(1 - 1/\eta) = v'(C^{mono})$$

First, suppose that the period 2 gross inflation rate Π^* is such that:

$$u'(C^{mono}) < \frac{R(P_1)}{\Pi^*} u'(Y) \tag{2}$$

for all values of P_1. Consider a putative equilibrium in the price-setting game in which all firms set their prices equal to some P_1. That price results in households demands, and the firms producing, $C_1^d(P_1)$, where:

$$u'(C_1^d(P_1)) = \frac{R(P_1)u'(Y)}{\Pi^*}$$

In order to hire that many workers, the nominal wage must equal:

$$W(P_1) = P_1 v'(C_1^d(P_1))/u'(C_1^d(P_1))$$

So far, there is no reason why P_1 (and therefore Π^*) can't survive as an equilibrium. But, given $W(P_1)$, let's calculate the typical firm's marginal profit (with respect to its price choice) at that putative equilibrium price P_1. It's proportional to:

$$\begin{aligned} & P_1^{-\eta}(1-\eta) + \eta W(P_1) P_1^{-\eta-1} \\ = \; & P_1^{-\eta}(1-\eta) + \eta (W(P_1)/P_1) P_1^{-\eta} \\ < \; & P_1^{-\eta}[(1-\eta) + \eta(1-1/\eta)] \\ = \; & 0 \end{aligned}$$

The key penultimate step follows from the observation that $C^d(P_1) < C^{mono}$ and so:

$$W(P_1)/P_1 < (1 - 1/\eta)$$

When (2) is satisfied, at any putative equilibrium price P_1, the typical firm would always gain by cutting its price still further.

Similarly, suppose that Π^* is such that:

$$u'(C^{mono}) > \frac{R(P_1)}{\Pi^*} u'(Y)$$

for all P_1. Then, for any putative equilibrium price level, a typical firm would be able to increase its profits by raising its price further.

These calculations explain the nature of the refinement embedded in dynamic equilibrium. Any gross inflation rate $\Pi^* > 0$ is an equilibrium in period 2. But some specifications of the period 2 gross inflation rate Π^* imply that the price-setting firms are in a race to pick the highest possible or lowest possible price in period 1. Because the firms' action sets are non-compact, this game has no equilibria, and so we discard those specifications of Π^* as equilibria.

But, as discussed in the introduction, this approach to equilibrium selection is unrealistic. Again, suppose households believe that, in period 2, agents will co-ordinate on equilibrium inflation Π^* in period 2, where:

$$u'(C^{mono}) < \beta R(P_1)u'(Y)/\Pi^*$$

for all P_1. We've seen that, for any putative equilibrium period 1 price level, firms would gain by cutting prices. *But how exactly does the presence of these gains induce agents in the future (period 2) to co-ordinate on a different equilibrium inflation rate*? The correct prediction here is that, if it is known in period 1 that agents in period 2 will co-ordinate on Π^*, then firms in period 1 will respond by setting their prices as low as they can. The problem is that this class of models doesn't define "as low as they can" because of the lack of compactness.

2.3 Price Bounds

In this subsection, I describe and characterize *dynamic bounded equilibria.*[4]

I impose an upper bound P^{UB} and a lower bound P^{LB} on the firms' price choices. Given those constraints, a dynamic bounded equilibrium is a specification of $(P_1^*, \Pi^*, W^*, \Phi^*, C_1^*, N_1^*, C_2^*, M_1^*)$ such that:

- $(C_1^*, C_2^*, N_1^*, M_1^*)$ solves the household's problem, given $(P_1^*, W_1^*, \Phi^*, \Pi^*)$.

[4]The terminology "bounded" is admittedly incomplete. If the firms' action sets were bounded but open intervals, we'd have the same equilibrium selection problem described in the prior subsection. So it is important that the firms' action sets also be closed.

- P_1^* solves the firm's bounded problem given W^*, so that:

$$P_1^* = argmax_{P_1}(P_1^{1-\eta} - P_1^{-\eta}W^*)$$
$$s.t.\ P^{LB} \leq P_1 \leq P^{UB}$$

  and the firm's nominal profits $\Phi^* = (P_1^* - W^*)N_1^*$.

- Markets clear, so that $C_1^* = N_1^*; M_1^* = M$; and $C_2^* = Y$.

We can then characterize dynamic bounded equilibrium outcomes as follows:

Proposition 2. *Given a monetary policy rule* R*, and a period 2 gross inflation rate* Π^**, an outcome* (N_1^*, P_1^*) *is part of a dynamic bounded equilibrium outcome if and only if:*

$$u'(C_1^*) = \frac{R(P_1^*)u'(Y)}{\Pi^*}$$

and one of the three following sets of conditions are satisfied:

1. $P_1^* = P^{LB}$ *and* $C_1^* \leq C^{mono}$
2. $P_1^* = P^{UB}$ *and* $C_1^* \geq C^{mono}$
3. $P^{LB} \leq P_1^* \leq P^{UB}$ *and* $C_1^* = C^{mono}$

Proof. I first show that these cases are, in fact, equilibria. Note first that the households' Euler equation is satisfied. We can define $W^* = v'(C_1^*)P_1^*/u'(C_1^*)$ so as to ensure that the households' labor supply conditions are satisfied. So, we need only check firm optimization.

In case 1: Since $C_1^* \leq C^{mono}$, $P_1^* \geq W^*(1 - 1/\eta)^{-1}$. The typical firm's optimization problem is:

$$max_P(P/P_1^*)^{-\eta}P - W^*(P/P_1^*)^{-\eta}$$

When $P > P_1^*$, the derivative of the profit function with respect to P is:

$$\begin{aligned} & (1-\eta)P^{-\eta} + \eta W_1^* P^{-\eta-1} \\ &= P^{-\eta}[(1-\eta) + \eta(W_1^*/P_1^*)(P_1^*/P)] \\ &< P^{-\eta}[(1-\eta) + \eta(W_1^*/P_1^*)] \\ &\leq 0 \end{aligned}$$

where the last step follows from (W_1^*/P_1^*) being less than $(\eta-1)/\eta$. The optimal choice of P is $P_1^* = P^{LB}$.

In case 2: We can use the same logic as above to prove that the optimal choice of P is P^{UB}.

In case 3: All firms are making monopoly profits, and so this is an equilibrium.

Are there other equilibria? It is straightforward to show that the Euler equation has to to be satisfied in any equilibrium. If $C_1^* < C^{mono}$, then the real wage is lower than $(1-1/\eta)$, and the above argument implies that it is optimal for firms to set their price as low as possible. If $C_1^* > C^{mono}$, the above argument implies that it is optimal for firms to their price as high as possible. □

Proposition 2 describes three kinds of dynamic bounded equilibria. In all of them, while they can substitute between consumption and interest-bearing money, the households end up spending their wage income and firm profits in period 1 to buy period 1 goods. Money plays no substantive role in the economy: households simply hold their initial money-holdings M into period 2 and then use that money to pay their taxes.

In the first kind of equilibria, output is even lower than in a purely real monopolistically competitive equilibrium, because households consume $C_1^* < C^{mono}$ and work $N_1^* < C^{mono}$. Given how low real wages are, the firms would like to cut their prices still further to expand demand and production, but can't. They end up making super-monopoly profits. In the second kind of equilibria (case 2), households produce and consume more than the monopolistic level C^{mono}. Given how high real wages are, firms would gain by raising their prices still further but cannot. They make sub-monopoly profits. (Indeed, it is possible that they make negative profits because exit is barred.) The final kind of equilibria (case 3) correspond to the equilibria in a moneyless economy.

2.4 Non-Selection in Dynamic Bounded Equilibrium

We have seen that, if we focus on period 2 in isolation, any gross inflation Π^* rate is an equilibrium. In this subsection, I prove that the same is true for bounded equilibrium, as long as the interest rate rule R is continuous.

Proposition 3. *For any continuous interest rate rule R, any period 2 gross inflation rate Π^*, price upper bound P^{UB}, and price lower bound P^{LB}, there exists a dynamic bounded equilibrium with that period 2 gross inflation rate.*

Proof. If there isn't an equilibrium of the case 1 form, then:

$$u'(C^{mono}) \geq \frac{R(P^{LB})}{\Pi^*} u'(Y)$$

If there isn't an equilibrium of the case 2 form, then:

$$u'(C^{mono}) \leq \frac{R(P^{UB})}{\Pi^*} u'(Y)$$

Since R is continuous, these two inequalities imply via the intermediate value theorem that there is some P_1^* in $[P^{LB}, P^{UB}]$ such that:

$$u'(C^{mono}) = \frac{R(P_1^*)}{\Pi^*} u'(Y)$$

so that there exists some equilibrium of the case 3 form. □

There is a dynamic bounded equilibrium for all (continuous) interest rate rules and all

Π^*. Note that the proof of existence is valid regardless of how large P^{UB} is or how small P^{LB} is.

2.5 Summary

In this section, I illustrated a problem with the standard concept of monopolistically competitive equilibrium with flexible prices: the requirement of existence of a dynamic equilibrium refines future equilibrium outcomes in an artificial fashion. I showed how to fix this problem by imposing bounds on firm price-setting.

Dynamic bounded equilibrium outcomes may be even less efficient than is implied by the monopolistic distortion. The extra inefficiency is created by monetary policy that makes the real return on money overly high, and so leads households to consume too little. Within this equilibrium, households would like to work and consume more. However, a given household can only trade its labor for consumption via firms that own the means of production. Those firms can't profitably expand their scale of operation because they can't cut prices.

3 Infinite Horizon Model with Pricing Bounds

In this section, I describe an infinite horizon monetary model in which all firms can costlessly adjust prices subject to upper and lower bounds. The bounds are defined relative to the prior period's price level. Hence, they end up serving as constraints on inflation rates. I define and characterize equilibria in this economy.

3.1 Model Setup

Consider an economy with a unit measure of households who live forever. Time is discrete and the households maximize the expected value of:

$$\sum_{t=1}^{\infty} \beta^{t-1}(u(C_t) - v(N_t)), 0 < \beta < 1$$

where C_t is the consumption of a composite good in period t and N_t is labor in period t. Here, I assume that:

$$u', -u'', v', v'' > 0$$

$$lim_{c \to 0} u'(c) = \infty$$

$$lim_{c \to \infty} u'(c) = 0$$

and that the functions u, v are bounded from above and below.[5]

The composite good consists of a measure ν of consumption goods, indexed by j, and is defined as:

$$C_t = (\int_0^\nu (c(j)^{1-1/\eta} dj)^{\frac{\eta}{\eta-1}}, \eta > 1$$

Each household's consumption of each good j is bounded from below by zero.

Each consumption good j is produced by a monopolistically competitive firm. A typical firm j has a technology at each date that converts x units of labor into x units of consumption good j, for any $x \geq 0$. The households own equal shares of all firms. Firms are not allowed to exit or enter. (I'll discuss the import of these entry/exit restrictions later.)

Labor markets are competitive, and so, at each date, firms all hire workers at the same wage W_t (denominated in terms of dollars). Given that wage, firms simultaneously set prices for their consumption goods in terms of dollars. The firms' problems are identical, and so they each choose the same price P_t in equilibrium; that price is also the aggregate price level. At date t, each firm j is constrained to choose its price in the interval:

$$\pi^{UB} P_{t-1} \geq p_t(j) \geq \pi^{LB} P_{t-1}.$$

where P_{t-1} is the price index in period $(t-1)$ (so these bounds are exogenous to the firm). I define the gross inflation rate π_t as P_t / P_{t-1}.

[5] The boundedness restrictions eliminate many commonly used utility functions, including log. I'll discuss how to generalize the results to the log case later.

Monetary policy works as follows. Each household is initially endowed with $\bar{M}_0$ dollars. Like reserves at many central banks, money is interest-bearing. Specifically, at the beginning of period $(t+1)$, a household that has M_t dollars is paid $(R(\pi_t)-1)M_t$ dollars. Here, R is an exogenous rule (function) that maps period t inflation into a period t gross nominal interest rate. I assume that $R(\pi) \geq 1$ for all inflation rates π; this restriction could be endogenized by explicitly including currency in the model (as is done in the two-period model in Appendix A).

Finally, fiscal policy works as follows. The government's only liability is interest-bearing money. Let $[\bar{M}_t\}_{t=1}^{\infty}$ be an arbitrary sequence of positive real numbers that converges to zero. At each date $(t+1)$, the government levies a lump-sum tax, in dollars, equal to:

$$\tau_t(\pi_t) = (R(\pi_t) - 1)\bar{M}_t + (\bar{M}_t - \bar{M}_{t+1})$$

This tax ensures that the per-household quantity of money at the end of period t is equal to $\bar{M}_t$. Accordingly, for any sequence of price levels, the government's per-household nominal liability is converging to zero. This specification of fiscal policy ensures that it plays no role in price level determination.

3.2 Equilibrium

In this subsection, I define an equilibrium in this economy. The definition is sufficiently general to allow for sunspot equilibria, even though fundamentals are date and state invariant.

Given the exogenous interest rate rule R and the exogenous fiscal policy $\bar{M}$, an equilibrium in this economy is a stochastic vector process $(C^*, N^*, M^*, P^*, W^*)$, where (C^*, N^*, M^*) represent per-household consumption, labor, and moneyholdings and (P^*, W^*) represent price levels and wages. Given this process, it's useful to define the implied inflation, taxes, and profits as:

$$\pi_t^* = P_t^*/P_{t-1}^*$$

$$\tau_t^* = \bar{M}_t(R(\pi_t^*) - 1) + \bar{M}_t - \bar{M}_{t+1}$$

$$\Phi_t^* = (P_t^* - W_t^*)N_t^*$$

The stochastic vector process satisfies the usual equilibrium conditions. First, (C^*, N^*, M^*) solve the household's optimization problem, given prices, wages, taxes, and profits that it treats as exogenous:

$$(C^*, N^*, M^*) = argmax_{(C,N,M)} E_0 \sum_{t=1}^{\infty} \beta^{t-1}(u(C_t) - v(N_t))$$

$$s.t.\ P_t^* C_t + M_t = M_{t-1}R(\pi_{t-1}^*) + W_t^* N_t - \tau_{t-1}^* + \Phi_t^* \ \forall t \geq 1, w.p.1$$

$$C_t, M_t, N_t \geq 0$$

Second, in any date and state, P_t^* solves firm j's pricing period t problem, given W_t^* and last period's price index (which shapes the bounds):

$$P_t^* = argmax_{P_t}(P_t^{1-\eta} - W_t^* P_t^{-\eta})$$

$$s.t.\ \pi^{LB} P_{t-1}^* \leq P_t \leq \pi^{UB} P_{t-1}^*$$

Finally, markets must clear in all dates and states:

$$C_t^* = N_t^*$$

$$M_t^* = \bar{M}_t$$

In what follows, I restrict attention to equilibria that are uniformly bounded from above

- that is, equilibria for which there exists some $\bar{C}$ such that for all t and with probability one:

$$C_t^* \leq \bar{C}.$$

This restriction rules out the possibility of equilibria in which economic activity converges to infinity over time. I rule out these equilibria because along any such path, firms are making losses that converge to infinity over time (because they are producing arbitrarily high quantities while having to pay the increasingly high-consumption workers ever higher real wages).

In contrast, I don't require equilibria to be uniformly bounded away from zero. As a result, equilibrium output can converge to zero over time. Along such paths, the Inada condition on u implies that the real wage is also falling to zero, so that firms' profit shares are converging to 1. This observation suggests that there are large gains to entry in the limit. I'll examine that possibility more closely later. For now, it suffices to say that, for any firm that must respect the price bounds, the gains to entry decline to zero with aggregate output. This statement can be strengthened to apply to potential entrants who don't face the price bound if the demand elasticity η is sufficiently close to 1.

3.3 A Simple Characterization of Equilibrium

In this economy, there are three decisions that get made each period: consumption-savings, consumption-labor, and price-setting. The first decision gives rise to the familiar Euler equation that leaves households marginally indifferent between consumption and money:

$$u'(C_t^*) = \beta R(\pi_t^*) E_t(\frac{u'(C_{t+1}^*)}{\pi_{t+1}^*})$$

If we exploit goods-market clearing, the consumption-labor decision gives rise to a standard intratemporal first order condition:

$$u'(C_t^*)w_t^* = v'(C_t^*)$$

Here, w_t^* represents the period t real wage:

$$w_t^* \equiv W_t^*/P_t^*.$$

Finally, the price-setting decision on the part of the firm gives rise to the following condition:

$$P_t^* = max(\pi^{LB}P_{t-1}^*, min(\pi^{UB}P_{t-1}^*, (1-1/\eta)^{-1}W_t^*)).$$

In words, the firm follows the usual markup formula unless doing so violates the price bounds. If we divide through by P_{t-1}^*, we can rewrite this price-setting condition as:

$$\pi_t^* = max(\pi^{LB}, min(\pi^{UB}, (1-1/\eta)^{-1}w_t^*\pi_t^*))$$

By combining these conditions together, we can conclude that:

Proposition 4. *A consumption-inflation-real wage process* (C^*, π^*, w^*) *is part of an equilibrium given an interest rate rule* R *if and only if it satisfies the following restrictions with probability one in all dates:*

$$\begin{aligned} u'(C_t^*) &= \beta R(\pi_t^*)E_t(u'(C_{t+1}^*)/\pi_{t+1}^*) \\ w_t^* &= \frac{v'(C_t^*)}{u'(C_t^*)} \\ \pi_t^* &= max(\pi^{LB}, min(\pi^{UB}, (1-1/\eta)^{-1}w_t^*\pi_t^*) \end{aligned}$$

and there exists some $\bar{C}$ *such that* $Pr(C_t \leq \bar{C}$ *for all* $t)$ *equals one.*

Proof. In Appendix B. □

The one subtlety in the proof of this proposition concerns the transversality condition on moneyholdings. The proof shows that it is automatically met because of the specification of fiscal policy, the non-negative nominal interest rate, and the boundedness of the utility function (u, v).

A key benchmark is the amount of output that would be produced in a (fictitious) moneyless version of the economy (in which labor, for example, is the numeraire). It is the unique level of consumption C^{mono} that satisfies:

$$(1 - 1/\eta)u'(C^{mono}) = v'(C^{mono})$$

Note that C^{mono} is inefficiently low because of the monopolistic distortion. Given an arbitrary level of consumption/output, I define the associated *output gap* to be:

$$\Delta = ln(u'(C^{mono})/u'(C)).$$

In any equilibrium, the output gap is uniformly bounded from above.

4 Results for the Infinite Horizon Model

In this section, I describe a number of properties of equilibria to the economy set forth in the prior section.

To establish a familiar benchmark, suppose there were no pricing bounds. Then, the price-setting condition in Proposition 1 would become:

$$\pi_t^* = w_t^*(1 - 1/\eta)^{-1}\pi_t^*,$$

If we assume that π_t^* is finite and positive, we can conclude that:

$$v'(C_t^*)/u'(C_t^*) = w_t^* = (1 - 1/\eta).$$

and that, in any equilibrium, the output gap is zero in any date and state. The only remaining issue is the behavior of inflation, which is shaped by the stochastic nonlinear difference equation:

$$E_t(1/\pi_{t+1}^*) = \beta^{-1} R(\pi_t^*)^{-1}$$

As Cochrane (2011) emphasizes, even if the monetary policy rule is active (so that the derivative $R' > 1$), there are a host of possible solutions to this equation.

So, without pricing bounds, the model exhibits real determinacy but nominal indeterminacy. Not surprisingly, the pricing bounds serve to restrict the range of nominal indeterminacy. But, as we shall see in this section, the pricing bounds also greatly expand the set of possible equilibrium real outcomes.

I begin with an obvious point based on the following assumption.

Assumption 1: There exists $\pi^{TAR} \in (\pi^{LB}, \pi^{UB})$ such that**:**

$$R(\pi^{TAR})/\pi^{TAR} = 1/\beta$$

Under this assumption, there is an equilibrium in which the output gap is zero and inflation equals (the central bank's target) π^{TAR}.

Proposition 5. *Suppose the interest rate rule R satisfies Assumption 1. Then, there exists an equilibrium in which:*

$$C_t^* = C^{mono} \textit{ for all } t$$

$$\pi_t^* = \pi^{TAR} \textit{ for all } t$$

with probability one.

Proof. Let $w^* = (1 - 1/\eta)$. Then it is readily verified that (C^*, π^*, w^*) satisfy the conditions of Proposition 4. $\square$

So, it is straightforward for the central bank to choose a rule that ensures that there is an equilibrium in which the output gap is zero and inflation stays at some target level forever. But what kinds of monetary policy rules can the central bank use to rule out other, less desirable, equilibria?

4.1 Ruling Out High Inflation and Positive Output Gaps

In this subsection, I show that a central bank can eliminate high inflation and high output as equilibrium outcomes by using an active monetary policy rule when inflation is high. The logic is a familiar one. If the monetary policy rule follows the Taylor Principle, then households can only be marginally indifferent between money and current consumption if either future inflation is sufficiently higher than current inflation or future consumption is sufficiently higher than current consumption. But, if we roll them forward in time, these two implications eventually run afoul of the upper bound on inflation and/or the uniform boundedness of consumption.

I formalize this logic through the next proposition, which exploits this assumption about the interest rate rule.

Assumption 2: $R(\pi)/\pi$ is strictly increasing as a function of π for all $\pi \geq \pi^{TAR}$.

Proposition 6. *Suppose that R satisfies Assumptions 1-2. If the output gap-inflation process (Δ^*, π^*) is part of an equilibrium, then with probability one and for all t:*

$$\pi_t \leq \pi^{TAR}$$

$$\Delta_t^* \leq 0$$

Proof. In Appendix B. □

This proposition stands in contrast with the conclusions of Cochrane (2011), who argues that active monetary policy rules do not serve to eliminate high-inflation equilibria. The difference in results arises primarily from the upper bound on inflation in this model.

The following proposition shows that Assumptions 1 and 2 also ensure that the combination of a zero output gap and target inflation is an absorbing outcome.

Proposition 7. *Suppose R satisfies Assumptions 1-2 and that the output gap-inflation process (Δ^*, π^*) is part of an equilibrium. In any event such that $\Delta^*_t = 0$ and $\pi^*_t = \pi^{TAR}$, $\Delta^*_{t+s} = 0$ and $\pi^*_{t+s} = \pi^{TAR}$ for all $s \geq 1$.*

Proof. In Appendix B. □

Later, we will see that the *combination* of a zero output gap and target inflation is critical in this proposition. In particular, if the output gap is zero, but inflation is below target, then the output gap can fall over time.

4.2 The L-Shaped Phillips Curve

Recall that the *output gap* in period t is the logged difference between the marginal utility of output and its moneyless benchmark.

$$\Delta_t = ln(u'(C^{mono})/u'(C_t))$$

We've seen that when there are no pricing bounds, the output gap has to be zero. It follows that, when the output gap is negative (positive), inflation has to be at its lower (upper) bound.

Proposition 8. *Suppose the output gap-inflation process (Δ^*, π^*) is part of an equilibrium.*

Then in any date and state:

$$\Delta_t^* > 0 \Rightarrow \pi_t^* = \pi^{UB}$$

$$\Delta_t^* < 0 \Rightarrow \pi_t^* = \pi^{LB}$$

Proof. If $\Delta_t^* > 0$, then:

$$w_t^* = \frac{v'(C_t^*)}{u'(C_t^*)} > \frac{v'(C^{mono})}{u'(C^{mono})} = (1 - 1/\eta).$$

Recall from Proposition 4 that in equilibrium:

$$\pi_t^* = max(\pi^{LB}, min(\pi^{UB}, w_t^*(1 - 1/\eta)^{-1}\pi_t^*)).$$

Suppose by way of contradiction that $\pi_t^* < \pi^{UB}$. Then:

$$\pi_t^* = max(\pi^{LB}, w_t^*(1 - 1/\eta)^{-1}\pi_t^*) > \pi_t^*,$$

which is a contradiction. We can conclude that if $\Delta_t^* > 0$, then:

$$\pi_t^* = \pi^{UB}.$$

The proof of the second part of the proposition is analogous. □

If we put together Propositions 6 and 8, we can conclude that if the interest rate rule R satisfies Assumptions 1-2, then the observed Phillips curve in any equilibrium is L-shaped. In some dates and states, the output gap may be negative. In all of those dates and states, inflation is at its lower bound. In those dates and states, inflation is indeterminate. Thus, we end up with a Phillips curve that has a vertical part (with a fixed output gap of zero) and a horizontal part (with a fixed inflation rate). Proposition 6 ensures that the output gap can't be positive.

4.3 Equilibria with a Negative Output Gap

We have seen that it is possible for the central bank to use an interest rate rule that eliminates equilibria with unduly high inflation and unsustainably high economic activity. In this subsection, we will see that it will not be possible to rule out the possibility of (expectation-driven) equilibria with extended periods of low economic activity. In this model, whenever output is below C^{mono}, inflation is at its lower bound. Hence, the nature of the low-output equilibria depends on the specification of $R(\pi^{LB})$. There are three cases that are distinguished by the magnitude of the real interest rate when inflation is at its lower bound:

- High real interest rate: $\frac{R(\pi^{LB})}{\pi^{LB}} > \beta^{-1}$.
- Natural real interest rate: $\frac{R(\pi^{LB})}{\pi^{LB}} = \beta^{-1}$
- Low real interest rate: $\frac{R(\pi^{LB})}{\pi^{LB}} < \beta^{-1}$

I consider each case in turn.

4.3.1 High Real Interest Rate

The first case is that the monetary rule is such that, when inflation is at its lower bound, the real interest rate is higher than the natural real interest rate (the shadow real interest rate in a moneyless economy: $\beta^{-1} - 1$). This is the kind of situation that might emerge if the zero lower bound is a binding constraint on the set of possible monetary policy rules. The following proposition shows that the elevated real interest rate means that it is possible for the output gap to be negative for arbitrarily long periods of time.

Proposition 9. *Suppose that the interest rate rule R satisfies Assumption 1 and that R is weakly increasing and continuous for $\pi \leq \pi^{TAR}$. Suppose too that:*

$$\beta R(\pi_{LB})/\pi_{LB} > 1$$

Then, for any horizon T, there is a deterministic equilibrium in which the output gap is negative for all $t \leq T$.

Proof. In Appendix B. □

The basic idea underlying this proposition is that, when the real interest rate is stuck at too high a level, then consumption exhibits super-normal growth. This super-normal growth can't continue forever, or the output gap would violate Proposition 6 by becoming positive. But the convergence process can continue for any arbitrarily long periods of time if the initial output gap is sufficiently negative.

The equilibria described in Proposition 9 are deterministic. There is also a rich set of stochastic equilibria under the hypothesis of the proposition. It is simple to show that, in any of these equilibria, whenever the output gap $\Delta_t^* < 0$, there is a positive probability that it will return to zero in finite time. However, the next example shows that there is an equilibrium in which the output gap follows a stationary two-state Markov chain.

Example 1. Suppose that R satisfies the hypotheses of Proposition 9, and that $R(\pi)/\pi$ is strictly increasing in a neighborhood of π^{TAR}. Then, there exists π_{normal} such that:

$$\pi^{LB} < \pi_{normal} < \pi^{TAR}$$

$$\beta R(\pi_{normal})/\pi_{normal} < 1$$

There exists some constant $\bar{c} < C^{mono}$ that satisfies:

$$\beta R(\pi^{LB}) u'(C^{mono})/\pi_{normal} < u'(\bar{c})$$

Note too that:

$$\beta R(\pi_{normal})/\pi^{LB} \geq \beta R(\pi^{LB})/\pi^{LB} > 1$$

It follows that:

$$\beta R(\pi_{normal})u'(\bar{c})/\pi^{LB} > u'(C^{mono}) > \beta R(\pi_{normal})u'(C^{mono})/\pi_{normal}$$

and so there exists p_1 in the interval $(0, 1)$ such that:

$$u'(C^{mono}) = \beta R(\pi_{normal})\{p_1 u'(C^{mono})/\pi_{normal} + (1 - p_1)u'(\bar{c})/\pi^{LB}\}$$

As well:

$$\beta R(\pi^{LB})u'(\bar{c})/\pi^{LB} > u'(\bar{c}) > \beta R(\pi^{LB})u'(C^{mono})/\pi_{normal}$$

and so there exists p_2 in the interval $(0, 1)$ such that:

$$u'(\bar{c}) = \beta R(\pi^{LB})\{p_2 u'(C^{mono})/\pi_{normal} + (1 - p_2)u'(\bar{c})/\pi^{LB}\}$$

Then, we can define an equilibrium as follows. Let $C_1^* = C^{mono}$ and:

$$\text{If } C_t^* = C^{mono},\ C_{t+1}^* = C^{mono} \text{ w.p. } p_1 \text{ and } C_{t+1}^* = \bar{c} \text{ w.p. } (1 - p_1)$$
$$\text{If } C_t^* < C^{mono},\ C_{t+1}^* = C^{mono} \text{ w.p. } p_2 \text{ and } C_{t+1}^* = \bar{c} \text{ w.p. } (1 - p_2)$$

Inflation is π_{LB} if $C_t^* < C^{mono}$ and is π_{normal} otherwise. The real wage $w_t^* = v'(C_t^*)/u'(C_t^*)$. It's straightforward to verify that this specification of (C^*, π^*, w^*) satisfies the conditions of Proposition 4. ■

In this example, the output gap fluctuates stochastically between 0 and a negative number $(ln(u'(C^{mono})/u'(\bar{c}))$. The inflation rate is close to, but below, the target when the output gap is zero and equals the lower bound when the output gap is negative.

4.3.2 Natural Real Interest Rate

We now turn to the case in which the central bank ensures that, when the inflation rate is at its lower bound, the real interest rate is at its natural level of $(1/\beta) - 1$. In this case, there is a host of equilibria with constant, but arbitrarily negative, output gaps.

Proposition 10. *Suppose that:*

$$\beta R(\pi^{LB})/\pi^{LB} = 1$$

Then, for any $\bar{\Delta} < 0$, *there exists a deterministic equilibrium in which, with probability one, the output gap equals* $\bar{\Delta}$ *for all* t.

Proof. Define (C^*, π^*, w^*) as follows for all t:

$$C_t^* = u'^{-1}(exp(-\bar{\Delta})u'(C^{mono}))$$

$$\pi_t^* = \pi^{LB}$$

$$w_t^* = v'(\bar{c})/u'(\bar{c}) < (1 - 1/\eta)$$

It is readily checked that this specification satisfies the conditions of Proposition 4. □

For this policy rule, the central bank ensures that the real interest rate, when inflation equals π^{LB}, is equal to the natural real rate. As a result, the economy grows at an efficient rate while inflation remains at its lower bound. However, the output gap can be arbitrarily negative.

The equilibria in Proposition 10 are all deterministic and all uniformly bounded from below. More generally, there is a large set of stochastic equilibria in which $\pi_t = \pi^{LB}$ for all t. The following example illustrates that these stochastic equilibria need not be uniformly bounded from below.

Example 2. Let R satisfy Assumption 1 and let p be any element of $(0, 1)$. For any $\bar{c}_1 <$

C^{mono}, we can define a strictly decreasing sequence $\{\bar{c}_n\}_{n=1}^{\infty}$ to satisfy:

$$u'(\bar{c}_n) = \beta R(\pi^{LB})[pu'(C^{mono})/\pi^{TAR} + (1-p)u'(\bar{c}_{n+1})/\pi^{LB}]$$

Then, define an equilibrium as follows. In any period t, if $C_t^* < C^{mono}$, then a coin is flipped with probability p of coming up heads. If the coin comes up heads, then the economy transits permanently to an equilibrium in which consumption equals C^{mono} and inflation equals π^{TAR}. If the coin comes up tails, then consumption falls to $\bar{c}_{t+1}$ and inflation remains at π^{LB}. ■

More generally, pick any martingale $\{m_t\}_{t=1}^{\infty}$ with the property that, for any t, $m_t \geq u'(C^{mono})$ with probability one. (These include the constant sequences described in Proposition 10.) Then, we can find an equilibrium by setting:

$$\begin{aligned} u'(C_t^*) &= m_t \\ w_t^* &= v'(C_t^*)/u'(C_t^*) \leq (1 - 1/\eta) \\ \pi_t &= \pi^{LB} \end{aligned}$$

for all t. In all of these equilibria, the output gap is always negative and the economy is always on the flat part of the Phillips curve.

4.3.3 Low Real Interest Rate

The final possibility is that the central bank's monetary policy rule specifies that the real interest rate is lower than the natural real interest rate when inflation is at its lower bound. Note that this case is not a by-product of the zero lower bound. Indeed, it is consistent with the monetary policy rule's obeying the Taylor Principle for all π.

The next proposition shows that, for any such monetary policy rule, there is a (large) set of equilibria in which:

- the economy is on the flat part of the Phillips curve.

- the output gap is negative and bounded away from zero in all dates with probability one.
- the output gap becomes arbitrarily negative over time.

Proposition 11. *Suppose that:*

$$\beta R(\pi^{LB})/\pi^{LB} < 1$$

For any $\bar{\Delta} \leq 0$, there exists a deterministic equilibrium in which the output gap sequence declines linearly over time:

$$\Delta_t^* = (t-1)ln(\frac{\beta R(\pi^{LB})}{\pi^{LB}}) + \bar{\Delta}, t \geq 1$$

Proof. Let $\gamma = \beta R(\pi^{LB})/\pi^{LB} < 1$. Define:

$$C_t^* = u'^{-1}(exp(-\Delta_t^*)u'(C^{mono})).$$

Then the sequence $(C_t^*)_{t=1}^{\infty}$ is part of an equilibrium in which $\pi_t^* = \pi^{LB}$ for all t and in which $w_t^* = v'(C_t^*)/u'(C_t^*)$ for all t. □

The proposition considers equilibria in which the output gap is becoming increasingly negative over time. Why is household demand so low, given that the real interest rate is below the natural real rate? The answer is that, in these equilibria, households believe that the output gap will be even lower next period. That belief leads them to demand less than C^{mono}, despite the low real interest rate. This basic mechanism is repeated over time, meaning that the households' beliefs are self-fulfilling: the output gap does in fact become more negative.

The equilibria in the preceding proposition are deterministic. However, it is possible to construct richer equilibria in which the output gap oscillates between 0 and lower levels. (Recall that, if the interest rate rule satisfies Assumptions 1 and 2, then the output gap can't be positive in equilibrium.)

Example 3. Suppose the interest rate rule R is weakly increasing and satisfies:

$$\frac{\beta R(\pi^{LB})}{\pi^{LB}} < 1$$

Let $\pi_0 > \pi_{LB}$ satisfy:

$$\frac{\beta R(\pi_0)}{\pi^{LB}} < 1$$

(Such a π_0 exists as long as the interest rate rule R is continuous.) Let $\bar{p}$ be an element of the open interval $(0, 1)$. Then, define a sequence $\{\bar{c}_n\}_{n=1}^{\infty}$ that satisfies the restrictions:

$$u'(C^{mono}) = \frac{\beta R(\pi_0)}{\pi_0}\{\bar{p}u'(\bar{c}_1)(\pi_0/\pi^{LB}) + (1-\bar{p})u'(C^{mono})\}$$
$$u'(\bar{c}_n) = \frac{\beta R(\pi^{LB})}{\pi^{LB}}\{\bar{p}u'(\bar{c}_{n+1}) + (1-\bar{p})u'(C^{mono})\frac{\pi^{LB}}{\pi_0}\}, n \geq 1$$

Because $\beta R(\pi_0)/\pi^{LB} < 1$, the first restriction implies that $\bar{c}_1 < C^{mono}$. Similarly, the second set of restrictions imply that $\bar{c}_{n+1} < \bar{c}_n$, for all $n \geq 1$.

Consider a Markov chain with a countably infinite state space indexed by the whole numbers. The transition probabilities are given by:

$$p_{n,n+1} = \bar{p} \text{ and } p_{n,0} = (1-\bar{p}), n \geq 0$$

We can define an equilibrium under which consumption equals $\bar{c}_n$ when the Markov state is $n \geq 1$ and equals C^{mono} when $n = 0$. In this equilibrium, inflation equals $\pi_0 > \pi_{LB}$ when the state $n = 0$ and is constant at π^{LB} in the other states. The real wage is given by $v'(C)/u'(C)$. ■

In the example equilibrium, the economy spends a fraction $(1-\bar{p})$ of its time in a "normal" state, in which the output gap is zero and inflation is above its lower bound. But even in that state, the real interest rate is below the natural rate. Households don't demand more than C^{mono}, despite the relatively low real interest rate, because they are concerned about the downside (sunspot-driven) risk to their future consumptions.

Notice that in the example, the output gap is not uniformly bounded from below. In the low real interest case, this turns out to be a general property of any equilibrium in which the output gap is negative.

Proposition 12. *Suppose that:*

$$\frac{\beta R(\pi^{LB})}{\pi^{LB}} < 1$$

In any equilibrium such that the output gap $\Delta_t^* < 0$*, and given any* $\bar{\Delta} < 0$*, there is some* T *such that* $Pr(\Delta_{t+s}^* < \bar{\Delta}$ *for all* $s \geq T) > 0$.

Proof. Given an equilibrium in which $C_t^* < C^{mono}$, we know that:

$$\begin{aligned} u'(C_t^*) &= \beta R(\pi^{LB}) E_t \frac{u'(C_{t+1}^*)}{\pi_{t+1}^*} \\ &\leq \gamma E_t u'(C_{t+1}^*) \end{aligned}$$

It follows that with positive probability, $u'(C_{t+1}^*) \geq \gamma^{-1} u'(C_t^*)$. So, define T such that:

$$exp(-\bar{\Delta}) u'(C^{mono}) < \gamma^{-T+1} u'(C_t^*).$$

There is a positive probability that $\Delta_{t+T}^* < \bar{\Delta}$. The proposition follows. □

Intuitively, a negative output gap is a sign of low aggregate demand. Households demand that little, despite the low real interest rate, because they are concerned about the possibility of an even lower output gap in the future.

The contrapositive of Proposition 12 implies that if:

- households believe that the output gap is uniformly bounded from below
- AND the real interest rate is low ($\beta R(\pi^{LB})/\pi^{LB} < 1$) when inflation is at its lower bound

then the output gap must be zero in all dates and states. This result underscores how low interest rates *can* eliminate low-output equilibria - but only if the government can somehow convince households that there is some limit to how negative the output gap will fall in the future. Without that kind of *expectations management*, low interest rate monetary policy can be consistent with increasingly inefficient equilibrium outcomes. How the government can manage expectations in this beneficial way is certainly beyond the scope of the paper.

4.3.4 Entry

Firm entry is not allowed in this model.[6] This restriction may seem artificial, especially over long horizons. In this subsection, I discuss how the potential gains to entry behave in the various low-output equilibria discussed above.

We have seen that, for any interest rate rule, the set of equilibrium output gaps is not bounded from below: that is, given any $\bar{\Delta} < 0$, we can find an equilibrium in which there is a positive probability of the output gap being less than $\bar{\Delta}$. These arbitrarily large negative output gaps might suggest that the gains to possible entry (by a monopolistically competitive firm who adds a new variety of good) are correspondingly large. But this isn't true. First, and most obviously, if output is close to zero, firm profits (denominated in consumption goods) must also be close to zero. This means that any potential entrant who faces a fixed cost in consumption goods has less incentive to pay that cost.

Admittedly, this calculation presumes that any potential entrant has to obey the same price-setting restrictions as the current firms. It might also be of interest to know the gains to entry for a firm that is allowed to ignore the lower bound on prices. Such a firm's maximal profits would equal:

$$Profits_{flex} = [(\frac{\alpha W_t}{P_t})^{1-\eta} P_t - (\frac{\alpha W_t}{P_t})^{-\eta} W_t] \frac{C_t}{P_t}$$

[6] Firm exit is also not allowed. However, firm profits remain positive in all of the low-output equilibria studied above.

where $\alpha = (1 - 1/\eta)^{-1}$ is the usual markup of price over marginal cost, given the elasticity of substitution η. We can rewrite this formula as:

$$Profits_{flex} = \psi(C_t)^{1-\eta} C_t (\alpha^{1-\eta} - \alpha^{-\eta})$$

where:

$$\psi(C_t) = v'(C_t)/u'(C_t)$$

The behavior of $Profits_{flex}$, as C_t converges to zero, is a race between aggregate demand C_t (which converges to zero) and $\psi(C_t)^{1-\eta}$ (which converges to infinity). If u' is well-approximated by a power function for C near zero, and $v'(0) > 0$, then we can show that the gains to entry without price bounds ($Profits_{flex}$) converge to zero with C as long as α (η) is sufficiently large (small).[7]

5 Discussion

In this section, I discuss several aspects of the above results, and provide connections to prior literature.

Enlarging the Class of Interest Rate Rules

I have restricted attention to interest rate rules that are functions only of current inflation. Suppose instead that the interest rate rule R is allowed to be a function of current inflation π_t, current consumption C_t, and m_t, where $m_t = \beta E_t u'(C_{t+1})/\pi_{t+1}$. Suppose too that there exists a consumption level C_{cap} such that:

$$R(\pi_{LB}, C_t, m_t) = 1$$

[7]These gains to entry are calculated in terms of goods. The conditions that ensure that the gains to entry, denominated in terms of hours, converge to zero with C are more stringent.

for all $C_t \leq C_{cap}$ and for all $m_t \geq \beta u'(C_{cap})/\pi_{LB}$. (Basically, the central bank cannot cut the interest rate any further once consumption hits C_{cap}.) It is straightforward to generalize the results about persistent downside macroeconomic risk in Section 4.3.2 (when $\beta/\pi_{LB} = 1$) and Section 4.3.3 (when $\beta/\pi_{LB} < 1$) to this kind of interest rate rule.

Flat Phillips Curve

The data from at least the past decade suggest that there is little connection between resource underutilization and inflation.[8] Thus, most measures of labor market slack rose sharply from 2008 to 2009 and inflation fell relatively little over that same time period. Similarly, inflation has remained essentially unchanged while most measures of labor market slack have fallen considerably over the past four years (2013-17).

These observations about inflation don't seem all that surprising when viewed through the lens of the bounded equilibrium models analyzed in this paper. As long as there is a negative output gap, the Phillips curve is flat: there is no connection between the magnitude of the gap and the inflation rate. The Phillips curve becomes vertical only when the output gap rises back to zero. And, when the Phillips curve is vertical, inflation is determined by the interaction of the nominal interest rate rule and expected inflation.

The above emphasizes the underutilization of labor when the output gap is negative. But the underutilization of labor is in fact due to excessive *product* market power created by the inflation lower bound. Hence, if the models included other inputs to production, these inputs would also be underutilized when the output gap is negative. In this sense, the notion of a negative output gap in this paper is consistent with the evidence in Bils, Klenow, and Malin (forthcoming).

[8] This lack of connection may well go back much further in time - see, for example, Stock and Watson (2009).

Limitations on Power of Forward Guidance

Krugman (1998) and Eggertsson and Woodford (2003) argue that central banks who find themselves constrained by the zero lower bound on nominal interest rates can stimulate the economy by committing to a temporary increase in the inflation target once the output gap returns to non-negative territory. But these commitments cannot eliminate equilibria of the kind described in Propositions 10 and 11, in which the output gap is permanently below zero.

Neo-Fisherian

How does a permanent increase in the nominal interest rate affect inflation and consumption outcomes in this class of models with bounded price-setting? Suppose that the nominal interest rate rule is a constant, so that $R(\pi) = \bar{R}$ for all π, where:

$$\pi^{LB} < \beta\bar{R} < \pi^{UB}$$

As is well-known from earlier work (Sargent and Wallace (1975)), the interest rate peg is consistent with a large set of equilibria. But it is usually thought that, with flexible prices, all of these equilibria have the same real interest rate and so the same expected inflation rate.

In the model with price bounds, we can first show, as in Proposition 5, that there is an equilibrium in which inflation is constant. Set:

$$C_t^* = C^{mono}$$

$$w_t^* = (1 - 1/\eta)$$

$$\pi_t^* = \beta\bar{R}$$

It is readily seen that this non-stochastic process satisfies the conditions in Proposition 4.

This class of equilibria satisfies what's been called the neo-Fisherian hypothesis (Cochrane (2016) and Schmitt-Grohe and Uribe (2017)). If we raise the nominal interest rate peg, then the constant inflation rate goes *up* one-for-one (in logs). This increasing relationship between the nominal interest rate and the inflation rate seems counterintuitive. But it is an inevitable consequence of the presumption that, in equilibrium, the real interest rate should always equal $(1/\beta) - 1$.

However, these equilibria are highly non-robust in the following sense. Keep the same interest rate rule and fix some horizon length T. Consider the following inflation sequences for $t > T$.

$$\begin{aligned} C_t^* &= C^{mono}, t > T \\ \pi_t^* &= \beta \bar{R}, t \geq (T+2) \\ \pi_{T+1}^* &= \beta \bar{R} - \epsilon \end{aligned}$$

In period $(T+1)$, inflation is slightly lower than in later periods. It follows that, in period T, the real interest rate is slightly higher than $1/\beta$, and so consumption is slightly lower than C^{mono} :

$$\begin{aligned} u'(C_T^*) &= \beta u'(C^{mono}) \bar{R} / \pi_{T+1}^* \\ &> u'(C^{mono}). \end{aligned}$$

This negative output gap implies that:

$$\pi_T^* = \pi^{LB}.$$

By rolling this argument backwards in time, we can conclude that, with probability one:

$$\pi_t^* = \pi^{LB}$$

for all $t \leq T$. It follows that with probability one:

$$u'(C_t^*) = (\frac{\beta \bar{R}}{\pi^{LB}})u'(C_{t+1}^*)$$
$$= (\frac{\beta \bar{R}}{\pi^{LB}})^{T-t}(\frac{\beta \bar{R}}{\pi_T^*})u'(C^{mono}), t \leq T$$

The (slightly) lower inflation expectations in period T imply that inflation is at its lower bound in all earlier periods. Similarly, consumption in period $t \leq T$ is (possibly much) lower than C^{mono}. Note that the output gap in these earlier periods is a decreasing function of $\bar{R}$.

This same argument can be readily applied to the case in which period $(T+1)$ inflation is known to be slightly *higher* than $\beta \bar{R}$. Then, inflation is at its upper bound, consumption is higher than C^{mono} in all prior periods, and the output gap is a decreasing function of $\bar{R}$.

To sum up: given that the nominal interest rate is pegged at $\bar{R}$, there is a neo-Fisherian equilibrium in which inflation is constant at $\beta \bar{R}$. However, the equilibrium is very delicate. Small perturbations in expected inflation, even far off in the future, push realized inflation to its upper or lower bounds over the intervening period. In this sense, extreme inflation realizations seem like the more natural outcomes of an interest rate peg.

The Misleading Predictions of Models Without Pricing Bounds

The results in Section 4 show that the predictions of an infinite horizon model without pricing bounds are a poor approximation to the infinite horizon model with bounds, even if those bounds are very loose. For example, suppose that the lower bound π^{LB} on the gross inflation rate is close to zero. In that case, if β is near 1 and the nominal interest rate rule is bounded below by 1:

$$\beta R(\pi^{LB})/\pi^{LB}$$

is very large. Proposition 9 demonstrates that for any such π^{LB} and any horizon length, there is a set of deterministic equilibria in which the output gap is negative over that horizon. Example 1 shows that there is a set of stochastic equilibria in which the output gap fluctuates between a negative value and zero. In contrast, if there is no bound at all, then the output gap necessarily equals zero in all dates and states. So, users of models without bounds would necessarily be unaware of the (highly) inefficient outcomes predicted by models with very loose bounds.

Heterogeneity in Bounds

In this paper, I've modeled all firms as facing the same bounds. Suppose instead that the firms differ in their (lower) bounds, so that the measure of firms with an inflation lower bound less than π is given by $\nu G(\pi)$. Recall that ν is the total measure of firms, so G is a cumulative distribution function; let g represent the corresponding density. The maximal lower bound is π_{max}^{LB}, so that $G(\pi_{max}^{LB}) = 1$.

This heterogeneity changes the Phillips curve. It is still true that there is a vertical portion of the curve (so that inflation is not pinned down if the output gap is zero). But the "flat" portion of the Phillips curve now has a positive slope. Suppose consumption $C_t < C^{mono}$ (defined as before). Then at least some firms are constrained by their inflation lower bound, so that inflation $\pi_t < \pi_{max}^{LB}$. For lower values of C_t, more firms are constrained, and inflation π_t is correspondingly lower. There is a one-to-one relationship between the output gap and inflation; the slope of this relationship is determined by G.

Nominal Rigidities

In the models that I study in Section 4, prices are completely flexible except for the bounds. There is considerable evidence that prices and wages are not completely flexible, although the degree of inflexibility remains a subject of much empirical study (see Nakamura and Steinsson (2013) for a recent survey of the relevant evidence). Appendix A shows that, even if some firms are unable to adjust their prices, there is a non-existence problem when the other firms can choose their prices from the entire real line. Hence, it would be of interest to extend Section 4 to consider the properties of models with conventional pricing frictions (like

Calvo stickiness or menu costs) as well as bounds on the (flexible) firms' pricing decisions.

Log Utility

The results in the paper can be extended to the case in which the utility function $u(C) = ln(C)$. All that we need to do is change the definition of equilibrium so that the households' consumption sets only consist of (C, N) bundles that satisfy the budget constraints and such that:

$$E_0 \sum_{t=1}^{\infty} \beta^{t-1} |ln(C_t) - v(N_t)| < \infty$$

This restriction ensures that households have an unambiguous ranking of all budget-feasible (C, N) bundles.

Growth

In the paper, I assumed that there is no growth. Suppose instead that at date t, each firm can transform n_t units of labor into $(1+g)^{t-1} n_t$ units of consumption, so that productivity is growing at a constant non-negative rate. As well, assume that agents have logarithmic utility over consumption (as above). Then, in the moneyless monopolistic competition equilibrium, labor is constant at N^{mono}, which solves:

$$(1 - 1/\eta)/N^{mono} = v'(N^{mono}).$$

Consumption grows at rate g and the natural real interest rate is given by:

$$\beta^{-1}(1 + g) - 1$$

Note that the results in the paper are stated in terms of output gaps. Hence, they can be extended as stated to this case with deterministic baseline growth (as long as we take this increase in the natural real interest rate in mind). But now consumption may be growing over time in an equilibrium in which the output gap is converging to negative infinity (such as those in Proposition 12).

Capital

Throughout, I've abstracted from accumulable capital. A thorough examination of its implications is outside the scope of the paper. However, the following point is worth noting.

Suppose $\beta R(\pi^{LB})/\pi^{LB} = 1$, so that, when inflation is at its lower bound, the real interest rate is equal to the natural real interest rate. Suppose too that firms have a constant-returns-to-scale production function F with capital and labor as inputs, and that capital is accumulable using the familiar linear technology:

$$K_{t+1} = (1-\delta)K_t + (Y_t - C_t)$$

Then, there is a continuum of steady-states defined by capital-labor pairs (K_{SS}, N_{SS}) that satisfy two restrictions:

$$\beta(1 + F_K(K_{SS}/N_{SS}, 1) - \delta) = 1$$
$$(1 - 1/\eta)u'(F(K_{SS}/N_{SS}, 1)N_{SS} - \delta(K_{SS}/N_{SS})N_{SS}) \geq v'(N_{SS})$$

The capital-labor ratio is pinned down by the real return to money, and is equal to its usual steady-state level. However, the level of steady-state labor can be arbitrarily low. A central bank can ensure that growth is efficient by pegging the real rate of interest at its natural level, but it cannot ensure that the resulting output gap is zero.

6 Conclusions

There are two main conclusions from this paper. The first conclusion is methodological. Models without price-setting bounds give highly unreliable predictions, even over the longer run, and so shouldn't be used. This recommendation is obviously relevant for monetary economists. But it is also relevant for macroeconomists who use non-monetary models. Presumably, their justification for ignoring money is that prices are so flexible that money has essentially no real effects. The point of this paper is that money does have possibly large real effects even when prices can be adjusted costlessly.[9]

The second conclusion is substantive. Secular stagnation (Summers (2013)) has been viewed as a phenomenon that requires a special modeling approach (a la Eggertsson and Mehrotra (2014)). But, once we take into account the presence of pricing bounds, it becomes clear that secular stagnation is an unavoidable macroeconomic risk. No matter what rule the central bank uses, there is a host of equilibria in which the output gap is negative for many years and possibly indefinitely.

There are at least three useful extensions that could be pursued. First, the model of consumer demand in this paper is a traditional one that relies heavily on intertemporal substitution. It would be useful to consider the implications of pricing bounds in the recent models of consumer demand that put more weight on redistribution (see, among others, Auclert (2017) and Kaplan, Moll, and Violante (2018)).

This paper is about the properties of what proves to be a plethora of rational expectations equilibria in models with pricing bounds. There has been a great deal of work in macroeconomics on how learning, bounded rationality, and/or higher order uncertainty can serve to limit indeterminacies of rational expectations equilibria. (See Christiano, Eichenbaum, and Johannsen (2018), Gabaix (2018), and Garcia-Schmidt and Woodford (2015) for recent examples.) It would be interesting in future work to explore the consequences of these considerations in models with pricing bounds.

Finally, I treat the pricing bounds as exogenous. It would be useful to endogenize them (perhaps as part of some kind of firm meta-game) so as to build an understanding of how they are influenced by monetary policy and economic outcomes.

[9]See Kocherlakota (2016) for a related argument.

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Appendix A

In this subsection, I consider the robustness of the selection result in Section 2.2 to five perturbations of the basic two-period model: sticky prices, incomplete markets for risk-sharing, other formulations of fiscal policy, the addition of non-interest-bearing currency with liquidity services, and the addition of a physical storage technology. In all of these models, if we consider period 2 in isolation, any positive gross inflation rate is an equilibrium. Dynamic equilibrium - including both periods 1 and 2 - serves to eliminate many of these outcomes.

Sticky Prices

In the two-period model in Section 2, all firms can adjust prices freely, and so equilibrium period 1 output is independent of the period 1 price level. We can generalize the equilibrium refinement result to allow for arbitrary amounts of price stickiness. Thus, suppose that a fraction ϕ of firms have prices fixed at $\bar{P}$ in period 1, while the other $(1-\phi)$ firms are allowed to adjust their prices freely. The price rigidity creates a relative price distortion between the two kinds of firms, and so equilibrium consumption in period 1 is a function of the period 1 price level. However, there is a limit to the damage that this distortion can do: the worst that can happen is that the fixed price firms produce nothing. This means that we know that equilibrium consumption in period 1 can be no lower than C_{LB}, where C_{LB} solves:

$$u'(C_{LB})(1-\phi)^{\frac{1}{\eta-1}}(1-1/\eta) = v'((1-\phi)^{\frac{1}{1-\eta}}C_{LB})$$

Here, C_{LB} is the level of consumption that would be produced if the period 1 price level were close to zero, so that nobody bought goods at the fixed price firms. So, even when almost all firms are unable to change their prices (ϕ near 1), there is no equilibrium for any interest rate rule R and any period 2 inflation Π^* such that:

$$u'(C_{LB}) < u'(Y)R(P_1)/\Pi^*$$

for all P_1.

Many applications of sticky price models use log-linear approximations. But such an approximation reveals no sign of this problem (the non-existence of dynamic equilibrium for many or most period 2 equilibrium inflation rates).[10] To see this, let $\Pi = \bar{\Pi}$, where:

$$u'(C^{mono}) = \beta R(\bar{P})u'(Y)/\bar{\Pi}.$$

It is easy to show that, in a log-linearized model, there is an equilibrium for any $\hat{\Pi} = ln(\Pi^*/\bar{\Pi})$. In this equilibrium, log-linearized period 1 consumption $\hat{C}_1 = ln(C_1/C^{mono})$ is given by:

$$\hat{C}_1 = \hat{\Pi}/CRRA^{mono}$$

(Here, $CRRA^{mono} = -C^{mono}u''(C^{mono})/u'(C^{mono})$.) Log-linearized period 1 labor is also equal to $\hat{N}_1$. Finally, the log-linearized price level in period 1 is proportional to $\hat{C}_1$, with a constant of proportionality that is linear in $(1-\phi)/\phi$. Thus, the log-linearized approximation is too inaccurate to capture the need for pricing bounds.

Incomplete Markets

We can generalize the selection result to allow for incomplete markets. Thus, suppose that a given agent i has random period 2 endowment Y_i, where Y_i is i.i.d. across agents with mean

[10] I thank Ivan Werning for making this point to me in a private communication.

Y, and that agents are unable to trade any other assets besides interest-bearing money. It is readily shown that, in this economy, there is no equilibrium for any Π^* and interest rate rule R such that:

$$u'(C^{mono}) \neq R(P_1)\frac{E(u'(Y_i))}{\Pi^*}$$

for all P_1.

Fiscal Policy

In the model in section 2.2, the government levies a period 2 purely nominal lump-sum tax equal to the average money-holdings in the economy. This fiscal policy makes the period 2 price level indeterminate.

Suppose instead that the government *pegs* the price of money in period 2 equal to $\Pi_2 P_1$, where P_1 is the endogenous period 1 price level and Π_2 is an exogenous gross inflation rate. This peg could be accomplished in a number of ways, including a restriction that specifies period 2 taxes to be $M(1+R(P_1))/(\Pi_2 P_1)$ units of consumption.[11] Such a peg would eliminate the indeterminacy in period 2 and restrict (rational) period 1 beliefs about period 2 inflation to be concentrated on Π_2.

Under these kinds of policies, the equilibrium selection issue becomes an equilibrium existence problem. In particular, suppose that the government specifies Π_2 and $R(P_1)$ so that:

$$u'(C^{mono}) < u'(Y)R(P_1)/\Pi_2$$

for all P_1. Then, there is no period 1 equilibrium and the model is uninformative about the implications of this (generic) set of policy choices. The equilibrium selection problem now becomes an equilibrium existence problem.

[11] This kind of fiscal policy, in which the government's intertemporal budget constraint is not satisfied for all price levels, is typically termed non-Ricardian (Woodford (1995)).

Currency

The discussion in section 2.2 treats all money as interest-bearing. Of course, in reality, some money (currency) is non-interest-bearing. and is held because it provides liquidity services over interest-bearing reserves. We can generalize the above discussion about equilibrium selection to a model in which agents can swap (utility-generating) currency in exchange for (interest-bearing) reserves with the government in the first period.

In particular, suppose that agents have preferences of the form:

$$u(C_1) + u(C_2) - v(N_1) + h(X_1'/P_2)$$

where X_1' represents the agent's (non-interest-bearing) currency-holdings at the end of period 1. Here, h is an increasing and concave function. Suppose too that agents can trade currency and reserves in period 1 with the government, so that their budget constraint in period 1 looks like:

$$P_1 c_1 + M_1' + X_1' \leq W_1 n_1 + M$$

and money market-clearing in period 1 is given by:

$$M_1' + X_1' = M$$

As before, the agents have to pay a lump-sum tax equal to average money-holdings at the end of period 2.

Suppose that the interest rate rule satisfies:

$$R_{min} \leq R(P_1) \leq R_{max}$$

for all P_1. Then, we can generalize the selection result in Section 2.2 to show that there is no period 1 equilibrium for any period 2 inflation rate Π^* such that:

$$\Pi^* u'(C^{mono}) > max(u'(Y)R_{max}, u'(Y) + h'(0)) \tag{3}$$
$$\Pi^* u'(C^{mono}) < u'(Y)R_{min}$$

The restriction (3) is a generalization of (1) that takes into account the new possibility that the agents can give their reserves to the government in exchange for an equivalent dollar amount of currency. In such an equilibrium, agents set their currency-holdings $X_1' = M$ and their reserve-holdings M_1' equal to 0. The nominal interest rate is $(1 + h'(\frac{M}{P_1 \Pi^*})/u'(Y)) > R(P_1)$. But, in this case, the nominal interest rate can be no larger than:

$$(1 + h'(0)/u'(Y))$$

Note that the restriction (3) only has bite if $h'(0) < \infty$, so that there is a limit to the willingness of agents to substitute between currency and consumption.

Storage

We can generalize the equilibrium selection result to allow for the presence of a technology that transforms current consumption into future consumption. In particular, suppose agents can store x units of consumption in period 1 to generate $(1 + \phi)x$ units of consumption in period 2, for any $x \geq 0$. I suppose that there exists $S^{mono} > 0$ such that:

$$u'(C^{mono} - S^{mono}) = (1 + \phi)u'(Y)$$

so that storage is positive in a non-monetary equilibrium allocation.

Let R be an interest rate rule that satisfies:

$$R_{min} \leq R(P_1) \leq R_{max}$$

for all P_1. We can generalize the refinement result in the presence of storage by noting that there is no equilibrium for any period 2 inflation rate Π^* such that:

$$(1+\phi)\Pi^* > R_{max}$$

or:

$$(1+\phi)\Pi^* < R_{min}$$

Appendix B

Proof of Proposition 4

The necessity of these conditions is straightforward. The sufficiency of the price-setting first order condition as a solution to the firm's problem is also obvious.

The sufficiency of these conditions for household optimality is somewhat deeper. Suppose (C', N', M') is budget-feasible and dominates $(C^*, N^*, \bar{M})$. That means:

$$\begin{aligned}
0 &< E_0 \sum_{t=1}^{\infty} \beta^{t-1}(u(C'_t) - v(N'_t)) - E_0 \sum_{t=1}^{\infty} \beta^{t-1}(u(C^*_t) - v(N^*_t)) \\
&= E_0 \sum_{t=1}^{T} \beta^{t-1}(u(C'_t) - v(N'_t)) - (u(C^*_t) - v(N^*_t)) \\
&+ \beta^T E_0 \sum_{t=T+1}^{\infty} \beta^{t-1-T}(u(C'_t) - v(N'_t)) - (u(C^*_t) - v(N^*_t))
\end{aligned}$$

If we take limits with respect to T and since (u, v) are bounded from above and below, we find that:

$$0 < lim_{T\to\infty} E_0 \sum_{t=1}^{T} \beta^{t-1}(u(C'_t) - v(N'_t)) - (u(C^*_t) - v(N^*_t))$$

Next, we can use the subgradient inequality for concave functions:

$$
\begin{aligned}
0 &< lim_{T\to\infty} E_0 \sum_{t=1}^{T} \beta^{t-1}(u'(C_t^*)(C_t' - C_t^*) - v'(N_t^*)(N_t' - N_t^*)) \\
&= lim_{T\to\infty} E_0 \sum_{t=1}^{T} \beta^{t-1}(u'(C_t^*)((C_t' - C_t^*) - W_t^*(N_t' - N_t^*)/P_t^*) \\
&= lim_{T\to\infty} E_0 \sum_{t=1}^{T} \beta^{t-1}(u'(C_t^*))((M_{t-1}' - \bar{M}_{t-1})R_t^*/P_t^* - (M_t' - \bar{M}_t)/P_t^*) \\
&= lim_{T\to\infty} E_0 \beta^{T-1}(u'(C_T^*))(\bar{M}_T - M_T')/P_T^*) \\
&\leq lim_{T\to\infty} \bar{M}_T(E_0\beta^{T-1}u'(C_T^*)/P_T^*)\bar{M}_T
\end{aligned}
$$

where the last step comes from the non-negativity of M' and the nonstochastic nature of $\bar{M}_T$.

Finally, note that:

$$
\begin{aligned}
& E_0\beta^{t-1}u'(C_t^*)/P_t^* \\
&= E_0 E_{t-1}\beta^{t-1}u'(C_t^*)/P_t^* \\
&= E_0 \frac{\beta^{t-2}u'(C_{t-1}^*)}{P_{t-1}^* R_{t-1}^*} \\
&\leq E_0\beta^{t-2}u'(C_{t-1}^*)/P_{t-1}^*
\end{aligned}
$$

and by iterating backwards:

$$
\begin{aligned}
& lim_{T\to\infty}\bar{M}_T(E_0\beta^{T-1}u'(C_T^*)/P_T^*) \\
&\leq (E_0 u'(C_1^*)/P_1^*) lim_{T\to\infty}\bar{M}_T \\
&= 0
\end{aligned}
$$

which is a contradiction.

Proof of Proposition 6

Define:

$$\lambda = \frac{\beta R(\pi^{UB})}{\pi^{UB}}$$

Assumptions 1-2 imply that $\lambda > 1$. Suppose there is a positive probability event such that for some t:

$$C_t^* > C^{mono}.$$

Then Proposition 4 implies that $w_t^* = v'(C_t^*)/u'(C_t^*) > (1 - 1/\eta)$, and that:

$$\pi_t^* = \pi^{UB}$$

It follows that:

$$\begin{aligned} u'(C^{mono}) &> u'(C_t^*) \\ &\geq \beta R(\pi^{UB}) E_t u'(C_{t+1}^*)/\pi^{UB} \\ &= \lambda E_t u'(C_{t+1}^*) \end{aligned}$$

Thus, there is a positive probability event such that:

$$u'(C_{t+1}^*) < \lambda^{-1} u'(C^{mono})$$

and that, for any s, there is a positive probability event such that:

$$u'(C_{t+s}^*) < \lambda^{-s} u'(C^{mono}).$$

But this contradicts the restriction that C^* is uniformly bounded from above.

Suppose instead that there is a positive probability event such that:

$$\pi_t^* > \pi^{TAR}$$

Define:

$$\lambda' = \beta R(\pi_t^*)/\pi_t^* > 1$$

In that event:

$$\begin{aligned} u'(C^{mono}) &= u'(C_t^*) \\ &= \beta R(\pi_t^*) E_t u'(C_{t+1}^*)/\pi_{t+1}^* \\ &\geq \beta R(\pi_t^*) u'(C^{mono}) E_t(1/\pi_{t+1}^*) \end{aligned}$$

and:

$$\begin{aligned} 1 &\geq \frac{\beta R(\pi_t^*)}{\pi_t^*} E_t \frac{\pi_t^*}{\pi_{t+1}^*} \\ &= \lambda' E_t \frac{\pi_t^*}{\pi_{t+1}^*} \end{aligned}$$

It follows that there is a positive probability event such that:

$$\pi_{t+1}^* \geq \lambda' \pi_t^*$$

By rolling forwards, we conclude that, for any s, there is a positive probability event such that:

$$\pi_{t+s}^* \geq (\lambda')^s \pi_t^*$$

But this contradicts the restriction that $\pi_t^* \leq \pi^{UB}$.

Proof of Proposition 7

Consider an event such that:

$$u'(C^{mono}) = \beta R(\pi^{TAR}) E_t \frac{u'(C^*_{t+1})}{\pi^*_{t+1}}.$$

Suppose $\pi^*_{t+1} < \pi^{TAR}$ with positive probability. Then:

$$\begin{aligned} u'(C^{mono}) &> \frac{\beta R(\pi^{TAR})}{\pi^{TAR}} E_t u'(C^*_{t+1}) \\ &= E_t u'(C^*_{t+1}) \end{aligned}$$

But this implies $C^{mono} < C^*_{t+1}$ with positive probability, which violates Proposition 6.

Now suppose $C^*_{t+1} < C^{mono}$ with positive probability. Then, from Proposition 6, we know that:

$$\begin{aligned} u'(C^{mono}) &> \beta R(\pi^{TAR}) u'(C^{mono}) E_t \frac{1}{\pi^*_{t+1}} \\ &= \frac{\beta R(\pi^{TAR})}{\pi^{TAR}} u'(C^{mono}) E_t \frac{\pi^{TAR}}{\pi^*_{t+1}} \\ &= u'(C^{mono}) E_t \frac{\pi^{TAR}}{\pi^*_{t+1}}. \end{aligned}$$

But this implies that $\pi^*_{t+1} > \pi^{TAR}$, which violates Proposition (6).

Proof of Proposition 9

Define $\gamma \equiv \frac{\beta R(\pi^{LB})}{\pi^{LB}}$ and define π^{fixed} to be the lowest inflation rate larger than π^{LB} that satisfies:

$$\beta R(\pi^{fixed}) = \pi^{fixed}$$

Note that $\pi^{fixed} \leq \pi^{TAR}$. Because R is continuous, we know that:

$$\beta R(\pi) > \pi$$

for all π in $[\pi^{LB}, \pi^{fixed})$ (or there would be a fixed point to βR that's smaller than π^{fixed}).

Pick some horizon T. Choose an initial level of consumption C_1^* so that:

$$u'(C_1^*)\gamma^{-T+1} > u'(C^{mono}) \geq u'(C_1^*)\gamma^{-T}$$

(Such a level exists because $u'(0)$ is infinity.) Define:

$$C_t^* = u'^{-1}(\gamma^{-t+1}u'(C_1^*)), 2 \leq t \leq T$$

and define:

$$C_t^* = C^{mono}, t \geq T+1$$

Then, define a corresponding inflation sequence as:

$$\pi_t^* = \pi^{LB}, 1 \leq t \leq T$$

and:

$$\pi_{T+1}^* = \beta R(\pi^{LB})u'(C^{mono})/u'(C_T^*)$$
$$\pi_{t+1}^* = \beta R(\pi_t^*), t \geq T+1.$$

In this sequence:

$$\pi_{T+1}^* \geq \pi^{LB}$$

because:

$$\pi_{T+1}^* = \frac{\beta R(\pi^{LB})u'(C^{mono})}{\gamma^{-T+1}u'(C_1^*)} \geq \frac{\beta R(\pi^{LB})}{\gamma} = \pi^{LB}$$

As well:

$$
\begin{aligned}
\pi^{fixed} &= \beta R(\pi^{fixed}) \\
&= \beta R(\pi^{LB}) \frac{u'(C^{mono})}{u'(C_T^*)} \frac{u'(C_T^*)}{u'(C^{mono})} \frac{R(\pi^{fixed})}{R(\pi^{LB})} \\
&\geq \beta R(\pi^{LB}) \frac{u'(C^{mono})}{u'(C_T^*)} \\
&= \pi_{T+1}^*
\end{aligned}
$$

Hence, we know that:

$$\pi^{fixed} \geq \pi_{T+1}^* \geq \pi^{LB}$$

and so that:

$$\beta R(\pi_{T+1}^*) \geq \pi_{T+1}^*$$

It follows that:

$$\beta R(\pi^{fixed}) = \pi^{fixed} \geq \pi_{T+2}^* = \beta R(\pi_{T+1}^*) \geq \pi_{T+1}^*$$

Thus, $(\pi_{T+s}^*)_{s=1}^{\infty}$ is a weakly increasing sequence that is bounded above by π^{fixed}; from the continuity of R, it converges and its limit point is π^{fixed}.

Finally, we define w^* to be:

$$w_t^* = v'(C_t^*)/u'(C_t^*)$$

It is readily checked that this specification of (C^*, π^*, w^*) satisfies the requirements in Proposition 4.