Abstract

We study the trade of indivisible goods and divisible assets in a frictional market. Indivisibility matters for equilibria along with the trading mechanism. Bargaining generates a good’s price that is not linked to dividend value of the asset or the number of active users of the asset, buyers. In contrast, price posting with competitive search generates a price as a continuous function of the dividend and the number of active users. We provide conditions under which stationary equilibrium exists. For positive dividend value on the asset, (generically) unique equilibrium exist, while for negative dividend value, multiple equilibria occur.

JEL: D51, G12.

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1 Introduction

Assets, just like money, can convey liquidity for consumption purposes. In reality, assets can be used as collateral for trade via credit in markets where there is a probability of default or imperfect credit. In general, only a fraction of the value of the asset can be pledged as a collateral (e.g., home equity loans, repo loans, capital loans based on business valuation) (see Kiyotaki and Moore (1997) for an early model). However, as argued in Lagos (2011) and Lagos et al. (2016), it is theoretically equivalent to have agents surrender the asset as a medium of exchange when credit is imperfect or not possible due to anonymity, lack of commitment and punishment devices. There is a literature studying asset as a medium of exchange for consumption in the environment of New Monetarism based on the original framework of Lagos and Wright (2005). Lagos and Rocheteau (2009), Rocheteau and Wright (2013), and Lagos et al. (2016) study this but for consumption of a perfectly divisible good.

In this paper we explore the consequences of having indivisible goods traded with divisible assets. Effectively, this is the reverse of the divisible good, indivisible money or asset model used in Shi (1995), and in Trejos and Wright (1995).\footnote{See Wright, Julien, Kircher and Guerrieri (2017) survey, He and Wright (2016) and Julien, Kennes and Schein (2016) for recent models based on Shi-Trejos-Wright using an asset instead of fiat money.} In that framework, agents cannot accumulate more than one unit of money. Here, we assume that buyers only want to consume one unit of the indivisible good but can hold any amount of asset. We show how indivisibility on the goods side matters for equilibria, and how they differ from equilibria with indivisible asset in trades.

We argue that goods in the economy are mostly indivisible, and the assumption of perfectly divisible goods is a convenient abstraction. While it is compelling to justify divisibility at the aggregate level, we believe that at level of pairwise trades, good divisibility is a stronger assumption. While indivisible assets and divisible goods have been extensively studied as evidenced in the extensive survey of Lagos et al. (2016), the consequences of indivisible goods traded for divisible assets has been neglected.\footnote{The notable exception is Han et al. (2016) who study indivisible goods traded in pure credit market and with fiat money.}

We consider an economy where a real asset is used as a medium of exchange. In
contrast to monetary models, the asset is in fixed total supply and it bears an exogenous dividend which could be positive or negative. The is a reason to consider real assets in trade, when credit is imperfect, money is not the only object that can serve as a medium of exchange.

The study of media of exchange is particularly tied to the way terms of trade are determined in the exchange process. The equilibrium asset price emerging from exchange process can differ significantly depending on the terms of trade mechanism used. We consider an environment where terms of trade are determined by generalized Nash bargaining and by price posting with competitive search. \(^3\) The former is an ex post mechanism mapping the amount of asset carried into the market into outcomes in the bilateral trade. The latter is an ex ante mechanism mapping the posted terms of trade into a the choice of asset to carry.

With indivisible goods, no adjustment can take place through the intensive margin and the available surplus is fixed. In particular, with asset, as we show, the bargained price does not depend on the return of the asset. Under bargaining, buyers can commit to bring the lowest amount of asset needed to make sellers indifferent between trading or not trading. This is driven by the loss of intensive margin adjustment with indivisibility. The ex post nature of the mechanism gives extra bargaining power to a buyer through the choice on the amount of asset to carry for trade. The solution is akin to a take-it-or-leave-it offer by buyers extracting the whole surplus for any level of exogenous bargaining power assumed under the Nash bargaining program. We show that with lotteries, the threat of not delivering the good, sellers are able to extract some of the surplus, even though lotteries end up not being used in equilibrium.

Price posting entails commitment by sellers prior to trade, and we assume that buyers choose their asset holding after observing prices. \(^4\) Under competitive search, unlike bargaining, the price of the indivisible good is a continuous function of the participation

\(^3\) The competitive search framework we use is based on Moen (1997) and Mortensen and Wright (2002). For the use of competitive search in monetary models see Rocheteau and Wright (2005) and Lagos and Rocheteau (2007).

\(^4\) Alternatively one can assume the choice of asset holding is made prior to observing prices. Although it can be rationalized by a budgetting argument, especially when there is a cost of carrying liquidity, this seems a less natural assumption.
rate and the cost of holding the asset. The dividend also has an indirect effect on the price through participation.

An important consequence of indivisibility is that in equilibrium, not all buyers may choose to participate in the indivisible good market. This is in contrast to models where the good is perfectly divisible and all buyers always participate as in Lagos and Rocheteau (2007). With divisible good, sellers have two instruments to always assure non-negative surplus for buyers. With indivisible good, only the price can adjust. Thus, for high cost of carrying the asset, if the equilibrium price is too high and all buyers participate, this can lead to a negative surplus for buyers from the congestion effect of the matching process. As a result, not all buyers participate. The participation decision is tied to the terms of trade but also to the asset price and the dividend from the asset.

For high enough positive values of the dividend, all buyers participate in the decentralized market, with unique equilibrium under bargaining and competitive search. When the dividend is low, even negative, the matching congestion effect comes into play and reduces buyers participation. The cost of carrying the asset for trade is indirectly determined by the liquidity premium on the asset price, which itself, depends on the dividend value and the number of active buyers. In bargaining and competitive search, a higher number of active buyers means a lower probability of trade and a lower cost of carrying the asset, which leads to the multiplicity of equilibria. With bargaining, there are exactly two equilibria: one with high buyers participation, and the other with low participation, although only the former has stability property. Under price posting, the continuous dependence of the good’s price on the dividend value and the number of active buyers and can lead to multiple equilibria.

We also show how the use of lotteries matter for equilibria under bargaining, but does not under competitive search and price posting. Finally, we perform comparative statics under stationary equilibrium changing the aggregate supply of the asset.

The paper is organized as follows. In Section 2 we describe the environment. In Section 3 we consider an economy with asset used as the medium of exchange, prove existence and uniqueness of asset equilibrium when bargaining or competitive search serves as the trading mechanism, and study the effects of changing asset supply. In Section 4 we
introduce lotteries, and Section 5 concludes.

2 Environment

The environment is based on the alternating markets framework of Rocheteau and Wright (2005).\textsuperscript{5} Time is discrete and goes on forever. A continuum of buyers and sellers, with measures \( N \) and 1, live forever. In each period, all agents participate in two markets consecutively. Agents discount between periods with factor \( \beta \in (0,1) \), but not across markets within a period, and \( r = 1/\beta - 1 \) is the discount rate. The first market to open is a decentralized market (DM), and the second is a frictionless centralized market (CM). Both buyers and sellers consume a divisible good in the CM, while only buyers consume an indivisible good in the DM.

Buyers’ preferences within a period are given by \( U(x) - h + u1 \), where \( x \) is CM consumption, \( h \) is CM labor, \( u \) is DM utility from consuming the indivisible good, and \( 1 \) is an indicator function giving 1 if trade occurs and 0 otherwise. Sellers’ preferences are \( U(x) - h - c1 \) with DM good produced at constant cost \( c \). We assume \( u > c \). Let \( x \) be the CM numeraire. We assume that \( x \) is produced one-to-one from labor \( h \).

Trade in the DM implies a price and quantity bundle \((p, q) \in P \times Q \) where \( P = \{0 \leq p \leq L\} \) and \( Q = \{0, 1\} \). \( L \) represents the available liquidity in the economy, with \( L = (\varphi + \rho)a \) being the value of asset holdings of a buyer in the DM. Here \( \rho \) represents the dividend of assets, which can be either positive or negative, and \( \varphi \) the CM price of assets in terms of \( x \).

In the DM, meetings occur according to a general meeting technology which is assumed homogeneous of degree one. Given the buyer-seller ratio \( n \leq N \), which is also the measure of participating buyers in the DM, the meeting rate for sellers and buyers are \( \alpha(n) \) and \( \alpha(n)/n \) respectively. Assume \( \alpha' > 0, \alpha'' < 0, \alpha(0) = 0, \lim_{n \to \infty} \alpha(n) = 1, \) and \( \lim_{n \to 0} \alpha'(n) = 1. \)

\textsuperscript{5}The original alternating markets framework by Lagos and Wright (2005) has agents receiving a preference shock in the CM revealing whether they will be a buyer or a seller in the DM. In Rocheteau and Wright (2005), buyers are always buyers and sellers are always sellers. All our results hold for both frameworks.
3 The Model

Assume that agents in the DM cannot commit and there are no enforcement or punishment mechanism. Hence, buyers must bring a medium of exchange into the DM to pay sellers. Let real assets be that medium of exchange. The total asset supply is fixed at $A_s$ and initially held by all buyers. Let $A = A_s / N$ be the average amount of assets held by each buyer.

Each buyer bringing back an amount of asset $a$ into the CM and solves

$$W_t^b(a) = \max_{x, h, \hat{a}} \left\{ U(x) - h + \beta V_{t+1}^b(\hat{a}) \right\} \quad \text{s.t.} \quad x = (\varphi_t + \rho) a + h - \varphi_t \hat{a},$$

where $\hat{a}$ is the asset holding carried into the following DM. Buyers participate in the DM if $V_{t+1}^b > 0$. Eliminating $h$ from the budget constraint and solving for optimal $x^*$ yields,

$$W_t^b(a) = \Sigma + (\varphi_t + \rho) a + \max_{\hat{a}} \left\{ \beta V_{t+1}^b(\hat{a}) - \varphi_t \hat{a} \right\}, \quad (1)$$

where $\Sigma = U(x^*) - x^*$ and $U'(x^*) = 1$. Similarly, for a seller with $a$ we have

$$W_t^s(a) = \Sigma + (\varphi_t + \rho) a + \max_{\hat{a}} \left\{ \beta V_{t+1}^s(\hat{a}) - \varphi_t \hat{a} \right\}. \quad (2)$$

The buyer’s value function in the DM is

$$V_t^b(a) = \frac{\alpha(n)}{n} \left[ u + W_t^b \left( a - \frac{p}{\varphi_t + \rho} \right) \right] + \left[ 1 - \frac{\alpha(n)}{n} \right] W_t^b(a), \quad (3)$$

where $p$ is the price paid by the buyer for the DM good, measured by the numeraire. Using $\partial W_t^b / \partial a = \varphi_t + \rho$,

$$V_t^b(a) = \frac{\alpha(n)}{n} (u - p) + W_t^b(a).$$

Similarly, for sellers, we have

$$V_t^s(a) = \alpha(n) (p - c) + W_t^s(a), \quad (4)$$

Alternatively, one can assume that agents use assets as collateral to get credit in the DM, as in Kiyotaki and Moore (1997) with the pledgeability parameter to be 1, and all the results still hold. Lagos et al. (2016) have elaborated on these two setups being mathematically equivalent.
since sellers do not need assets for trading purposes in the DM, but they can still use assets as a store of value. A necessary condition for sellers to hold assets is \( \varphi_t \leq \beta(\varphi_{t+1} + \rho) \), i.e., the asset is priced at most at its fundamental value, where \( \varphi_{t+1} + \rho \) is the anticipated real return on assets measured by CM goods. In any time \( t \), Buyers pay \( \varphi_t \hat{a} \) with a fully anticipated future value of \( (\varphi_{t+1} + \rho) \hat{a} \) at the time of transaction in the DM. This is used as payment for the indivisible good as long as \( p \leq (\varphi_t + \rho) \hat{a} \). After successful trades, sellers end up with assets which they can sell in the subsequent CM of period \( t + 1 \) at price \( \varphi_{t+1} \). The equilibrium price \( \varphi_{t+1} \) is determined by supply and demand in the CM. At time \( t \), agents anticipate fully this equilibrium price for the asset. Given discounting, for buyers the anticipated total cost (gain) from carrying the assets for trade in the DM is

\[
s_t \beta(\varphi_{t+1} + \rho) \hat{a} = [\varphi_t - \beta(\varphi_{t+1} + \rho)] \hat{a} > 0 \quad (< 0),
\]

where we refer to \( s_t = \frac{\varphi_t}{\beta(\varphi_{t+1} + \rho)} - 1 \) as the spread on the assets or liquidity premium.\(^7\) If \( s_t < 0 \) both buyers and sellers would demand the asset for store of value since they can make a capital gain holding the asset for store of value. Given the fixed assets supply \( A^s \), this would lead to excess demand leading to an increase in \( \varphi_t \) until it hits \( s_t = 0 \) in which case the asset is priced at its fundamental value. Therefore, in any equilibrium path, including at the stationary equilibrium, we have \( s_t \geq 0 \). If when priced at the fundamental there is still an excess demand, then \( s_t > 0 \). Sellers would no longer hold the assets and buyers would hold all the assets for transaction purpose only given that the assets are costly to hold. In any period, buyers pay \( \varphi_t \hat{a} \) to acquire the assets for transactions, and they get a real return of \( (\varphi_{t+1} + \rho) \hat{a} \) at the beginning of the subsequent DM which they use for payment. The total cost for each buyer carrying the asset is given by \( s_t \beta(\varphi_{t+1} + \rho) \hat{a} \) made of the liquidity premium paid on acquiring the asset for transactions \( s_t \), the discounted opportunity cost of each asset \( \beta(\varphi_{t+1} + \rho) \) and the amount of assets \( \hat{a} \). Thus, in any equilibrium including stationary ones we have \( s_t \geq 0 \) and with equality when the asset is priced at its fundamental value.

\(^7\)Using \( \beta = \frac{1}{1+r} \), rewrite the spread equation as \( 1 + s = (1 + r) \frac{\varphi_t}{\varphi_{t+1} + \rho} \). This is reminescent of the Fisher equation used in monetary models where \( 1 + i = (1 + r) \frac{\phi}{\pi_{t+1}} \) with \( i \) the nominal interest rate set by monetary policy and \( \phi \) the price of money in terms of the numeraire good in the CM. There \( i \) is exogenous while with fixed assets the spread \( s \) is endogenous. Thus \( i \) is the spread on money.
For any given value of $s_t$ along any equilibrium, from (5), the assets price at time $t$ is given by

$$\varphi_t = \beta(\varphi_{t+1} + \rho)(1 + s_t).$$

(6)

Given the anticipated any $(\varphi_{t+1}, \beta, \rho)$ there is bijection between $s_t$ and $\varphi_t$. This comes handy in solving for stationary equilibrium from supply and demand for liquidity which, as shown below, are both function of $s$. Once solved for $s$, we can back out the equilibrium price $\varphi$.

Since $s_t \geq 0$ along any equilibrium including stationary ones, for any price of the indivisible good $p$, anticipated under bargaining and observed under competitive search with price posting, the asset is weakly costly to carry. Buyers have no incentives to bring more than the price. Therefore, under any trade mechanism, the feasibility constraint $p = (\varphi_t + \rho)a$ is binding in any period.

### 3.1 Bargaining

The generalized Nash bargaining problem is

$$\max_p (u - p)^\eta (p - c)^{1-\eta} \text{ s.t. } p \leq (\varphi_t + \rho)a, \; u - p \geq 0, \; p - c \geq 0.$$ 

Since the feasibility constraint $p = (\varphi_t + \rho)a$ is binding and $c \leq (\varphi_t + \rho)a \leq \bar{p}^B$, where $\bar{p}^B = (1 - \eta)u + \eta c$ is the unconstrained bargaining solution. Any negotiated price $p \in [c, \bar{p}^B]$ is a potential bargaining solution.\(^8\) Substituting $V_{t+1}^b$ into $W_t^b$ and a buyer’s CM value function is:

$$W_t^b(a) = \Sigma + (\varphi_t + \rho)a + \beta W_{t+1}^b(0) + \max_{\tilde{a}} \left\{ \beta \frac{\alpha(n)}{n} (u - p) + [\beta (\varphi_{t+1} + \rho) - \varphi_t] \tilde{a} \right\}.$$

The buyer’s problem can be rewritten as

$$\tilde{V}^b(\tilde{a}) \equiv \max_{\tilde{a} \in [\underline{a}, \bar{a}]} \left\{ \beta \frac{\alpha(n)}{n} [u - (\varphi_{t+1} + \rho) \tilde{a}] - s_t \beta (\varphi_{t+1} + \rho) \tilde{a} \right\},$$

(7)

where $\underline{a} = c/(\varphi_{t+1} + \rho)$ and $\bar{a} = [(1-\eta)u + \eta c]/(\varphi_{t+1} + \rho)$. It is apparent that $\tilde{V}^b(\tilde{a})$ is strictly

\(^8\)Using Proportional bargaining (Kalai (19XX) for terms of trade yields the same results.
decreasing in \( \hat{a} \) for all values of \( s_t \geq 0 \). The optimal solution satisfies \( \hat{a}^*(\varphi_{t+1} + \rho) = c \).

With bargaining, a buyer can commit to not paying more than the seller’s reservation price \( c \). The buyer’s expected net value from participating in the DM is

\[
\tilde{V}^b(n, s_t) = \beta \left[ \frac{\alpha(n)}{n} (u - c) - s_t c \right] \geq 0, \tag{8}
\]

which is the discounted expected total benefit net of the cost of carrying the asset. The measure of DM buyers \( n^* \leq N \) is determined by a free entry condition, \( \tilde{V}^b(n^*, s_t) \geq 0 \). When \( s_t = 0 \), the asset is priced at its fundamental value and \( \tilde{V}^b(n^*, s_t) > 0, \forall n^* \leq N \), so all buyers participate. However, since \( s_t \) is independent of \( n \), and \( \frac{\alpha(n)}{n} \) is strictly decreasing in \( n \) (a congestion effect for buyers), larger \( s_t \) leads to lower \( n \) when \( \tilde{V}^b(n^*, s_t) = 0 \). This leads to \( n^*(s_t) \) with \( n^*(s_t) < 0 \). Note that \( s_t \) is function of asset prices itself endogenous.

As mentioned previously, we use the supply and demand for liquidity as function of \( s \), and back out the stationary equilibrium asset price \( \varphi \) and participation \( n^* \).

The asset prices in stationary equilibria satisfy \( \varphi_t = \varphi \), and \( s_t = s, \forall t \). To establish equilibrium existence and uniqueness, we use the demand and supply for liquidity. Under bargaining, they are \( L^d(s) = n^*(s)c \) and \( L^s(s) = (\varphi(s) + \rho)A^s \) with \( n^*(s) \) determined from \( \tilde{V}^b(n^*, s_t) = 0 \), and \( \varphi(s) \) is solved from (6),

\[
\varphi(s) = \frac{\beta \rho (1 + s)}{1 - \beta (1 + s)} = \frac{\rho (1 + s)}{r - s}. \tag{9}
\]

When \( s = 0 \), \( \varphi = \frac{\rho}{r} \equiv \varphi^F \) is the fundamental asset price in stationary equilibrium. However, if \( s \neq r \),

\[
L^s(s) = \frac{\rho (1 + r)}{r - s} A^s = n^*(s)c = L^d(s), \tag{10}
\]

and when \( r = s \), the supply of liquidity is infinitely elastic. The stationary equilibrium \( s^* \geq 0 \) depends on critical values of parameters \((\rho, r, c, A^s)\) and from (10),

\[
s^* = r - \frac{\rho A^s (1 + r)}{n^* c}, \tag{11}
\]

with \( n^* \) determined when (8) holds with equality.

**Lemma 1** There exist \( \bar{s}^B \geq r \) and \( s^N \leq \bar{s}^B \), such that: (i) for \( s \leq s^N \), \( \exists ! L^d \) with \( n^* = N \),
and \(L^d = Nc\); (ii) for \(s \in (s^N, \bar{s}^B]\), \(\exists! L^d\) with \(n^* < N\), \(L^d = n^*c\) and \(dL^d/ds < 0\); (iii) for \(s > \bar{s}^B\), \(\nabla n^* > 0\) and \(L^d\) is not well-defined.

**Lemma 2** (i) For \(\rho < 0\) (\(r < s\)), \(L^s\) is convex and \(dL^s/ds < 0\); (ii) for \(\rho = 0\), \(L^s\) is perfectly elastic at \(s = r\); (iii) for \(\rho \in (0, \rho^F)\) (\(0 < s < r\)), \(L^s\) is concave and \(dL^s/ds > 0\); (iv) for \(\rho \geq \rho^F\), \(L^s\) is perfectly elastic at \(s = 0\).

Define \(\rho^N\) to be the dividend value corresponding to \(s^N\). Let \(\rho\) be the cutoff for the existence of DM trades. And let \(\rho^F\) again denote the cutoff value above which assets are priced fundamentally. Next lemma characterizes the aggregate supply of liquidity. We summarize the equilibria in the following proposition.

**Proposition 1** In the model with bargaining: (i) for \(\rho \geq \rho^F > 0\), \(\exists!\) stationary equilibrium (SE) with \(\varphi = \varphi^F\) and \(n^* = N\); (ii) for \(\rho \in [\rho^N, \rho^F)\) and \(\rho^N > 0\), \(\exists!\) SE with \(\varphi = \varphi^N > \varphi^F\) and \(n^* = N\); (iii) for \(\rho \in (0, \rho^N]\) and \(\rho^N > 0\), \(\exists!\) SE with \(\varphi = \varphi^* > \varphi^F\) and \(n^* < N\). (iv) for \(\rho \in [\rho^N, 0]\) and \(\rho^N < 0\), \(\exists\) two SE, one with \(\varphi = \varphi^N > \varphi^F\) and \(n^* = N\); and the other with \(\varphi = \varphi^* > \varphi^F\) but \(\varphi^* < \varphi^N\) and \(n^* < N\); (v) for \(\rho \in (\rho, \rho^N)\) and \(\rho < 0\), \(\exists\) two SE, with \(\varphi_h = \varphi^* > \varphi_i = \varphi^* > \varphi^F\) and \(n^*_h < n^*_i < N\); (vi) for \(\rho = \rho < 0\), \(\exists!\) SE with \(\varphi = \varphi^* > \varphi^F\) and \(n^* < N\); (vii) for \(\rho < \rho\), \(\nexists\) SE.

Basically, for all positive dividend value there exist a unique stationary equilibrium, while for negative dividend value there are two equilibria.

For case (i) if \(\rho \geq \rho^F = (1 - \beta)cN/A^s > 0\) where the value of \(\rho^F\) solves (10) with \(n^* = N\), a buyer does not need to carry many assets for trading purposes in the DM, and the marginal holder of assets is a seller. Assets are priced at the fundamental value \(\varphi^F = \rho/r\). In this case, the spread \(s = 0\), the participation constraint (8) becomes \(\bar{V}^b(n) = \beta(u - c)\alpha(n)/n \geq 0\), which is positive for all \(n \leq N\). All buyers participate and the liquidity needs of all buyers are satisfied.

For case (ii), if \(\rho \in [\rho^N, \rho^F) \subset \mathbb{R}_+\), \(s^* > 0\) and from (11) into the participation constraint (8),

\[
\bar{V}^b(n^*) = \beta \frac{\alpha(n^*)}{n^*}(u - c) - (1 - \beta)c + \frac{\rho A^s}{n^*} \geq 0.
\]
Define \( B(n^*) = \beta \alpha \frac{n^*}{n^*} (u - c) - (1 - \beta)c \) as the discounted total benefit minus the flow payment in stationary equilibrium for a buyer. The per buyer participation value in the DM becomes

\[
\tilde{V}^b(n^*) = B(n^*) + \frac{\rho A^s}{n^*},
\]

with \( B(n^*) \) strictly decreasing in \( n^* \), sharing the same properties as \( \frac{\alpha(n^*)}{n^*} \). For all \( \rho \geq 0 \), \( \tilde{V}^b(n^*) \) is strictly decreasing in \( n^* \). Buyers receive an extra benefit from carrying the asset with positive dividend. However, it can be that \( B(n^*) < 0 \), and buyers still participate given the positive benefit of carrying the asset with positive dividend. Hence, \( \tilde{V}^b(N) = 0 \) defines a \( \rho^N = -\frac{N}{A^s} B(N) \equiv f(N) \) (that we use in the proof). For all \( \rho^N \leq \rho \), \( \tilde{V}^b(n^*) > 0 \) for all \( n^* \leq N \), and thus \( n^* = N \) with all buyers participating. The marginal holder of assets is a buyer, who cares about liquidity. The liquidity demand drives up the asset price to be above its fundamental value, and sellers no longer hold assets. Using (11) evaluated at \( n^* = N \) gives

\[
\varphi = \frac{Nc}{A^s} - \rho \equiv \varphi^N > \varphi^F.
\]

Now the asset price is strictly decreasing in \( \rho \) and \( A^s \). The liquidity premium on the asset decreases with total liquidity \( \rho A^s \). For case (iii) \( \rho \in [0, \rho^N) \), \( s^* > 0 \), we have that \( \tilde{V}^b(N) < 0 \). If all buyers were to participate, the congestion effect due to the properties of the matching technology is too strong to generate positive value. Hence, the unique equilibrium entails \( n^* < N \), and \( \varphi^{n^*} = \frac{n^* c}{A^s} - \rho \). These three cases are illustrated in Figure 1.a. and 1.b.

For the other cases, when \( \rho < 0 \), there exist two stationary equilibria up to a cut off value \( \underline{\rho} < 0 \) at which the stationary equilibrium is unique again (case (vi)). These are illustrated in Figure 1.c and 1.d. From the participation constraint:

\[
\tilde{V}^b(n^*) = B(n^*) + \frac{\rho A^s}{n^*} = 0 \text{ or } f(n^*) \equiv \frac{B(n^*) n^*}{A^s} = -\rho.
\]

We show in Appendix that \( f(n^*) \) is strictly concave when \( \rho < 0 \) for \( n^* \in [0, N] \). In addition, we show that there is a unique value of \( \underline{\rho} = -\max_{n^* \in [0, N]} f(n^*) \). This extreme cut-off negative dividend value is such that, while participation is maximizing a buyer’s participation benefit, the cost of carrying liquidity, \( \underline{\rho} A^s < 0 \), provides zero net surplus.
Interestingly, the solution to $\max_{n^* \in [0,N]} f(n^*)$ is characterized by

$$\alpha'(n^*) = \frac{(1 - \beta)c}{\beta(u - c)}$$

which is unique and corresponds to the well-known Hosios (1990) condition for efficient entry.\(^9\) The marginal contribution to the matching process by a buyer equals the flow cost of entry $(1 - \beta)c$, over the discounted total surplus $\beta(u - c)$. For all $\rho < \rho < 0$, there is no equilibrium (case (vii)).

For other values of $\rho \in [\rho^N, 0)$ and $\rho \in (\rho, \rho^N)$, cases (iv) and (v), the concavity of $f(n^*)$ implies two values of $n^*$ and $\varphi$ for each $\rho$. In the first region, the equilibria entail one with full participation $n^*_h = N$ with $\varphi^N > \varphi^F$ and one with $n^*_l < N$ and $\varphi^F < \varphi^{n^*_l} < \varphi^N$. In the second region $(\rho, \rho^N)$, the equilibria entail one with low participation $n^*_l < N$ with $\varphi^F < \varphi^{n^*_l}$, and one with high participation $n^*_l < n^*_h < N$ and $\varphi^{n^*_l} < \varphi^{n^*_h}$. Both have less than full participation. To understand the intuition, note that from (9), the asset price is negatively related to the spread $s$ when $\rho < 0$. As such, a low asset price implies a high spread, making participation more costly. The congestion effect in the matching process leads to lower participation, until free entry condition is restored. This is easily represented by

$$\tilde{V}^b(n^*_l) = B(n^*_l) + \frac{\rho A^s}{n^*_l} = B(n^*_h) + \frac{\rho A^s}{n^*_h} = \tilde{V}^b(n^*_l) = 0$$

with $B(n^*_l) > B(n^*_h)$ and $\frac{\rho A^s}{n^*_l} > \frac{\rho A^s}{n^*_h}$ since $\rho < 0$. When $\rho < 0$, there is a coordination issue among buyers at the participation stage weighting the congestion effect of the matching process with the cost total cost of carrying liquidity. Yet, unless the negative dividend value hits the cut-off $\rho$, the Hosios’ entry condition fails. It is as if the cut-off value removes the coordination problem and generates uniqueness.

\(^9\)The Hosios’ entry condition is most commonly expressed as $\varepsilon(n) = \frac{(1 - \beta)c}{\beta u (u - c)}$ with $\varepsilon(n) = \frac{\alpha'(n)n}{\alpha(n)}$. 

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Figure 1 completely depicts the equilibrium values of $n$ and the asset price $\varphi$ w.r.t. the dividend $\rho$, depending on different values of $\varphi$. When $\rho < 0$, the negative dividend is similar to a storage cost, akin to Kiyotaki and Wright (1989). There are two prices and two different levels of buyer’s participation in the DM, high and low, corresponding to two stationary equilibria. When the equilibrium participation is high, a larger liquidity demand drives up the asset price $\varphi$, which implies a smaller asset spread $s$, since $\partial s/\partial \varphi < 0$ when $\rho < 0$. Buyers now face a low probability of trade in the DM, but they are compensated by a small cost of holding assets. Similarly, when the equilibrium participation is low, buyers receive a high probability of trade with a large holding cost. A similar trade-off, between the probability of trade and the trading price, generates multiple equilibria in Rocheteau and Wright (2005), while in their monetary model, the cost
of holding liquidity is exogenously given at $i$, the nominal interest rate. When $\rho \geq 0$, this coordination problem does not exist, since $\partial s / \partial \varphi \geq 0$. Less participation implies a higher probability to trade and a lower spread of holding assets, and hence the equilibrium is unique.

When $\rho = 0$, assets are equivalent to money with a constant supply, and the results are comparable to Han et al. (2016). In a monetary economy, a higher participation in the DM implies lower probability of trade for buyers. Since the cost of holding money is not adjusting with $n$, buyers are strictly worse off, and the above coordination does not exist. For $\rho > 0$, an equilibrium with assets always exists. However, monetary equilibrium may not exist with deflation, if the surplus from trade is small enough, since the cost of holding money is independent of $n$. When $s = 0$, carrying assets becomes costless, the asset economy is the same as a credit economy.

When $\rho$ is large enough, assets are priced at the fundamental value and they are not affected by the buyer’s participation in the DM. For any dividend between $\rho^N$ and $\rho^F$, the asset price is above its fundamental price and decreasing in $\rho$. When $\rho$ becomes small enough, i.e. $\rho < \rho^N$, the asset dividend and the buyer’s participation $n^*$ affect the asset price in opposite directions, and $\varphi$ is increasing in $\rho$ in the stable equilibrium. For $\rho < 0$, the asset prices are still positive due to the liquidity premium in facilitating DM trade.

At this point, one may wonder what happens to the stationary equilibrium if we change the total asset supply $A^s$. To understand the effects, we define $\mathcal{A} = \rho A^s$ to be the amount of total liquidity in the market. One can notice that the asset price and the buyers’ participation are affected by both the dividend value $\rho$ and the supply of assets $A^s$. If $\rho = 0$, we are back to the money case and intuitively, $\partial \varphi / \partial A^s < 0$ and $\partial n / \partial A^s = 0$. More interesting cases are when $\rho \neq 0$. In particular, when $\rho > 0$, an increase in either $\rho$ or $A^s$ will make liquidity $\mathcal{A}$ more abundant, but if $\rho < 0$, i.e., assets bear a storage cost, an increase in $\rho$ or a decrease in $A^s$ actually implies more costly liquidity. The following proposition summarizes the effects of changing $\mathcal{A}$ under bargaining, and wlog we fix the value of $A^s$.

**Proposition 2** In the stable SE with bargaining: (i) for $\mathcal{A} \geq \rho^F A^s$, we have $\partial \varphi / \partial \mathcal{A} > 0$ and $\partial n / \partial \mathcal{A} = 0$; (ii) for $\mathcal{A} \in [\rho^N A^s, \rho^F A^s)$, $\partial \varphi / \partial \mathcal{A} < 0$ and $\partial n / \partial \mathcal{A} = 0$; (iii) for
\( A \in (\rho A^s, \rho N A^s) \), \( \partial \varsigma / \partial A > 0 \) and \( \partial n / \partial A > 0 \). In the unstable SE with bargaining, for \( A \in (\rho A^s, 0) \), \( \partial \varsigma / \partial A < 0 \) and \( \partial n / \partial A < 0 \).

When \( A \) is large, i.e., liquidity is abundant in the economy, all transactional needs of buyers are satisfied, and hence assets have no liquidity premium and are priced at the fundamental value. All buyers participate in the DM due to zero holdup cost.

When liquidity is relatively scarce, i.e., in case (ii) of Proposition 2 and \( A \) is not very large, buyers start to pay a liquidity premium for holding assets. While all the buyers still participate in the DM, the asset price increases with a drop in \( A \). For simplicity, we fix the value of \( A^s \) and a drop in \( A \) implies a decrease in \( \rho \). Alternatively, a decrease in asset supply \( A^s \) when \( \rho > 0 \) or an increase in \( A^s \) when \( \rho < 0 \) may also cause a shortage of liquidity. In order to satisfy the demand of assets for transactional purposes in the DM, asset price actually increases facing the shortage of liquidity, which implies an even higher liquidity premium. This result echoes a key point in many New Monetarist based literature: liquidity plays a key role in determining the price of an asset.

As \( A \) decreases further, liquidity becomes even more scarce, and buyers start to drop out of the DM. An adjustment in the extensive margin leads to two opposite effects. As \( \rho \) decreases, the asset price also drops and more buyers choose not to enter the DM, while the participating buyers are compensated by a larger probability to trade. This is the standard “hot potato” effect: people trade faster as the cost of transaction increases. On the other hand, when \( \rho < 0 \), the asset is toxic, i.e., it has a storage cost. If a decrease in \( A \) is caused by more toxic assets in the economy, there is an incentive for more buyers to participate in the DM. More buyers lead to a higher demand for assets in DM transactions, which drives up liquidity premium, offsets the negative dividend, increases the asset price, and lowers the cost of carrying liquidity. This channel works as if more buyers get involved to share the toxic nature of the assets, and we call it the “poison apple” effect. Both effects are present when \( \rho < 0 \). While the “hot potato” effect dominates at the stable equilibrium, the “poison apple” effect prevails at the unstable equilibrium.

Notice that we need both adjustable extensive margin and endogenous liquidity cost to have the “poison apple” effect. Lagos and Rocheteau (2005) generates the “hot potato”
effect but not the second one, since agents do not have participation decision. Liu et al. (2011) In fact, the “poison apple” effect does not exist in any monetary models, because the cost of holding liquidity, i.e., money is always exogenous and independent of $n^*$. 

3.2 Competitive Search

We study competitive search equilibrium with price posting. As in Moen (1997) and Rocheteau and Wright (2005), instead of a single DM, there exist a continuum of sub-markets, each identified by masses of sellers posting the same terms of trade. Sellers post DM prices before buyers enter the DM. All sellers commit to their posted prices. After observing all the posted prices, each buyer chooses the one that gives him the maximum surplus. Each seller can only produce for one buyer in each period. If a seller is visited by multiple buyers, he chooses one with equal probability. Let $n$ represent the expected queue length for any seller in a submarket offering price $p$. As before, the meeting rate for sellers is $\alpha(n)$, and $\alpha(n)/n$ for buyers in the submarket featuring $p$. By posting a lower price, a seller attracts more buyers and increases his trading probability.

The buyer’s DM value function is now

$$V^b_t(p,a) = \frac{\alpha(n)}{n} (u - p) + W^b_t(a),$$

where $p$ is the price posted by the chosen seller. From (1) and (12), buyers’ value is

$$W^b_t(a) = \Sigma + (\varphi_t + \rho) a + \beta W^b_{t+1}(0) + \max_{a,p,n} \left\{ \frac{\alpha(n)}{n} (u - p) + \left[ \beta \left( \varphi_{t+1} + \rho \right) - \varphi_t \right] \hat{a} \right\}.$$

Let $\Omega$ be the equilibrium expected utility of a buyer in the DM. A seller solves

$$\max_{p,n} \alpha(n) (p - c) \text{ s.t. } \frac{\alpha(n)}{n} (u - p) - sp \geq \Omega, p \leq (\varphi_t + \rho) a.$$

The seller’s price posting problem after substituting $p$ from the constraint yields

$$\max_n \pi(n) = \alpha(n) \left[ \frac{\alpha(n) u - n\Omega}{\alpha(n) + ns} - c \right],$$

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and $p^e$ is the seller’s optimal price

$$
p^e = \frac{\alpha (n^*) \{[1 - \varepsilon (n^*)] u + \varepsilon (n^*) c\} + \varepsilon (n^*) n^* s c}{\alpha (n^*) + \varepsilon (n^*) n^* s}, \tag{14}
$$

where $\varepsilon (n) = \alpha'(n) n / \alpha(n)$ is the elasticity of the matching function and $\varepsilon (n) < 1$. In equilibrium, the optimal queue length is consistent with free entry

$$
\frac{\alpha (n^*)}{n^*} (u - p^e) - s p^e = \Omega \geq 0. \tag{15}
$$

Equations (14) and (15) generate $(p^e(s), n^*(s))$. We study the existence and uniqueness of equilibrium by equating the aggregate demand and supply of liquidity, taking the price $\varphi$ as given for now, since it is an endogenous variable. Then, we back out asset price $\varphi$ and participation $n^*$ in equilibrium. The aggregate demand of liquidity $L^d(s) = n^*(s)p^e(s)$ is a function of the spread $s$. From $s = (r\varphi - \rho)/(\varphi + \rho)$, a one-to-one mapping from the asset price $\varphi$ to $s$, we can solve for $\varphi = (1 + s)\rho/(r - s)$, and the aggregate supply is $L^s(s) = (\varphi + \rho)A^s = (1 + r)\rho A^s/(r - s)$.

The aggregate demand and supply of liquidity are characterized by the following two lemmas.

**Lemma 3** There exist $s^C \geq r$ and $s^N \leq s^C$, such that: (i) for $s \leq s^N$, $\exists!$ $L^d$ with $n^* = N$, and $dL^d/ds < 0$; (ii) for generic $s \in (s^N, s^C)$, $\exists!$ $L^d$ with $n^* < N$, and $dL^d/ds < 0$; (iii) for $s > s^C$, $\forall n^* > 0$ and $L^d$ is not well-defined.

Define $\rho^N$ and $\bar{\rho}^C$ to be the dividend values corresponding to $s^N$ and $s^C$. Recall $s$ is the spread of assets and $\partial s / \partial \rho < 0$. As shown in Lemma 3, if the asset dividend is low enough, i.e., $\rho < \bar{\rho}^C$, and the cost of holding assets is high enough, the DM will shut down. As long as the DM operates and $L^d$ is well-defined, it is monotonically decreasing in $s$. The DM participation of buyers varies depending on different values of $\rho$ hence $s$. Now let $\rho^F$ again denote the cutoff value of dividend under competitive search, above

\[10\] Interestingly, if we had divisible goods $q \in \mathbb{R}_+$ and sellers posting $(p, q)$ we can obtain an additional equilibrium condition $u'(q)/c'(q) - 1 = s/\alpha(n)$ as in Lagos and Wright (2005) with money. Thus when asset is value at its fundamental value, we get $q^*$, the efficient quantity. Furthermore, solving this equation for $s$ and substituting it into (14) gives $p^e = ((1 - \varepsilon (n^*)) u(q)c(q) + \varepsilon (n^*) c(q)u'(q))/\varepsilon (n^*) u'(q) + (1 - \varepsilon (n^*)) c'(q)$ the standard pricing equation in monetary environment, and also with bargaining replacing $\varepsilon (n^*)$ by the bargaining power parameter of buyers.
which assets are priced fundamentally. Next lemma characterizes the aggregate supply of
liquidity.

**Lemma 4** (i) For \( r < s \), \( L^s \) is convex and \( dL^s/ds < 0 \); (ii) for \( \rho = 0 \), \( L^s \) is perfectly elastic at \( s = r \); (iii) for \( \rho \in (0, \rho^F) \) \( (0 < s < r) \), \( L^s \) is concave and \( dL^s/ds > 0 \); (iv) for \( \rho \geq \rho^F \), \( L^s \) is perfectly elastic at \( s = 0 \).

Notice that the spread of assets can be rewritten in two parts, \( s = r - (1+r)\rho/\varphi \). If \( \rho = 0 \) and assets have no dividend return, the second term vanishes and only the discount factor is left. Notice that now all the cutoff values of asset dividend are defined under competitive search, different from the bargaining case.

**Proposition 3** In the model with competitive search, there exist \( \rho^N < \rho^F \), and \( \rho \), such that: (i) for \( \rho \geq \rho^F \), \( \exists! \) symmetric SE with \( \varphi = \varphi^F \) and \( n^* = N \); (ii) for \( \rho \in (\rho^N, \rho^F) \), \( \exists! \) symmetric SE with \( \varphi = \varphi^N > \varphi^F \) and \( n^* = N \); (iii) for (generic) \( \rho \in [\rho, \rho^N] \), \( \exists! \) symmetric SE if \( \rho > 0 \), and \( \exists! \) symmetric SE if \( \rho < 0 \), with \( \varphi = \varphi^N > \varphi^F \) and \( n^* \leq N \) \( (\leq N \rho N) \); (iv) for \( \rho < \rho \), \( \exists! \) equilibrium with an active DM.

For \( \rho > 0 \), there is a unique equilibrium under both competitive search and bargaining, since there does not exist a trade-off between the probability to trade and the cost of holding liquidity. For \( \rho < 0 \), multiple equilibria exist with bargaining, but the equilibrium is still unique for \( \rho \in (\rho^N, 0) \). This is because buyers search randomly and the equilibrium price in the bargaining game is the seller’s reservation value, independent of the market tightness in the DM. With competitive search, the prices posted by sellers direct the buyers’ search behavior and serve as a coordination device.

Proposition 3 also shows uniqueness for \( 0 < \rho < \rho^N \), i.e., \( r > s > s^N \). While in Han et al. (2016), when money is the medium of exchange, there may still exist multiple equilibria for \( r > i > i^N \). The different result is due to the cost of holding assets being endogenously determined while the cost of holding money is an exogenous policy variable. For \( i > i^N \), the liquidity demand for money may not be unique for a countable number of interest rates. For these exogenous \( i \), there are multiple equilibria featuring different real money balances. With assets, the liquidity demand can have multiple values at a
countable number of $s$ as well, but the spread is endogenously determined by $L^d = L^s$. According to Lemma 4, $L^s$ is monotonically increasing in $s$, and there is a unique asset spread given $n$. For $\rho > 0$, the asset spread is increasing in $n$. With more buyers entering the DM, they face a higher cost of holding assets and a lower probability of trade. There is no coordination problem and a unique equilibrium $n^*$ with a unique asset spread exists.

We also obtain uniqueness if the cost of holding assets is a constant and independent of $n$, such as $s = r$ with $\rho = 0$. When holding assets is costless, i.e., $s = 0$, and the asset is priced at its fundamental value, the equilibrium has the same price and participation in the DM as a credit economy.

When $\rho < 0$ and $n^* < N$, we lose generic uniqueness, and this result is also different from the monetary economy. For a given spread $s$, one can show that equilibrium is generically unique, just as the monetary case, and sellers post only one price given the spread. Because $s$ is endogenously determined, there may exist equilibria with different spreads, each generating the same payoff for both buyers and sellers. For example, one equilibrium may feature a higher probability to trade and a higher spread of carrying assets, while another has a lower probability and a smaller cost of liquidity, making buyers indifferent in terms of equilibrium payoff and giving sellers the same profit. In this situation, competitive search still serves as a coordination device under the same spread, but there may exist equilibria with different values of $s$. 
Figure 2 shows the relationship between equilibrium participation $n^*$ and dividend $\rho$ by the dashed curves below the horizontal axis. Above the horizon, the solid curves represent asset price $\varphi$ as a function of $\rho$. As long as the dividend is high enough, all buyers participate in the DM and assets are priced at the fundamental value. If $\rho$ is smaller than $\rho^N$, not all buyers enter the DM. Since a larger dividend implies a smaller spread $s$, i.e., a lower cost of holding assets, the buyers’ participation is monotonically increasing in $\rho$. However, the asset price $\varphi$ may change in a non-monotonic way with respect to $\rho$. Equating the demand and supply of liquidity, we get the asset price $\varphi = L^d/A^s - \rho$, which is the difference between the dividend and the return of holding assets. As $\rho$ gets larger, the asset return also increases due to a higher demand induced by $\rho$. Then, the change in asset price depends on how much the liquidity demand responds to $\rho$, which is ambiguous.
under general parameter values.

Let $A$ again denote the amount of total liquidity in the market, and wlog we fix asset supply $A^*$. The next proposition summarizes the effects of changing $A$ under competitive search.

**Proposition 4** In the SE with competitive search: (i) for $A \geq \rho^F A^*$, we have $\partial \varphi / \partial A > 0$ and $\partial n / \partial A = 0$; (ii) for $A \in [\rho^N A^*, \rho^F A^*)$, $\partial n / \partial A = 0$ and $\partial \varphi / \partial A$ is ambiguous; (iii) for $A \in (\rho A^*, \rho^N A^*)$, $\partial \varphi / \partial A$ and $\partial n / \partial A$ are ambiguous.

These findings are similar to the bargaining case. When liquidity is abundant in the economy, all buyers participate in the DM and assets are priced at the fundamental value, and hence $\varphi$ increases with $A$. As the amount of liquidity decreases in the economy, the asset price exceeds its fundamental value due to liquidity premium, and $\varphi$ may either increase or decrease with $A$. Buyers start to stop participating in the DM. For $\rho < 0$ and $n^* < N$, both the “hot potato” effect and the “poison apple” effect exist. Under general parameter values, either one may be the dominant force, and the effect of changing $A$ on $\varphi$ and $n$ is ambiguous at different equilibrium.

### 3.3 Discussion

Indivisibility matters mainly from losing an intensive margin of adjustment. It makes the available surplus fixed without endogenous participation of buyers in the DM. In addition, different pricing mechanisms yield different results, and it matters if assets are used compared to pure credit or money. In this section, we compare the results in an asset economy with a pure credit economy and a monetary economy, which are studied in Han et al. (2016).

To ease the comparison, we catalog the different cases. Let $n \leq N$ be the active measure of buyers in the DM. Let $B^j_L(n)$ be buyers’ benefit from participation, $L \in \{a, c, m\}$ be the three types of liquidity, assets, credit, and money, and $j \in \{b, c\}$ the type of pricing mechanisms, bargaining or competitive search. Let $p^j$ be the equilibrium price under the mechanism $j$. 
In the asset economy, we find

\[B^b_a(n) = (u - p^b) \frac{\alpha(n)}{n} \geq s p^b,\]
\[B^c_a(n) = (u - p^c) \frac{\alpha(n)}{n} \geq s p^c,\]

where the spread \(s(n, \rho)\) is decreasing in \(n\) and increasing in \(\rho\). With bargaining, we find unique \(n\) when \(\rho > 0\), but when \(\rho < 0\), there are two equilibrium \(n\) for a range of \(\rho\). Similarly with competitive search, the asset equilibrium is unique or generically unique for \(\rho > 0\), and multiple equilibria exist for \(\rho < 0\). The price reacts to the dividend value and endogenous participation.

In the credit economy, Han et al. (2016) finds that buyers participate in the DM if

\[B^b_c(n) = (u - p^b) \frac{\alpha(n)}{n} \geq 0,\]
\[B^c_c(n) = (u - p^c) \frac{\alpha(n)}{n} \geq 0.\]

The main difference is the bargained price being independent of \(n\), but not under competitive search. As long as \(B^c_c(N) > 0\), all potential buyers participate in the DM.

For the monetary economy, Han et al. (2016) finds

\[B^b_m(n) = (u - p^b) \frac{\alpha(n)}{n} \geq i p^b,\]
\[B^c_m(n) = (u - p^c) \frac{\alpha(n)}{n} \geq i p^c.\]

Since \(\alpha(n)/n\) is decreasing in \(n\), under bargaining, for large enough \(i\), \(B^b_m(N) < i p^b\) and not all buyers participate in the DM. With competitive search, \(p^c\) is increasing in \(n\) and decreasing in \(i\). Higher \(i\) reduces \(p^c\), which increases \(B^c_m(n), \forall n\). But it also increases \(i p^c\). This generates the potential for multiple equilibria with \(n < N\). However, for generic values of \(i\), these possibilities have zero measure. Thus, monetary equilibrium is generically unique.
4 Lotteries

In an environment with indivisible goods, one can consider lotteries. To do so, let $E = \mathcal{P} \times \{0, 1\}$ denote the space of trading events, and $\mathcal{E}$ the Borel $\sigma$-algebra. Define a lottery to be a probability measure $\omega$ on the measurable space $(E, \mathcal{E})$. We can write $\omega(p, q) = \omega_q(q)\omega_{p|q}(p)$ where $\omega_q(q)$ is the marginal probability measure of $q$ and $\omega_{p|q}(p)$ is the conditional probability measure of $p$ on $q$. Without loss of generality, as shown in Berentsen et al. (2002), we restrict attention to $\tau = \Pr\{q = 1\}$ and $1 - \tau = \Pr\{q = 0\}$, and $\omega_{p|0}(p) = \omega_{p|1}(p) = 1$. Randomization is only useful on $q$ because $Q$ is non-convex. Thus, $\tau \in [0, 1]$ is the probability that the good is produced and traded.

When introducing lotteries, the generalized Nash bargaining problem becomes

$$\max_{p, \tau} (\tau u - p)^\eta (p - \tau c)^{1-\eta} \text{ s.t. } p \leq (\varphi + \rho) a, \tau \leq 1,$$

with $\tau u \geq p$ and $p \geq \tau c$.

**Lemma 5** The solution to the bargaining problem is

$$ (p^B, \tau^B) = \begin{cases} \left(\bar{p}^B, 1\right) & \text{if } (\varphi + \rho) a > \bar{p}^B \\ ((\varphi + \rho) a, 1) & \text{if } p^B \leq (\varphi + \rho) a \leq \bar{p}^B \\ ((\varphi + \rho) a, (\varphi + \rho) a/p^B) & \text{if } c \leq (\varphi + \rho) a < p^B \\ (0, 0) & \text{if } (\varphi + \rho) a < c \end{cases} $$

where $\bar{p}^B = (1 - \eta) u + \eta c$ and $p^B = uc / (\eta u + (1 - \eta)c)$.

The buyer’s CM value is

$$ W^b_t(a) = \Sigma + (\varphi_t + \rho)a + \beta W^b_{t+1}(0) + \beta \max_{\hat{a}} v(\hat{a}) $$

where $v(\hat{a}) = (\tau^B u - p^B) \alpha(n)/n - s(\varphi_{t+1} + \rho)\hat{a}$.

**Proposition 5** In the model with bargaining and lotteries: (i) for $\rho \geq \rho^F$, $\exists!$ SE with $\varphi = \varphi^F$ and $n^* = N$; (ii) for $\rho \in [\rho^N, \rho^F)$, $\exists!$ stable SE with $\varphi = \varphi^N > \varphi^F$ and $n^* = N$; (iii) for $\rho \in [\rho, \rho^N)$, $\exists!$ stable SE with $\varphi = \varphi^{n^*} > \varphi^F$ and $n^* < N$; (iv) for $\rho \in (\rho, 0)$, $\exists!$ unstable SE; (v) for $\rho < \rho$, $\exists!$ equilibrium with an active DM; (vi) $p^B = \bar{p}^B$ and $\tau^B = 1$ hold for (i)-(iii).
Lotteries are not used in equilibrium, and buyers bring enough assets to achieve the maximum expected surplus from trade at $\tau^B = 1$. $p^B$ and $\tau^B$ do not change with $\rho$. The buyer’s asset holding is always just enough to pay for the DM transaction, which is not affected by the spread $s$. Compared to the case of bargaining without lotteries, we still get a continuum of equilibria for $\rho \in (\underline{\rho}, 0)$, since the coordination problem still exists. Lotteries do not lead to the uniqueness of equilibrium.

Finally, introducing lotteries to competitive search, the price posting problem becomes

$$\max_{p, \tau, n} \alpha(n)(p - \tau c) \text{ s.t. } \frac{\alpha(n)}{n}(\tau u - p) - sp \geq \Omega, p \leq (\varphi_i + \rho)a, \tau \leq 1$$

The following proposition shows that lotteries are not used in competitive search equilibrium, either.

**Proposition 6** In the model with competitive search and lotteries, there exist $\rho^F$, $\rho^N$, and $\underline{\rho}$, such that: (i) for $\rho \geq \rho^F$, $\exists!$ symmetric SE with $\varphi = \varphi^F$ and $n^* = N$; (ii) for $\rho \in (\rho^N, \rho^F)$, $\exists!$ symmetric SE with $\varphi = \varphi^N > \varphi^F$ and $n^* = N$; (iii) for (generic) $\rho \in [\rho, \rho^N]$, $\exists!$ symmetric SE if $\rho > 0$ ($\rho \leq 0$), with $\varphi = \varphi^\rho > \varphi^F$ and $n^* \leq N$ ($< \text{ if } \rho < \rho^N$); (iv) for $\rho < \rho$, $\not\exists$ equilibrium with an active DM; (v) $\tau^C = 1$ holds for (i)-(iii).

## 5 Conclusion

In this paper, we use a general equilibrium model to study the trade of indivisible goods in financial markets. Indivisibility matters, especially when terms of trade are determined by bargaining. The bargained price gives sellers no surplus and is independent of the dividend on assets. Introducing lotteries does not change the independence on dividend, but sellers are able to extract a positive surplus. Under competitive search, the trading price depends on the dividend of assets and the number of buyers in the market. Lotteries do not matter under competitive search.

Under bargaining, the equilibrium is unique as long as the asset dividend is non-negative. With a negative dividend we find two equilibria, with low and high participation. The congestion nature of the matching technology generates a concave net benefit for buyers in the number of active buyers leading to a coordination problem and two equilibria.
With competitive search and price posting, we find a unique equilibrium for positive dividends, and multiple equilibria exist for negative dividends. While in a monetary economy, using price posting as a coordination device can solve the problem present under bargaining, it cannot eliminate multiplicity when assets are the medium of exchange. This is because the cost of holding assets is endogenously determined but the cost of holding money is exogenous.

Overall, the consequences of indivisibility on the goods side matter and differ from indivisibility on the asset side. Indivisibility affects the bargaining outcome because it isolates the price of goods from dividend value and the number of buyers. Price posting with competitive search reestablishes the link and generically produces a unique equilibrium with certain dividend values. While we have focused on stationary equilibrium, the model can easily be used to study asset price dynamics. We leave this for future research.
References


Appendix

Proof of Lemma 1. Given buyers’ participation constraint, \( n^* = N \) if \( \frac{\alpha(n^*)}{n^*} (u - c) - sc > 0 \). Define \( s^N = \frac{\alpha(N)}{N} \frac{u - c}{c} \), then given \( s \leq s^N \), for all \( n^* < N \), \( \frac{\alpha(n^*)}{n^*} (u - c) - sc > 0 \), contradiction. Hence \( n^* = N \), and hence (i). Define \( s^B = \frac{u - c}{c} \), then \( \forall n^* \in (0, N] \), for \( s > s^B \), \( \not\exists n^* \) st \( \frac{\alpha(n^*)}{n^*} (u - c) - sc \geq 0 \), hence (iii). For \( s \in (s^N, s^B] \), \( \frac{\alpha(n^*)}{n^*} (u - c) - sc = 0 \). Then \( dn^*/ds = \frac{\partial[\alpha(n^*)/n^*]}{\partial n^*} \frac{c}{u - c} < 0 \), therefore \( dL^d/ds < 0 \), and hence (ii). ■

Proof of Lemma 2. If assets are priced at the fundamental value, then all buyers participate in the DM and \( s = 0 \). Let \( \rho^F = (1 - \beta)c/A \). If \( \rho \geq \rho^F \), then \( \forall n \) st \( (\varphi + \rho)A_s/n \geq (\varphi^F + \rho)A_s/n \geq \rho^F A/(1 - \beta) = c \). The liquidity need for assets is satisfied and the marginal holders of assets only care about the store of value function of assets. Hence, \( \varphi = \varphi^F \) and \( s = 0 \), hence (iv). If \( \rho = 0 \), the cost of holding assets is \( s = r \), hence (ii). Otherwise, \( \varphi = (1 + s)\rho/(r - s) \), then substitute \( s \) into the liquidity supply and \( L_s = (1 + r)\rho A_s/(r - s) \), with \( \partial L_s/\partial s = (1 + r)\rho A_s/(r - s)^2 \) and \( \partial^2 L_s/\partial s^2 = -2(1 + r)\rho A_s/(r - s)^3 \). It is easy to check \( \partial L_s/\partial s > 0 \) and \( \partial^2 L_s/\partial s^2 < 0 \) for \( \rho \in (0, \rho^F) \), i.e., \( 0 < s < r \), hence (iii) and for \( \rho < 0, \partial L_s/\partial s < 0 \) and \( \partial^2 L_s/\partial s^2 > 0 \). Hence (i). ■

Proof of Proposition 1. (i) is straightforward from Lemma (2). In this case, \( L^s \) and \( L^d \) don’t have an intersection for all \( s > 0 \), \( L^s \geq L^d \). Therefore sellers hold some assets too. We have \( s = 0, \varphi = \varphi^F \), and \( n^* = N \). The equilibrium is unique. For all \( \rho < \rho^F \), all equilibria satisfy \( L^s = L^d \) and the buyers’ participation constraint. Rewrite the constraint we get \( -\rho \leq \frac{n}{A^\prime} \left[ \frac{\alpha(n)}{n} \beta(u - c) - (1 - \beta)c \right] = \frac{n}{A^\prime} B(n) \equiv f(n) \). Notice \( f''(n) < 0 \), then \( f(n) \) has a unique global maximum point on the support \([0, N]\). Now define \( \rho = -\max_{n \in [0, N]} f(n) \). For \( \rho < \rho \leq 0 \), \( f(n) < -\rho \ \forall n > 0 \), then the buyers’ participation constraint doesn’t hold and the DM shuts down, \( L^s = L^d \) will never hold, hence (vii). For (ii), (iii), (iii), (v), and (vi); we need to examine the uniqueness and the number of active buyers.

We establish the uniqueness first. For \( \rho = 0 \), the asset case is equivalent to the fiat money case with zero money growth rate, and we show the uniqueness in Han et al. (2016) proposition 3. For \( \rho > 0 \), we have \( dL^s/ds > 0 \) and \( dL^d/ds \leq 0 \), hence the equilibrium is
unique. For $\rho = \rho$, the equilibrium is unique because of the unique $n$ which maximizes $f(n)$. For $\rho \in (\rho, 0)$, there are two roots s.t. $f(n) = -\rho$, call them $n^*_l$ and $n^*_h$. With the lose of generality, let $n^*_l < n^*_h$. Then it is easy to show $n^*_l < N$, then $n^* = n^*_l$ which satisfies $f(n^*_l) = -\rho$ and $L^s = L^d$ is an equilibrium. We focus on the other root $n^*_h$. If $n^*_h \geq N$, we have $f(N) \geq f(n^*_h) = -\rho$. Then $n^* = N$ and $L^s = L^d$ is the other equilibrium; otherwise, $n^* = n^*_h < N$ and $L^s = L^d$ is the other equilibrium. In sum, for $\rho \in (\rho, 0)$, the two equilibria with $n^* = n^*_l$ and $n^* = \max\{n^*_h, N\}$, and it is easy to show the equilibrium with a higher $n^*$ has a higher $\varphi$. Then we examine the number of active buyers. Define $\rho^N = \frac{N}{\alpha} \left[ \frac{\alpha(N)}{N} \beta (u-c) - (1 - \beta) c \right]$, then for $\rho \in [\rho^N, \rho^F)$, $f(n) > -\rho \ \forall \ n < N$, hence $n^* = N$ is a possible candidate equilibrium. If $\rho < \rho^N$, all equilibria should satisfy $n^* < N$. After considering uniqueness and the number of buyers, we have (ii), (iii), (iii), (v), and (vi).

**Proof of Proposition 2.** Wlog, we take $A^s$ as given. For the stable equilibrium, we have shown that there exists a cutoff $\rho^F$ such that $\rho^F A^s = (1 - \beta) cN$. Then, $\forall A > (1 - \beta) cN$, we have $\varphi = \varphi^F$ and $n = N$, and then $\partial \varphi / \partial A > 0$ and $\partial n / \partial A = 0$, hence (i). For $\rho \in [\rho^N, \rho^F)$, i.e., $A \in [\rho^N A^s, \rho^F A^s)$, we have $A \geq (1 - \beta) cN - \beta (u-c) \alpha (N)$, and then $n = N$ and $\partial \varphi / \partial \rho < 0$, implying $\partial n / \partial A = 0$ and $\partial \varphi / \partial A < 0$, hence (ii). For $\rho \in [\rho, \rho^N)$, i.e., $A \in [\rho A^s, \rho^N A^s)$, $\partial \varphi / \partial \rho > 0$ and $\partial n / \partial A > 0$, and thus $\partial n / \partial A < 0$ and $\partial \varphi / \partial A > 0$, hence (iii). For the unstable equilibrium with $\rho \in (\rho, 0)$, i.e., $A \in (\rho A^s, 0)$ we have $\partial \varphi / \partial \rho < 0$ and $\partial n / \partial A > 0$, and hence $\partial n / \partial A < 0$ and $\partial \varphi / \partial A < 0$.

**Proof of Lemma 3.** To prove that $L^d$ is a well-defined function for $s \leq s^C$, it is sufficient to show $n^* > 0$ exists and is unique. Substituting $p^e$ into (15) gives $\alpha \varepsilon (u-c) s + \alpha^2 \varepsilon (u-c) / n^* = \alpha [(1-\varepsilon) u + \varepsilon c] s + \varepsilon n^* c^2$. Define $h(n^*, s) = \alpha \varepsilon (u-c) s + \alpha^2 \varepsilon (u-c) / n^* - \alpha [(1-\varepsilon) u + \varepsilon c] s - \varepsilon n^* c^2$. Given any $n \in (0, N]$, $h(n, s) = 0$ is a quadratic function in $s$, which has two real solutions with opposite signs. The positive solution $s_+$, satisfying $h(n, s_+) = 0$, is an implicit function of $n$, $s_+(n)$. Let $s_+(0) = \lim_{n \to 0} s_+(n) < \infty$, and $s_+(0)$ is continuous on $[0, N]$. Define $s^N$ by $h(N, s^N) = 0$ and $s^C = \max_{n \in [0, N]} s_+(n)$. For $s < s^N$, $h(N, s) > 0$ hence $n^* = N$. Then $L^d = N p^e(N, s)$ is unique, and $dL^d / ds = N dp^e(N, s) / ds < 0$, hence (i). For $s > s^C$, $h(n^*, s) < 0 \ \forall n^*$, and the free-entry condition does not hold due to $\alpha (n^*) (u - p^C) / n^* - sp^c < 0$, hence (iii).

Regarding (ii), for $s \leq s^C$, $h(n^*, s) = 0$ always holds for some $n^* > 0$, and $L^d$ exists.
To show that $L^d$ is generically unique and monotone, consider $L^d = n^*pC$ and $dL^d/ds = \partial L^d/\partial s + (\partial L^d/\partial n^*) (\partial n^*/\partial s)$. Given $h(n^*, s) = 0$, we have $L^d = \alpha(n^*)n^* u/[\alpha(n^*) + sn^*]$, hence $\partial L^d/\partial s < 0$ and $\partial L^d/\partial n^* > 0$. Then, it is sufficient to show that $n^*$ is generically unique and $\partial n^*/\partial s < 0$. We claim that although there might be multiple $n^*$ which maximize $\pi(n, s)$, $n^*$ is still unique and $\partial n^*/\partial s < 0$ for generic $s$. To see this, suppose $\pi(n_1^*, s) = \pi(n_2^*, s) = \max_n \pi(n, s)$ and $n_2^* > n_1^*$. Then, $n_1^*$ is the minimum $n$ maximizing $\pi(n, s)$, and $\pi(n_1^*, s) > \pi(n, s), \forall n < n_1^*$. For $\epsilon > 0$ small enough, $\pi(n_1^*, s + \epsilon) > \pi(n, s + \epsilon)$ also holds for $n < n_1^*$ due to continuity. If $\partial^2 \pi/\partial s \partial n^* < 0$, then $\pi(n_1^*, s + \epsilon) > \pi(n_2^*, s + \epsilon)$, and the global maximizer is a unique $n$ in the neighborhood of $n_1^*$. Next, we need to show $\partial^2 \pi/\partial s \partial n^* < 0$. Derive $\partial \pi/\partial n$ from (13),

$$\frac{\partial \pi}{\partial n} = \frac{(\alpha + sn)[(u - c)\alpha' - sc] - s(1 - \varepsilon)[(u - c)\alpha - snc]}{(\alpha + sn)^2/\alpha}.$$ 

Define $T(s) = (\alpha + sn)[(u - c)\alpha' - sc] - s(1 - \varepsilon)[(u - c)\alpha - snc]$, and $T'(s) = n[(u - c)\alpha' - sc] - (\alpha + sn)c - (1 - \varepsilon)[(u - c)\alpha - snc] + snc(1 - \varepsilon)$. Since $T_{n=n^*} = 0$, $\partial^2 \pi/\partial s \partial n^* = T'(s)/[(\alpha + sn^*)^2/\alpha]$. With $\alpha(u - c) - snc > 0$, we have

$$T'_{n=n^*}(s) = -\frac{[\alpha(u - c) - snc](1 - \varepsilon)\alpha - c(\alpha + sn^*)(\alpha + sn^*)}{\alpha + sn^*} < 0.$$ 

Therefore, $\partial^2 \pi/\partial s \partial n^* < 0$ holds. In addition, $\arg\max_n \pi(n, s)$ might have more than one solution for some $s \geq s^{NC}$, but the set of such asset spreads has measure zero, hence (ii). Finally, we prove $s^C \geq r$ by contradiction. Suppose $s^C < r$, then for $s_1 = (r\varphi_1 - \rho_1)/\varphi_1 + \rho_1 \in (s^C, r)$, $\rho_1 > 0$ and $n_1^* = 0$. Hence, $\varphi_1 = \varphi^F_1$ and $s_1 = 0$, contradicting $s_1 > s^C > 0$. ■

**Proof of Lemma 4.** If assets are priced at the fundamental value, then all buyers participate in the DM and $s = 0$. Let $\rho^F = (1 - \beta)p^s_{N, s=0}/A$. If $\rho \geq \rho^F$, the average asset holding $(\varphi + \rho)A^s/n \geq (\varphi^F + \rho)A^s/n \geq \rho^FA/(1 - \beta) = p^s_{N, s=0}$. The liquidity need for assets is satisfied and the marginal holders of assets only care about the store of value function. Hence, $\varphi = \varphi^F$ and $s = 0$. If $\rho = 0$, the cost of holding assets is $s = r$. If $\rho < \rho^F$ and $\rho \neq 0$, substitute $s$ into the liquidity supply and $L^s = (1 + r)\rho A^s/(r - s)$, with $\partial L^s/\partial s = (1 + r)\rho A^s/(r - s)^2$ and $\partial^2 L^s/\partial s^2 = -2(1 + r)\rho A^s/(r - s)^3$. It is easy to check $\partial L^s/\partial s > 0$ and $\partial^2 L^s/\partial s^2 < 0$ for $\rho \in (0, \rho^F)$, and for $\rho < 0$, $\partial L^s/\partial s < 0$ and
\[ \frac{\partial^2 L^s}{\partial s^2} > 0. \]

**Proof of Proposition 3.** For \( \rho \geq \rho^F \), a downward-sloping \( L^d \) and a perfectly elastic \( L^s \) ensure the existence and uniqueness of equilibrium \( s^* \) with \( n^* = N \), hence (i). For \( \rho = 0 \), assets are equivalent to money with zero inflation, and the proof follows Proposition 5 in Han et al. (2016). For \( \rho \in (0, \rho^F) \), \( L^d \) and \( L^s \) intersect once and there exists a unique equilibrium. For \( \rho < 0 \), \( \bar{s}^C \geq r \) according to Lemma 3. If \( \bar{s}^C = r \), \( \frac{\partial}{\partial \rho} \) non-degenerate equilibrium; if \( \bar{s}^C > r \), \( L^d \) and \( L^s \) may have more than one intersection, hence more than one candidate equilibrium. Given \( n^* \) being a function of \( s \), we can rewrite the seller’s problem (13) as

\[
\max_s \alpha(n^*(s)) \left[ \frac{\alpha(n^*(s)) u - n^*(s) \Omega}{\alpha(n^*(s)) + n^*(s) s} - c \right].
\]

Given different values of \( s^* \) satisfying the first-order condition, there could be more than one \( s^* \) which maximize seller’s profit. Hence \( \frac{\partial}{\partial \rho} \) uniqueness in this region. Next is to show the existence of \( s^* \). If \( \bar{s}^C = r \), \( \frac{\partial}{\partial \rho} = 0 \). Consider \( \bar{s}^C > r \). \( s \leq r \) implies \( \rho \geq 0 \), and thus (iii).

For \( s \in (r, \bar{s}^C) \), \( \rho < 0 \), \( \partial L^s/\partial \rho = (1 + r) A^s/(r - s) < 0 \), and \( L^d \) is constant. Hence, \( \exists! \left( \rho^*(s) \right) \) such that \( L^s(\rho^*) = L^d \), and define \( \rho = \min_{s \in [r, \bar{s}^C]} \rho^*(s) < 0 \). For \( \rho < \rho^* \), \( L^s(\rho) > L^d \), and there exists no equilibrium, hence (iv).

For the rest of the proposition on participation and asset prices, first consider \( \rho \geq \rho^F \). According to Lemma 4, the cost of holding assets \( s = 0 \), implying \( \varphi = \varphi^F \) and \( n^* = N \). Let \( \rho^N = (r - s_N)p^c/(1 + r)A \). If \( \rho \in (\rho^N, \rho^F) \), then \( s^N > s > 0 \). The buyer’s participation constraint is slack, and \( (\varphi + \rho)A^s/N = p^c \). Hence, \( n^* = N \) and \( \varphi = \varphi^N = (1 + s)p^c/(1 + r)A > \varphi^F \). If \( \rho \in [\rho^*, \rho^N] \), the buyer’s participation constraint is binding, and \( s > 0 \) and \( (\varphi + \rho)A^s/n^* = p^c \). Therefore, \( \varphi = \varphi^N = n^*(1 + s)p^c/N(1 + r)A > \varphi^F \).

**Proof of Proposition 4.** Similar to bargaining, \( A^s \) is taken as given. There exists a cutoff \( \rho^F \) satisfying \( \rho^F A^s = (1 - \beta) N[1 - \varepsilon(N)]u + (1 - \beta) N\varepsilon(N)c \). Then, \( \forall A > \rho^F A^s \), we have \( \varphi = \varphi^F \) and \( n = N \), and then \( \partial \varphi / \partial A > 0 \) and \( \partial n / \partial A = 0 \), hence (i). For \( \rho \in (\rho^N, \rho^F) \), i.e., \( A \in (\rho^N A^s, \rho^F A^s) \), we have \( A \geq (1 - \beta) cN - \beta (u - c) \alpha (N) \), then \( n = N \) and \( \partial \varphi / \partial \rho \) is ambiguous, implying \( \partial n / \partial A = 0 \) and \( \partial \varphi / \partial A \) is ambiguous, hence (ii). For \( \rho \in [\rho^*, \rho^N] \), i.e., \( A \in [\rho A^s, \rho^N A^s] \), \( \partial \varphi / \partial \rho \) and \( \partial n / \partial \rho \) are ambiguous, and thus \( \partial \varphi / \partial A \) and \( \partial n / \partial A \) are ambiguous, hence (iii).

**Proof of Lemma 5.** Using \( \lambda_1 \) and \( \lambda_2 \) for the multipliers on the asset constraint and
the lotteries constraint gives the following Kuhn-Tucker conditions.

\[ 0 = -\eta (\tau u - p)^{n-1} (p - \tau c)^{1-n} + (1 - \eta) (\tau u - p)^n (p - \tau c)^{-n} - \lambda_1 \]  
\[ 0 = \eta u (\tau u - p)^{n-1} (p - \tau c)^{1-n} - c (1 - \eta) (\tau u - p)^n (p - \tau c)^{-n} - \lambda_2 \]  
\[ 0 = \lambda_1 ((\varphi + \rho) a - p) \]  
\[ 0 = \lambda_2 (1 - \tau). \]  

It is straightforward to check that if \( \lambda_1 = 0, \rho = \tau^B \bar{p}^B \). Substituting this into (17) implies \( \lambda_2 > 0 \), and hence \( \tau^B = 1 \). In order to support \( \tau^B = 1 \), buyer needs to bring enough asset to the DM trade, i.e. \( (\varphi + \rho) a > \bar{p}^B \). On the other hand, if \( \lambda_2 = 0, \tau^B = p^B / \bar{p}^B \). Substituting this into (16) implies \( \lambda_1 > 0 \) and \( p^B = (\varphi + \rho) a \) and \( \tau^B = 1. \) \( \lambda_1 > 0 \) implies \( (\varphi + \rho) a < \bar{p}^B \), and \( \lambda_2 > 0 \) implies \( (\varphi + \rho) a > \bar{p}^B \). Finally, the seller certainly does not trade if he meets a buyer with \( (\varphi + \rho) a < c \).

**Proof of Proposition 5.** First, buyers do not want to bring \( (\varphi_{t+1} + \rho) \hat{a} > \bar{p}^B \), since additional assets do not affect the surplus from trade. Second, they do not bring \( (\varphi_{t+1} + \rho) \hat{a} < c \), for no trade. Next, for \( (\varphi_{t+1} + \rho) \hat{a} \in (\bar{p}^B, \bar{p}^B) \), \( v'(\hat{a}) = -(\varphi_{t+1} + \rho)[s + \alpha(n)/n] < 0 \), and buyers want to choose \( (\varphi_{t+1} + \rho) \hat{a} = \bar{p}^B \). For \( (\varphi_{t+1} + \rho) \hat{a} \in (c, \bar{p}^B) \), \( v'(\hat{a}) = (\varphi_{t+1} + \rho)[\alpha(n)\eta(u - c)/nc - s] \), and the sign of \( v'(\hat{a}) \) depends on the value of the spread \( s \). Since \( \alpha(n)(u - \bar{p}^B)/n - sp^B = \bar{p}^B[\alpha(n)\eta(u - c)/nc - s], \) \( v'(\hat{a}) \) shares the same sign as \( \alpha(n)(u - \bar{p}^B)/n - sp^B \). Suppose \( v'(\hat{a}) < 0 \), buyers choose \( \tau^B = 0 \) and there is no equilibrium with an open DM. If \( v'(\hat{a}) > 0 \), buyers of measure \( n \) in the DM choose \( (\varphi_{t+1} + \rho) \hat{a} = \bar{p}^B \). The cutoff spread satisfying \( v'(\hat{a}) = 0 \) is given by \( \alpha(n)(u - \bar{p}^B)/n - sp^B = 0 \), which is equivalent to the participation constraint \( n[\alpha(n)\beta(u - \bar{p}^B)]/n - (1 - \beta)\bar{p}^B]/A = g(n) \geq -\rho. \) Since \( g^\nu(n) < 0 \), let \( \rho = -\max g(n) \), \( \rho^F = (1 - \beta)\bar{p}^B/A \), and \( \rho^N = [(1 - \beta)\bar{p}^B - \beta\alpha(N)(u - \bar{p}^B)/N]/A. \) For \( \rho \geq \rho^F \), all equilibria feature \( p^B = \bar{p}^B \) and \( \tau^B = 1. \) If \( \rho \geq \rho^F \), then \( \varphi = \varphi^F \); otherwise \( \varphi > \varphi^F. \) If \( \rho \geq \rho^N \), then \( n^* = N; \) otherwise \( n^* < N. \) The rest of the proof on equilibrium stability follows directly from Proposition 1.

**Proof of Proposition 6.** We need to check that sellers always post \( \tau^C = 1 \) and the
rest of the proof follows Proposition 3. Let $\lambda$ be the multiplier for $\tau$, and the FOCs are

\[
0 = \varepsilon(n)(p - \tau c) - \frac{\alpha(n)[1 - \varepsilon(n)](\tau u - p)}{\alpha(n) + ns}, \quad (18)
\]

\[
0 = \tau \left[ \frac{\alpha^2(n)u}{\alpha(n) + ns} - \alpha(n)c - \lambda \right], \quad (19)
\]

\[
0 = \lambda (1 - \tau).
\]

Given the buyer’s optimal participation $n = n^*$ and (18), we have

\[
p^c = \frac{\alpha(n^*)\{[1 - \varepsilon(n^*)]\tau u + \varepsilon(n^*)\tau c\} + \varepsilon(n^*)n^*s\tau c}{\alpha(n^*) + \varepsilon(n^*)n^*s}.
\]

Solve for $\lambda$ from (19), and we need $\lambda = \alpha(n^*)(u - c) - cn^*s > 0$ to assure $\tau^C = 1$. Since $p^c/\tau > c \forall \tau$, $\alpha(n^*)(u - c) - cn^*s > \alpha(n^*)(u - p^c/\tau) - n^*sp^c/\tau \geq 0$. The last inequality is the buyer’s participation constraint in the DM, which holds if $\rho \geq \underline{\rho}$ and $n^* > 0$. ■