Optimal Joint Bond Design∗

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February 29 2016

Abstract

We study the optimal design of a joint borrowing arrangement among countries. In our framework, a safe country, which never defaults, and a risky country, which may default, participate in a joint borrowing scheme, through which they allocate a predetermined fraction of their bond issuance to a joint bond. The joint borrowing scheme is flexible, and can feature pooled issuance, in which countries share the funds raised through the joint bond, and joint liability, in which one country guarantees the obligations of another one. We show that a change in the scale of the joint borrowing scheme affects social welfare through five different channels: the risk sharing channel, the default deadweight loss channel, the free riding channel, the joint liability channel and default spillover channel. We also show that only the first two channels are non-zero up to a first-order when the joint bond issuance is small. We develop a simple test based on pricing data to determine whether a joint borrowing scheme is socially desirable. We show that our main insights remain valid through a number of extensions. We provide a quantitative analysis of the different policies (in progress).

JEL numbers: F38, F42, G18

Keywords: sovereign debt, sovereign default, joint borrowing agreements, stability bonds, Eurobonds, ESBies, safe assets

Work in progress: please do not distribute
(a complete draft, including the dynamic quantitative model, will be ready by June)

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1 Introduction

The European sovereign debt crisis, which began to develop at the end of 2009 and whose effects are still felt today, dramatically changed the outlook of sovereign debt markets in Europe. As shown in figure 1, starting in 2009, the interest rate spreads paid by most European countries widened to unprecedented levels since the creation of the eurozone. The inability of several eurozone member states to repay or refinance their government debt called for the introduction of mechanisms of mutual support among countries. For instance, a number of stability funds were put in place. Among those, the EFSF, European Financial Stability Facility, and the EFSM, European Financial Stabilization Mechanism, played important roles.

![Figure 1: Sovereign bond yields (nominal yields in percentage points, 10-year bonds) for eurozone members (1990-2014). The blue line (EZ) plots the GDP-weighted average of bond yields for eurozone members.](image)

Nonetheless, the policy proposal that caused the most heated debate in the public sphere was the creation of some form of permanent joint borrowing agreement among eurozone countries. A number of different proposals were openly endorsed by prominent economists and policy experts: blue/red bonds, eurobonds, eurobills, ESBies, and synthetic bonds, among others, received widespread attention. Even the European Commission circulated a green paper assessing the feasibility of common issuance of sovereign bonds among the member states of the euro area. Somewhat surprisingly, despite the strong interest by policymakers and the general public, the formal analysis of this issue remains underdeveloped.

In this paper, we seek to provide a systematic treatment of the positive and normative consequences of joint borrowing agreements between sovereigns. In particular, we focus on characterizing under which conditions a joint borrowing scheme is desirable from a welfare standpoint. To carry out this task, we
develop a framework in which a safe country, which never defaults on its debt, and a risky country, which may default, have the ability to enter into a joint borrowing scheme. Within the scheme, both countries allocate a predetermined share of their total bond issuance to a joint bond, while the rest of their debt remains individually issued by the countries. We consider flexible schemes that allow us to separate the role of pooled issuance, through which countries split ex-ante the funds raised by the issuance of the joint bond, from the role of joint liability, through which one country guarantees ex-post the obligations of another country.

Our first main result shows that there are five different channels through which changes in the scale of the joint borrowing scheme affect social welfare. We refer to them as the risk sharing channel, the default deadweight loss channel, the free riding channel, the joint liability channel and default spillover channel.

First, the risk sharing channel concerns the ability to redistribute resources through pooled issuance towards those countries with relatively higher marginal utility. A joint issuance arrangement will improve welfare when it increases the funds received by those countries who need them the most in marginal utility terms.

Second, the default deadweight loss channels captures the reduction in the likelihood of sovereign default associated with an increase in the joint bond issuance. A joint issuance arrangement will improve welfare when it reduces the number of states in which countries default, avoiding the resource losses that defaulting entails.

Third, the free riding (or moral hazard) channel is a side effect of the introduction of the joint borrowing scheme that emerges when countries are free to make their unconstrained borrowing choices. In our case, the risky country does not take into account how the marginal unit of borrowing a) increases the borrowing costs of the safe country through the joint bond and b) forces the safe country to pay for the joint liability promise more often.

Fourth, the joint liability channel captures the net effect in welfare terms associated with the promise by the safe country to guarantee a fraction of the pooled debt through the joint bond issued by the risky country. On the one hand, increasing the issuance of joint bonds increased the ex-post liability of the safe country. On the other hand, this increases the amount raised ex-ante. The net effect, using the appropriate country specific discount factors, determines the total welfare effect of the joint liability channel.

The fifth and final channel is the default spillover channel. The direct change in the default region caused by varying the degree of joint issuance impacts directly the welfare of the safe country, because there is no guarantee that the safe country is indifferent between the default and no default decisions of the risky state, in particular, due to the changes in overall borrowing conditions.

In general, all five channels are important to determine the optimal magnitude of the joint borrowing scheme. However, our next main result shows that only two of the five channels have a first-order impact in welfare when the amount of joint bond issued is small. Formally, we show that only the risk sharing channel and the default deadweight loss reduction channel affect are in general non-zero for small issuance levels of
the joint bond. Crucially, the free-riding effects caused by changing the scale of the joint borrowing scheme are zero for small issuance levels. Despite the complexity of the environment, we provide a simple test, based on bond pricing data, to determine the sign of both remaining first-order terms and, consequently, to determine whether the introduction of a joint borrowing agreement is socially beneficial.

An important contribution of this paper is to define a new measurable object, the pricing wedge of the joint bond, that turns out to be crucial to understand the effects of a joint borrowing scheme. The pricing wedge of the joint bond is defined as the difference in prices between the joint bond and the (properly weighted) sum of the individually issued bonds. In our baseline model with joint liability, the pricing wedge simply captures the present discounted value of the joint liability embedded into the joint bond. However, when we move to an environment in which lending markets cease to be frictionless, for instance, when the joint bond enjoy a safety or a liquidity premium but not all of the individually issued bonds, the pricing wedge can become positive even without joint liability, creating a new first order welfare effect. Importantly, as long as the pricing wedge is positive and the joint borrowing scheme involves pooled issuance, our results highlight that the joint borrowing scheme must also determine how to split the pricing wedge among the participating countries.

We study additional implications of our framework. First, we determine under which conditions introducing a joint borrowing scheme can create actual Pareto improvements — in the baseline model, we focus in Kaldor-Hicks efficiency. We conjecture that this maybe the case for sufficiently open economies (for now, this remains a conjecture). Second, we study the problem of the optimal determination of the weights of the different countries in the joint borrowing scheme. Third, we characterize the set of optimal corrective policies that eliminate the free-riding behavior by modifying the borrowing behavior of the participating countries.

We also study several extensions. First, we allow for bonds from different countries, and the joint bond, to be potentially priced according to different stochastic discount factors. This can account for different frictions in lending markets. This is a very important extensions, since the ability to capture some form of safety premium has been a major argument in policy discussions. Second, by slightly changing our timing assumptions, we can introduce multiple equilibria into the model caused by rollover risk, as in Calvo (1988) or Cole and Kehoe (2000). We show that this form of rollover risk can be seen as a microfoundation for the default deadweight loss reduction defined in our first proposition. Third, we study the case with symmetric countries, which generates a stronger link between countries. Although our baseline model features time separable expected utility, exogenous output and risk neutral lenders, our insights remain unchanged if we allow for more general utility specifications, production or lenders with non trivial pricing kernels.

Finally, although we do not emphasize it throughout the paper, our results can provide insights to understand the interactions through borrowing decisions between federal and state or local governments.
Literature

To our knowledge, this paper provides the first formal systematic analysis of joint borrowing schemes among sovereigns. By doing so, it contributes to the well-developed and growing literature that studies sovereign borrowing and sovereign default. The recent survey by Aguiar and Amador (2013) provides a detailed review of this literature. Most of this work focuses on the problem a single sovereign, usually a small open economy. There are a few exceptions that incorporate linkages among borrowers. Kim and Zhang (2012) is one exception that studies the problem of decentralized entities (within a given economy) that borrow independently but that default in a centralized way. The nature of the free-riding problem studied in their paper is similar to the one that emerges in our paper. Arellano and Bai (2013), which model interlinkages across sovereign markets due to joint renegotiation with risk averse lenders, is another exception. In their work, the feedback between the default decisions of countries through the behavior of their common lenders may give rise to multiple equilibria. In our baseline model, countries are linked to being with through the joint borrowing scheme. In our version with symmetric countries in section 5, a multiplicity phenomenon of similar nature to theirs arises. In a more abstract setup, Korinek (2014) provides conditions under which global cooperation among policymakers is beneficial.

Several policy proposals regarding joint borrowing agreements emerged during the recent European debt crisis. The European Commission proposal of stability bonds, as well as the synthetic bond proposal by Beck, Uhlig and Wagner (2011) only involve pooling of individually issued bonds. The Blue Bond/Red Bonds proposal of Delpla and Von Weizsacker (2011) involves both pooling and tranching. In this proposal, the whole bulk of national sovereign debt up to 60% of GDP would be mutualized and would back the joint bond called Blue Bond, through joint and several guarantees. The residual sovereign debt (Red Bonds) would still be issued nationally. The Eurobills proposal of Hellwig and Philippon (2011) only involves pooling and focuses on short-term instruments. They argue that these maturities have the highest potential for reaping a liquidity premium and can also impose a discipline through the need to rollover the debt. Other policy proposals revolve around the generic idea of joint bond issuance to reap the benefits of a liquidity and safety premia. The European Safe Bonds (ESBies) proposal by Brunnermeier and al. (2011) does not involve joint guarantees. They argue that such arrangement optimally trades off the benefits of pooling some national sovereign debt against the risk of creating moral hazard, as the red bond is priced nationally and provides fiscal discipline.

Despite the abundance of policy proposals, there is very little formal research on this area. To our knowledge, only the recent work by Zhiguo He, Arvind Krishnamurthy and Konstantin Milbradt (2015) studies the possibility of pooled issuance of sovereign debt within a formal model. They do so in an environment with risk neutral countries, rollover risk, imperfect common knowledge, and rule out joint liability agreements. Our framework, which is more general in a number of dimensions, allows us to have a more complete picture of the effects and the convenience of joint borrowing schemes.
Beyond the sovereign debt literature, this paper also relates to the literature on the macroeconomic shortage of safe assets. Papers such as Caballero and Farhi (2014), Krishnamurthy and Vissing-Jorgensen (2012), Gorton and Ordonez (2013) and Gourinchas and Jeanne (2012), highlight the mismatch between a high demand and a low supply for safe assets. In section 5, we incorporate the possibility that some bonds may carry a safety premium. Taking such premium as given, we explore its implications for the joint borrowing scheme. Although it is not the focus of our paper, we should also reference the work that emphasizes the interactions between sovereign and banking risk as to provide rational for intervention in sovereign debt markets. For instance, Bolton and Jeanne (2011) analyze how the level of bond issuance of one country impacts the financial fragility in financially integrated economies. Gennaioli, Martin and Rossi (2014) explicitly models endogenous sovereign default and bank default to characterize the vicious circle between the two. Weymuller (2013) shows that safe sovereign debt enables banks to create safe bank debt, inducing a safety multiplier.

Outline Section 2 describes our baseline model and characterizes the equilibrium of the economy. Section 3 carries out the welfare analysis, introducing the first main set of results. Section 4 deepens the analysis of the baseline model, while section 5 analyzes several major extensions. Section 6 quantitatively assess the mechanisms studied in the paper and section 7 concludes. The appendix contains all proofs and derivations.

2 Baseline model

We study the problem of two small open economies that borrow from competitive foreign creditors. One economy, which we call risky, may default on its sovereign debt, while the other one, which we call safe, never defaults. By assumption, these economies are unable to transfer funds to each other directly through fiscal transfers. However, they have the ability to design a joint borrowing agreement to issue sovereign debt. In this paper, we first analyze under which conditions such a joint borrowing arrangement is welfare improving and then thoroughly study its properties and its optimal design. We initially introduce our main results in a transparent three-period formulation and subsequently study a number of extensions.

2.1 Environment

Time is discrete. There are three dates, \( t = 0, 1, 2 \) and two countries that differ along a number of dimensions. We denote countries by \( i = \{S, R\} \), where \( S \) designates the safe country, which always pays back its debt, and \( R \) designates the risky country, which may default on its debt. There is a single type of consumption good, which serves as numeraire.

Preferences Each country \( i \) behaves as to maximize the welfare of a risk averse representative agent with time separable expected utility and a rate of time preference \( \beta_i \). Flow utility \( U_i(\cdot) \) satisfies standard
regularity conditions for both countries: \( U'_i (\cdot) > 0, U''_i (\cdot) < 0 \) and \( \lim_{C \to 0} U'_{\text{avg}} (C) = \infty \). Preferences can differ across countries. We denote by \( C^i_t \) the consumption of country \( i \) at date \( t \). Hence, country \( i \) maximizes:

\[
E_0 \sum_{t=0}^{\infty} \beta^t U_i (C^i_t)
\]

**Endowments** We denote by \( Y^i_t > 0 \) country \( i \)'s endowment of consumption good of at date \( t \). The date 0 endowments \( Y^i_0 \) are given.\(^1\) Endowments at dates 1 and 2 are stochastic. Endowment risk is the only source of uncertainty. We use the function \( F_t (\cdot, \cdot) \) to denote the joint distribution of endowments at date \( t \). We do not restrict the serial correlation or the cross-country correlation of shocks: endowment shocks can be correlated across countries and over time. We denote by \( \bar{Y}_t \) and \( \tilde{Y}_t \) the upper and lower limits for the realizations of endowment shocks.

**Market structure** Both countries can issue short-term (one period) debt at dates 0 and 1. We denote by \( B^i_t \) the total number of bonds with unit face value issued by country \( i \) at date \( t \). A fraction of these bonds may be issued as part of a joint bond, as described below.

Country \( R \) lacks commitment and decides whether to repay or not its debt at dates 1 and 2. If country \( R \) defaults, two things happen: a) it is excluded from financial markets in future periods and b) it losses a fraction \( \delta \) of its endowment from then on. Both are canonical assumptions in the sovereign debt literature. Country \( S \) is fully committed to repay its liabilities and never considers the possibility of defaulting. Any debt issued individually by country \( S \) trades at the risk-free rate.

**Joint borrowing scheme** Both countries participate in a joint borrowing scheme that works as follows. The first \( \theta^i B^i_t \) bonds issued by country \( i \) are pooled into a joint bond, whose price we denote by \( \tilde{q}^i_J \) (the index \( J \) stands for joint bond). The remaining \( B^i_t - \theta^i B^i_t \) bonds are individually issued by country \( i \), at a price \( \tilde{q}^i_i \). Naturally, the shares of bonds issued by both countries in the joint bond must add up to unity, so \( \sum_i \theta^i = 1 \). Hence, \( \bar{B} \geq 0 \) denotes the face value of the joint bond. Countries commit to not know which particular bonds form part of the joint bond. Therefore, country \( R \) cannot default selectively on the pooled bonds and repay the individually issued ones and vice versa.

We make two further assumptions. First, country \( i \) receives at issuance a price \( q^i_{IP} \) per unit of bond pooled into the joint bond (the index \( P \) stands for pooled issuance). This price \( q^i_{IP} \) corresponds to a linear combination, with weights \( \kappa \) and \( 1 - \kappa \), between a) the price of the joint bond and b) the sum of the price of the individually issued bond and a share \( \chi^i \) of the term \( \Omega^i_t = \tilde{q}^i_J - \sum_i \theta^i \tilde{q}^i_i \), which we define below as the pricing wedge of the joint bond. The (weighted) shares \( \chi^i \) must add up to unity, that is, \( \sum_i \theta^i \chi^i = 1 \). This first assumption introduces the possibility of pooled issuance, so that both countries split ex-ante the funds raised by the issuance of the joint bond.\(^2\)

\(^1\)The initial endowment \( Y^i_0 \) can be seen as the available resources of country \( i \) net of debt payments, that is \( Y^i_0 = \check{Y}^i_0 - B^i_{-1} \), where \( \check{Y}^i_0 \) denotes the actual endowment realization.

\(^2\)We could have assumed that countries split the pricing wedge equally by setting \( \chi^i = 1 \). There is no a priori reason to do...
Second, when country $R$ defaults, country $S$ becomes liable for a fraction $\lambda$ of the bonds of country $R$ that were pooled into the joint bond. This second assumption introduces the possibility of joint liability, so that country $S$ guarantees ex-post promises made by country $R$.

Formally, the unit price of bonds issued by country $i$ at date $t$, which we denote by $q^i_t$, can be written as follows

$$q^i_t = \phi^i_t q^P_t + (1 - \phi^i_t) \tilde{q}^i_t,$$  \hspace{1cm} (1)

where $\phi^i_t$ denotes the fraction of bonds issued by country $i$ that are pooled into the joint bond. That is,

$$\phi^i_t \equiv \frac{\theta^i B_t}{B^i_t}$$  \hspace{1cm} (2)

Because it is the relevant case in practice, and to ease the exposition, we work under the assumption that $\theta^i B_t < B^i_t$, so $\phi^i_t \in [0, 1)$. For completeness, we study the more general case in the appendix. We denote by $\tilde{q}^i_t$ the unit price of the bonds individually issued by country $i$ and by $q^P_t$ the price per unit of bond pooled by country $i$ into the joint bond, formally given by

$$q^P_t = \kappa \tilde{q}^J_t + (1 - \kappa) \left( \tilde{q}^i_t + \chi^i \Omega_t \right),$$  \hspace{1cm} (3)

where $\tilde{q}^J_t$ denotes price of the joint bond and $\Omega_t$ is formally defined as

$$\Omega_t \equiv \tilde{q}^J_t - \sum_i \theta^i \tilde{q}^i_t$$  \hspace{1cm} (4)

We refer to $\Omega_t$ as the *pricing wedge* of the joint bond. It corresponds to the difference between the price of the joint bond and the (appropriately weighted) sum of the prices of the individually issued bonds. Depending on the assumptions, $\Omega_t$ may be positive or zero throughout the paper. As it will become clear, the behavior of the

In particular, in our baseline model, given that the price of all bonds is frictionlessly determined by the same set of lenders, as described below, the pricing wedge corresponds to the value of the new liability assumed by country $S$ through the joint bond. It takes the value

$$\Omega_t = \lambda \theta^R \left( \tilde{q}^S_t - \tilde{q}^R_t \right)$$  \hspace{1cm} (5)

Therefore, the pricing wedge $\Omega_t$ will be strictly positive if and only if there is joint liability ($\lambda > 0$), but will be zero when $\lambda = 0$. Consequently, the price of the joint bond must then satisfy $\tilde{q}^J_t = \sum_i \theta^i \tilde{q}^i_t + \lambda \theta^R \left( \tilde{q}^S_t - \tilde{q}^R_t \right)$.

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3 As we show in section 5, the pricing wedge can also take positive values when there are frictions in the lending market. For
The parameter $\kappa$ modulates the degree of pooled issuance. When $\kappa \to 1$, $q^i_P = \tilde{q}^i$, $\forall i$, so every country receives the price of the joint bond per unit of bond pooled. When $\kappa \to 0$, there is no pooled issuance at all, so countries effectively borrow at the unit price of their individual bond, $\tilde{q}^i$, augmented by a share $\chi^i$ of the pricing wedge $\Omega_t$, which must be allocated between both countries. The parameter $\lambda$ modulates the degree of joint liability. When $\lambda \to 1$, $\tilde{q}^J = \tilde{q}^S$, so the joint bond is issued at the risk-free rate. When $\lambda \to 0$, there is no pricing wedge, so the price of the joint bond equals the (weighted) sum of the individually issued bonds. When $\overline{B} \to 0$, there is no joint borrowing scheme at all, so both countries operate independently. For that reason, we focus on the behavior of $\overline{B}$ as the main design parameter. Note that the pricing wedge $\Omega_t$ is endogenous object, while the shares $\chi^i$ that distribute such wedge between countries are parameters of the joint borrowing scheme.

For consistency, funds raised from the issuance of the joint bond must equal the weighted sum of the funds received by both countries. Formally, $\sum_i \theta^i q^i_P = \tilde{q}^J$. Note that the prices $q^i$ and $q^i_P$, in equations (1) and (3), are determined by the joint borrowing scheme. However, the prices $\tilde{q}^i$ and $\tilde{q}^J$ are the prices of actually traded securities — we use the tilde to emphasize that. Equations (2) and (4) are definitions, while equation (5) is an arbitrage condition.

Summing up, the joint borrowing scheme is fully characterized by five parameters: the number of pooled bonds $\overline{B} \geq 0$, the fraction of bonds of each country in the pooled bond $\theta^i \in [0, 1]$, the degree of pooled issuance $\kappa \in [0, 1]$, the degree of joint liability $\lambda \in [0, 1]$, and the share of the pricing wedge assigned to each country $\chi^i \in [0, 1]$.

We study the properties of the time-invariant joint borrowing design under commitment. In particular, we focus on the optimal determination of $\overline{B}$, and then study how the choice of the other four variables effect the joint borrowing scheme. We can interpret our welfare analysis as the problem solved ex-ante under commitment by a supra-national authority without access to fiscal transfers. We seek to understand the sources of the frictions that a joint borrowing scheme addresses and generates, as opposed to implementing first-best allocations. Unrestricted transfers can trivially recover the first-best in our model.

Remark. (Interpretation of the environment) Given that our analysis is motivated by the European experience, it is natural to start from a benchmark with countries that are ex-ante asymmetric. This is important, for instance, in section 4, when we study the possibility of finding Pareto improvements. In section 6, we discuss how our results extend to the case with ex-ante symmetric countries: the same mechanisms appear there, with the additional complexity that the strategic behavior among countries plays a bigger role. Our formulation with one safe and one risky country is both more parsimonious and relevant.

Remark. (Relevance of joint borrowing scheme) Our formulation allows us to show that pooled issuance and joint liability have different effects. Pooled issuance allows to split the funds raised by issuing the instance, when some assets can be used as collateral and carry a safety premium, or when there are liquidity differences among sovereign debt markets.

\footnote{We use lower case greek letters to denote the (exogenously given) parameters of the joint borrowing scheme with the exception of $\overline{B}$, which we keep for consistency with $B^i$.}
joint bond. It is effectively equivalent to agreeing to a set of implicit transfers at issuance determined by the prices of the individually issued bonds and the joint bond. Joint liability corresponds to the promise of paying the debt of another country when this one defaults on its obligations. It is equivalent to the promise of transferring resources to lenders in default states. Moreover, our formulation forces to determine how to share the pricing wedge, through $\chi^i$, and the degree of involvement of each country, through $\theta^i$. Many policy proposals implicitly entail pooled issuance, joint liability, or both, often without being explicit about it. They often fail to highlight how the distribute the pricing wedge. This paper provides a framework to dissect along these dimensions the core attributes of the different proposals.

\[
\begin{array}{|c|c|}
\hline
\kappa = 0 & \kappa = 1 \\
\hline
\lambda = 0 & \text{No Arrangement} \\
\lambda = 1 & \text{Joint Liability (Government Guarantees)} \\
\kappa = 1 & \text{Pooled Issuance (ESBies/Synthetic Bonds)} \\
\hline
\kappa = 1 & \text{Both (Eurobonds, Blue/Red Bonds)} \\
\hline
\end{array}
\]

**Table 1: Joint Bond Configurations**

Table 1 relates our parameters to known proposals. When $\kappa \to 0$ and $\lambda = 1$, countries borrow independently, but country $S$ is liable for $\lambda \theta^R B^R$ bonds issued by country $R$. When $\kappa \to 1$ and $\lambda \to 1$, countries pool the funds of the joint bond at issuance and maintain the joint liability. These two formulations are closest to the original Eurobonds proposal and the Blue bond/Red bond proposal. When $\kappa \to 1$ and $\lambda = 0$, countries split the proceeds from the issuance of the joint bond, but there is no joint liability ex-post — this is similar to the ESBies proposal.

Varying $\theta^R/\theta^S$ changes the degree of involvement of both countries. Importantly, to avoid further behavioral responses, the composition of the joint bond must be independent of the choices made by the governments. For instance, $\theta^i$ could correspond to the long-run share of output of country $i$, that is, $\theta^i = \frac{\mathbb{E}[Y^i]}{\sum \mathbb{E}[Y^i]}$.

**Lenders** There is a unit measure of risk neutral and perfectly competitive lenders who require a given rate of return $1 + \rho$ on every bond they hold. They offer competitive price schedules to both countries at dates 0 and 1. At date 1, country $R$ makes its default decision before lenders offer a new price schedule.\(^5\) The small open economy assumption guarantees that the borrowing behavior of both countries does not affect the world interest rate $1 + \rho$.

Lenders do not recover anything from country $R$ in case of default. However, they do recover a fraction $\lambda$ of the fraction $\theta^R$ of pooled bonds, paid by country $S$. Allowing lenders to partially recover a fraction of their debts in default is straightforward and does not affect our results.

**Budget constraints** Given our assumptions, we can write the date 0 budget constraint of country $i$ as

\[
C^i_0 = Y^i_0 + q^i_0 (\cdot) B^i_0,
\]

\(^5\)This timing assumption eliminates multiplicity problems caused by rollover risk. We explore the alternative timing assumption in which in section 5. See, for instance, Aguiar and Amador (2013) for a clear comparison of both formulations.
where we make explicit the fact that countries understand how their behavior affects the price of new borrowing through the function $q_i^t(\cdot)$, determined in equilibrium. We choose the indicator $D_1^R = \{0, 1\}$ to denote whether country $R$ decides to default at date 1. We equivalently define $D_2^R = \{0, 1\}$ for date 2. If $D_1^R = 1$, it has to be that $D_2^R = 1$.

The budget constraints of country $S$ at dates 1 and 2 respectively are

$$C_1^S = Y_1^S + q_1^S B_1^S - B_0^S - \lambda \theta^R BD_1^R$$

$$C_2^S = Y_2^S - B_1^S - \lambda \theta^R B (1 - D_1^R) D_2^R$$

The last term in both equations corresponds to the additional liability of country $S$ with respect to the bonds pooled with country $R$. When $\lambda = 0$ and there is no joint liability, the last term in both equations (7) and (8) disappears. When $\lambda > 0$, country $S$ must pay back the face value of the $\theta^R B$ bonds issued by country $R$.

The budget constraints of country $R$ at dates 1 and 2 respectively are

$$C_1^R = Y_1^R + [q_1^R(\cdot) B_1^R - B_0^R] (1 - D_1^R) - \delta Y_1^R D_1^R$$

$$C_2^R = Y_2^R - B_1^R (1 - D_1^R) (1 - D_2^R) - \delta Y_2^R [D_1^R + (1 - D_1^R) D_2^R]$$

When country $R$ does not default, it repays the amount owed and borrows, taking into account the price schedule offered. When country $R$ defaults at date 1, $C_1^R = (1 - \delta) Y_1^R$ and $C_2^R = (1 - \delta) Y_2^R$. When it defaults at date 2, $C_2^R = (1 - \delta) Y_2^R$.

For reference, figure 2 illustrates the timeline of choices.

**Equilibrium definition/regularity conditions** An equilibrium, for given levels of the joint borrowing scheme parameters $\{B, \theta^i, \kappa, \lambda, \chi^i\}$ is defined as a set of consumption allocations $C_i^t$ for every date and state, borrowing choices $B_i^t$, default decisions $D_i^t$, and bond prices schedules $q_i^t(\cdot)$ such that countries borrow and default optimally internalizing how their borrowing choices affect debt prices, and lenders competitively

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6If country $S$ decided to guarantee a fraction $\lambda$ of all the debt issued by country $R$, not only the one in the pooled bond, the last term in equations (7) and (8) would read as $\lambda B_0^R D_1^R$ and $\lambda B_1^R D_2^R (1 - D_2^R)$. We do not consider that possibility, and exclusively focus on the joint liability through the pooled bond.
offer price schedules making zero profit.\footnote{Importantly, countries take as given the parameters of the joint borrowing scheme: there is scope to further understand the substantially more complex problem in which countries behave strategically with respect to the choice of $B$ and other parameters by a supranational authority, in a Stackelberg fashion. This is a natural topic for future research.}

As usual at this level of generality, the problem solved by an individual country is not necessarily convex and equilibrium uniqueness is not guaranteed. We provide sufficient conditions for convexity and uniqueness in the appendix and proceed throughout, without further discussion, assuming that the model is well-behaved. We work under the assumption that countries borrow at all dates/states: a sufficiently low discount factor is sufficient condition for this to be the case.

### 2.2 Equilibrium characterization

We now characterize the equilibrium of this economy for a given joint borrowing scheme. We proceed backwards, by sequentially studying default decisions at date 2, borrowing decisions, default decisions and bond pricing at date 1, and borrowing decisions and bond pricing at date 0. We first characterize the behavior of the safe country, then the behavior of the risky country, and finally combine the optimal behavior of both to establish a number of properties of the equilibrium that will be useful for the welfare analysis. Because country $S$ does not face default risk and it issues bonds at the risk-free rate, the problem of country $R$ becomes independent of the problem of country $S$, which greatly simplifies the analysis. To ease notation, we omit the arguments of pricing functions and other functions for the rest of the paper, unless we want to make a special emphasis.

#### 2.2.1 Safe country $S$

Country $S$ faces only two choices: how much to borrow at dates 0 and 1. Given that the bond prices faced by country $S$ do not depend on its own borrowing choices, country $S$ is effectively a price taker. Therefore, two Euler equations characterize its behavior. These are

\begin{align}
q^S_0 U' \left( C^S_0 \right) &= \beta S E_0 \left[ U' \left( C^S_1 \right) \right] \\
q^S_1 U' \left( C^S_1 \right) &= \beta S E_1 \left[ U' \left( C^S_2 \right) \right],
\end{align}

where the consumption of country $S$ is given by equations (6), (7) and (8).\footnote{We could have alternatively assumed that country $S$ has access to a set of complete markets. In that case, there would be as many Euler equations as shock realizations and the consumption of country $S$ would be constant across states. At times, we will refer to that case as a benchmark to interpret some of the results.}

The bond price faced by country $S$ if no default has occurred can be written as

\[ q^S_t = \phi^S_t q^P_t + \left( 1 - \phi^S_t \right) \frac{1}{1 + p}. \]
where the price received per unit of bond pooled is given by
\[ q^{Sp}_{Rt} = \kappa \overline{q}_t + (1 - \kappa) \left( \frac{1}{1 + \rho} + \chi S \Omega_t \right) \],
the pricing wedge is given by
\[ \Omega_t = \lambda \theta R \left( \frac{1}{1 + \rho} - \tilde{q}_R\right) \]
and the joint bond is issued at \( \tilde{q}_t = \theta S \frac{1}{1 + \rho} + \theta R \tilde{q}_R + \Omega_t \). The debt pricing schedule of country \( R \), \( \tilde{q}_R(B^R_t) \) is determined below. When country \( R \) has defaulted, country \( S \) simply borrows at the riskless rate, so \( q_t^S = \frac{1}{1 + \rho} \).

2.2.2 Risky country \( R \)

Date 2 default decision  Assume that country \( R \) did not default at date 1. At date 2, for a given level of debt \( B^R_1 \), country \( R \) defaults for low realizations of \( Y^R_2 \) and repays otherwise. Formally, its optimal default decision at date 2 is
\[
D^R_2 = \begin{cases} 
1, & \text{if } Y^R_2 < \frac{B^R_1}{\delta}, \text{ Default (} D^R_2 = 1) \\
0, & \text{if } Y^R_2 \geq \frac{B^R_1}{\delta}, \text{ No Default (} D^R_2 = 0) 
\end{cases}
\] (13)

Figure 3 illustrates graphically the default decision of country \( R \) at date 2. Country \( R \) can default strategically, when it has enough resources to repay but it chooses not to do so, or be forced to default, when it simply lacks the funds to pay back its debt.

Date 1 default decision, borrowing decision and bond pricing  If country \( R \) does not default at date 1, taking as given the optimal default decision of country \( R \) at date 2, lenders offer the following bond price schedule
\[
\tilde{q}_R(B^R_1) = \frac{\int \int_{Y^R_2} dF_{2|1}(Y^S_2, Y^R_2)}{1 + \rho} = \frac{1 - F_{2|1}(\frac{B^R_1}{\delta})}{1 + \rho},
\] (14)
where \( F_{2|1} (y^S_2, y^R_2) \) denotes the date 1 conditional distribution of the date 2 endowment shocks and \( F^R_{2|1} (\cdot) \) denotes the date 1 conditional marginal cdf of country \( R \) shocks. To ease notation, we do not write the arguments of distribution functions \( F \) when it is clear to which variable we refer to.

The bond price schedule faced by country \( R \) can be written as

\[
q^R_1 = \phi^R_1 q^{RP}_1 + (1 - \phi^R_1) q^R_1 (B^R_1)
\]

where \( q^{RP}_1 = \kappa q'_1 + (1 - \kappa) (q^R_1 + \chi^R \Omega_1) \), the pricing wedge is given by \( \Omega_1 = \lambda \theta^R \left( \frac{1}{1 + \rho} - q^R_1 \right) \) and the joint bond is issued at \( q'_1 = \theta^S \frac{1}{1 + \rho} + \theta^R q^R_1 + \Omega_1 \), where \( q^R_1 (B^R_1) \) is given by equation (14).

Given the bond pricing schedules offered by lenders, country \( R \) must decide how much to borrow at date 1 if it has not defaulted. The optimal level of \( B^R_1 \) is given by the solution to

\[
V^R_{\text{No Default}} (y^R_1, B^R_0) = \max_{B^R_1} \left\{ U_R (y^R_1 + q^R_1 (\cdot) B^R_1 - B^R_0) + \beta_R \int_{y^R_1}^{n^R_1} U_R (y^R_2 - B^R_1) dF^R_{2|1} + \beta_R \int_{y^R_1}^{n^R_1} U_R (y^R_2 (1 - \delta)) dF^R_{2|1} \right\}
\]

(15)

For a well behaved problem, equation (16) characterizes the optimal choice of \( B^R_1 \) as

\[
U'_R (C^R_1) \left( q^R_1 + \frac{\partial q^R_1}{\partial B^R_1} B^R_1 \right) = \beta \int_{y^R_1}^{n^R_1} U'_R (y^R_2 - B^R_1) dF^R_{2|1}
\]

(16)

If country \( R \) defaults at the beginning of date 1, it has no choices to make, so the indirect utility of country \( R \) takes the form

\[
V^R_{\text{Default}} (y^R_1) = U_R (y^R_1 (1 - \delta)) + \beta \int U_R (y^R_2 (1 - \delta)) dF^R_{2|1}
\]

(17)

At the beginning of date 1, country \( R \) compares the indirect utility of repaying, in equation (15), with the indirect utility of defaulting, in equation (17). It then follows that, for a given realization of \( Y^R_1 \) and a predetermined choice of \( B^R_0 \), there exists a threshold boundary, which we define by \( \hat{Y} (B^R_0; \overline{B}) \), such that country \( R \) defaults for realizations of \( Y^R_1 \) such that \( Y^R_1 < \hat{Y} (B^R_0; \overline{B}) \) and vice versa. Formally,

\[
D^R_1 = \begin{cases} 
1, & \text{if } Y^R_1 < \hat{Y} (B^R_0; \overline{B}) \quad \text{Default } (D^R_1 = 1) \\
0, & \text{if } Y^R_1 \geq \hat{Y} (B^R_0; \overline{B}) \quad \text{No Default } (D^R_1 = 0)
\end{cases}
\]

(18)

We fully characterize \( \hat{Y} (B^R_0; \overline{B}) \) in the appendix and show that

\[
\frac{\partial \hat{Y}}{\partial B^R_0} > 0 \quad \text{and} \quad \frac{\partial \hat{Y}}{\partial \overline{B}} < 0
\]

(19)

Intuitively, holding everything else constant, country \( R \) defaults more often when it has borrowed more.
Holding everything else constant, country R defaults less often when the level of joint borrowing is higher. This result relies on the fact that $\frac{\partial V_{1R}}{\partial B} > 0$, which is a crucial feature for the desirability of implementing a joint borrowing agreement.

Figure 4 illustrates the shape of the default region and the comparative statics in equation (19).

**Date 0 borrowing decision and bond pricing** Taking as given the optimal default decision of country R at date 1, lenders offer the following bond price schedule:

$$q_{0}^{R} (B_{0}^{R}; B) = \frac{\int_{Y_{1}^{R}}^{Y_{1}^{R}} \left( \bar{Y} (B_{0}^{R}; B) \right) dF_{1} (Y_{1}^{S}, Y_{1}^{R})}{1 + \rho} = \frac{1 - F_{1}^{R} (\hat{Y} (B_{0}^{R}; B))}{1 + \rho},$$

(20)

The bond price schedule faced by country R at date 0 can be written as

$$q_{0}^{R} = \phi_{0}^{R} q_{0}^{RP} + \left( 1 - \phi_{0}^{R} \right) q_{0}^{R},$$

where $q_{0}^{RP} = \kappa q_{0}^{I} + \left( 1 - \kappa \right) \left( q_{0}^{R} + \chi^{R} \Omega_{0} \right)$, the pricing wedge is given by $\Omega_{0} = \lambda \theta^{R} \left( \frac{1}{1 + \rho} - q_{0}^{R} \right)$ and the joint bond is issued at $q_{0}^{J} = \theta^{S} \frac{1}{1 + \rho} + \theta^{R} q_{0}^{R} + \Omega_{0}$, where $\hat{q}_{0}^{R} (B_{0}^{R}; B)$ is given by equation (14).

Given the bond pricing schedules offered by lenders, country R must decide how much to borrow at date 0. The optimal level of $B_{0}^{R}$ is given by the solution to

$$V_{0}^{R} = \max_{B_{0}^{R}} \left\{ U_{R} (Y_{0}^{R} + q_{0}^{R} B_{0}^{R}) + \beta_{R} \left[ \int_{Y_{1}^{R}}^{Y_{1}^{R}} V_{1D}^{R} (B_{0}^{R}, Y_{1}^{R}) dF_{R} (Y_{1}^{R}) + \int_{\hat{Y} (B_{0}^{R}; B)}^{\hat{Y} (B_{0}^{R}; B)} V_{1N}^{R} (B_{0}^{R}, Y_{1}^{R}) dF_{R} (Y_{1}^{R}) \right] \right\},$$

(21)

where $V_{1N}^{R}$ and $V_{1D}^{R}$ are defined in equations (15) and (17) For a well behaved problem, equation (16)
characterizes the optimal choice of $B^R_0$ as follows

$$U'_R(C^R_0) \left[ q^R_0 + \frac{\partial q^R_0}{\partial B^R_0} B^R_0 \right] = \beta_R \int_{\hat{Y}(B^R_0,B)}^{\hat{Y}_1} U'_R(y^R_1 + q^R_1 B^R_1 - B^R_0) \, dF^R_1,$$

where we use the envelope theorem to be able to write $\frac{\partial V}{\partial B^R_0} = -U'_R(\cdot)$.

### 2.2.3 Comparative statics

An equilibrium is fully characterized by the optimal default thresholds for country $R$, determined by equations (13) and (18), the four Euler equations that determine the borrowing behavior of both countries $R$ and $S$, in equations (11), (12), (16) and (22), and the two individual bond price schedules offered to country $R$, given by equations (14) and (20).

The equilibrium of the model has a tractable structure. The values of $B^R_0$, $B^R_1$ and $\hat{Y}$ are determined in equilibrium independently of the choices of country $S$. Under appropriate conditions on primitives, in particular, for sufficiently large values of the elasticity of intertemporal substitution (EIS) of both countries, we show that, in equilibrium

$$\frac{dB^R_1}{dB} > 0, \quad \frac{dB^R_0}{dB} > 0, \quad \text{and} \quad \frac{d\hat{Y}}{dB} \gtrless 0$$

Country $R$ borrows more at both dates: this is essentially a consequence of the cheaper interest rates faced by country $R$ combined with a high EIS — see the appendix for a full description of all effects. The direction of the default threshold in equilibrium is uncertain. On the one hand, as shown in equation (19), an increase in $B$ reduces the default region at date 1 holding all else constant. On the other hand, an increase in $B^R_0$ makes the event of default more likely. However, it will become clear that it is the partial derivative $\frac{\partial \hat{Y}}{\partial B}$ — and not the total one — the one that matters to assess the desirability of varying the level of $\bar{B}$.

Similarly, under appropriate conditions on primitives, in particular, for sufficiently large values of the elasticity of intertemporal substitution of both countries, we show that, in equilibrium

$$\frac{dB^S_0}{dB} < 0 \quad \text{and} \quad \frac{dB^S_1}{dB} < 0$$

Country $S$ borrows less at both dates: this is essentially a consequence of the higher interest rates faced by country $S$ combined with a high EIS — see again the appendix for a full description of all effects. It will become clear that the effect of varying $\bar{B}$ on the level of borrowing by country $R$ is the one that matters for welfare, but not the effect on the borrowing of country $S$.

**Remark. (Free riding)** The fact country $R$ borrows more when $\bar{B}$ increases is a form of free riding. We establish next how this change in behavior by country $R$ causes welfare losses, although we will also show
that these losses are second order in the vicinity of $\bar{B} = 0$.

**Pricing schedules**

In addition to the changes in the borrowing behavior of countries due to the introduction of the joint bond, the changes in prices caused by changes in the joint borrowing scheme are important inputs for the welfare analysis. We now characterize several comparative statics.

First, we can show that the value of $\frac{\partial q_i^t}{\partial B^i_t}$ is given by

$$
\frac{\partial q_i^t}{\partial B^i_t} = \frac{\partial \tilde{q}_i^t}{\partial B^i_t} + \theta^i \left( \kappa (\tilde{q}_i^t - \tilde{q}_i^J) + (1 - \kappa) \chi^i \Omega_t \right) + \theta^i \bar{B} \left( \kappa \left( \frac{\partial \tilde{q}_i^J}{\partial B} - \frac{\partial \tilde{q}_i^t}{\partial B} \right) + (1 - \kappa) \chi^i \frac{\partial \Omega_t}{\partial B} \right)
$$

(23)

This expression captures the change in total revenue raised by a unit increase in the total issuance of the joint bond, holding constant the level of individual issuance. It is easy to show that $\frac{\partial q^R_t}{\partial B^R_t}$ is always positive for the risky country and that $\frac{\partial q^S_t}{\partial B^S_t}$ negative for the safe country for most choices of $\chi^i$. The sum across countries of $\frac{\partial q_i^t}{\partial B^i_t}$ has an intuitive value and plays an important role in the welfare analysis. Because of its importance, we express the following result as a lemma.

**Lemma 1. (Price impact decomposition)** An increase in the total issuance of joint bonds, holding constant the borrowing choices of both countries, causes a change in the total amount raised of

$$
\sum_i \frac{\partial q_i^t}{\partial B^i_t} = \sum_i \frac{\partial \tilde{q}_i^t}{\partial B^i_t} + \Omega_t + \bar{B} \frac{\partial \Omega_t}{\partial B}
$$

(24)

**Proof.** See appendix.

The induced change in the total amount issued has three components which do not cancel out when adding up. First, an increase in $\bar{B}$ changes the individually issued prices. Second, an increase in $\bar{B}$ increases the amount raised by the pricing wedge. Third, any change in the pricing wedge caused by an increase in $\bar{B}$ increased the amount raised by all the existing units of the joint bond. Note the last two terms can be written as the pricing wedge corrected by a price impact elasticity as $\Omega_t \left( 1 + \frac{\partial \Omega_t}{\partial B^i_t} \right)$. Note also that while $\kappa$ plays an important role in equation (23), it fully cancels out in equation (24).

We also show in the appendix that the direct effect of an increasing in borrowing by country $R$ on the unit price per unit of bond issued by country $S$ is given by

$$
\frac{\partial q^S_t}{\partial B^R_t} = \phi^S_t \left( \kappa \frac{\partial \tilde{q}_i^J}{\partial B^R_t} + (1 - \kappa) \chi^S \frac{\partial \Omega_t}{\partial B^R_t} \right)
$$

Importantly, note that $\frac{\partial q^S_t}{\partial B^R_t} = 0$ when $\bar{B} = 0$. This fact is crucial to show in proposition 2 that the free-riding effects that arise through pooled issuance are second order in the vicinity of $\bar{B} = 0$. Finally, we show in the appendix that $\frac{\partial q^R_t}{\partial B^R_t} > 0$ and $\frac{\partial q^S_t}{\partial B^R_t} = 0$. Because country $S$ never defaults, it is trivial to show that $\frac{\partial q^S_t}{\partial B^S_t} = 0$. 

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3 Welfare analysis

We now study how varying the scale $\overline{B}$ of the joint borrowing scheme affects ex-ante welfare. We initially use a Kaldor-Hicks welfare criterion: we aggregate indirect utilities using a social welfare function with arbitrary Pareto weights but allowing for ex-ante transfers between countries. This approach is equivalent to maximizing the sum of certainty equivalents. In section 4.3, we analyze under which conditions actual Pareto improvements are possible.\(^9\)

We denote by $W^i(\overline{B})$ the equilibrium indirect utility of country $i$ as a function of $\overline{B}$. It can be written as

$$W^i(\overline{B}) \equiv U_i(Y^i_0 + q^i_0B^i_0) + \beta_iE_D[V^i_{1D}(\cdot)] + \beta_iE_N[V^i_{1N}(\cdot)],$$

where pricing, default, and borrowing choices are equilibrium outcomes and date 1 indirect utilities are defined in equations (15) and (17). We use $E_D[\cdot]$ and $E_N[\cdot]$ to respectively denote expectations over the default and no default regions at date 1.

Because lenders are perfectly competitive and make zero profit, they drop out of the social welfare calculation. Therefore, social welfare for arbitrary Pareto weights $\xi^i$, denoted by $W(\overline{B})$, is given by

$$W(\overline{B}) = \sum_i \xi^i W^i(\overline{B}).$$

We now establish the desirability of varying the scale of the joint borrowing scheme by studying the value of $\frac{dW}{d\overline{B}}$. Under the sustained regularity conditions, social welfare is everywhere differentiable.

**Proposition 1. (Welfare effects of varying the scale $\overline{B}$ of the joint borrowing scheme)** a) The change in indirect utility, measured in date 0 dollars, induced by a marginal change in the number of bonds pooled $\overline{B}$ for countries $S$ and $R$ is respectively given by

$$\frac{dW^S}{d\overline{B}} \bigg|_{U^S_i(C^S_0)} = \sum_t \left( \frac{\Pi^S_i \left( \frac{\partial q^S_i}{\partial \overline{B}} B^S_i + \frac{\partial q^S_i}{\partial \overline{B}} B^R_i \frac{d\overline{B}}{d\overline{B}} \right)}{\Pi^S_i \Delta^S_i \Delta_{i,N}} - \lambda \theta^R E_D \left[ \Pi^S_i \right] \right) \right)$$

(25)

$$\frac{dW^R}{d\overline{B}} \bigg|_{U^R_i(C^R_0)} = \sum_t \left( \frac{\Pi^R_i \frac{\partial q^R_i}{\partial \overline{B}} B^R_i}{\Pi^R_i \Delta^R_i \Delta_{i,N}} \right)$$

(26)

where $\Pi^i_j \equiv \frac{(\beta^i_j)U^i_j(C^i_j)}{U^i_j(C^i_0)}$, $\Pi^i_{i,N} \equiv \frac{(\beta^i_j)U^i_j(C^i_j,D^i_j=0)}{U^i_j(C^i_0)}$ and $\Delta^S_i \Delta_{i,N} \equiv \frac{V^S_i}{U^S_i(C^S_j,D^S_j=0)}$.

b) Consequently, the change is social welfare induced by a marginal change in $\overline{B}$ can be decomposed in

\(^9\) A Kaldor-Hicks improvement is a necessary condition for a Pareto improvement.
five channels, as follows

\[
\frac{dW}{dB} = \sum_t E_N \left[ Cov_i \left[ \Pi_i^t, \frac{\partial q_i^t}{\partial B_i^t} \right] \right] + \sum_t E_N \left[ E_i \left[ \Pi_i^t \right] E_i \left[ \frac{\partial q_i^t}{\partial B_i^t} \right] \right]
\]

\[
\text{Risk Sharing} + \text{Default DWL Reduction}
\]

\[
+ \sum_t E_N \left[ \Pi_i^S \frac{\partial \tilde{q}_i^s}{\partial B_i^R} dB_i^R \right] + \sum_t (1 + \rho) E_D^S \left[ \Pi_{i,N,D,Y,D_N}^S \frac{\partial \tilde{q}_i^R}{\partial B_i^R} dB_i^R \right]
\]

\[
\text{Free Riding} + \text{Joint Liability (inframarginal)}
\]

\[
+ \sum_t E_N \left[ \frac{\partial \Omega_i}{\partial B} \right] - \sum_t \lambda \theta R E_D \left[ \Pi_i^S \right] + \sum_t E_N \left[ \frac{\partial \Omega_i}{\partial B} \right] - \sum_t (1 + \rho) E_D^S \left[ \Pi_{i,N,D,Y,D_N}^S \frac{\partial \tilde{q}_i^R}{\partial B} \right]
\]

\[
\text{Joint Liability (marginal) + Default Spillover}
\]

\[
\text{Proof.} \text{ See appendix. Equation (27) follows from combining equations (25) and (26), imposing the Kaldor-}
\]

\[
\text{Hicks assumption and using lemma 1.}
\]

Proposition 1 allows us to provide a sharp description of the effects first-order effects associated with changing the scale of the joint borrowing scheme. We now proceed to describe the five components that determine \( \frac{dW}{dB} \).

We refer to the first term in equation (27) as the risk sharing component. It is formally given by

\[
\sum_t E_N \left[ Cov_i \left[ \Pi_i^t, \frac{\partial q_i^t}{\partial B_i^t} \right] \right]
\]

It corresponds to the expected sum of the cross sectional covariances of individual stochastic discount factors with the pricing impact of the increase on \( B \) on the amount of debt total debt outstanding by country \( i \). This term is positive when the direct change in prices (without accounting for behavioral responses) induced by the change in \( B \) favors the country with relatively higher marginal utility in a given date/state. Note that this term would appear as longs as both countries are not perfectly insured and the terms of their pricing change with the \( B \).

We refer to the second term in equation (27) as the default deadweight loss (DWL) reduction component. It is formally given by

\[
\sum_t E_N \left[ E_i \left[ \Pi_i^t \right] E_i \left[ \frac{\partial q_i^t}{\partial B_i^t} \right] \right]
\]

It corresponds to the expected sum of the (shadow) average riskless discounted — this is where the term \( E_i \left[ \Pi_i^t \right] \) appears — of the cross sectional average of the direct change in prices (without accounting for behavioral responses) induced by the change in \( B \). Because in our model, \( \frac{\partial q_i^t}{\partial B_i^t} = 0 \), only the reduction on the default probability, enters into the term \( E_i \left[ \frac{\partial q_i^t}{\partial B_i^t} \right] \). This term captures the direct reduction in the deadweight loss of default caused by change in the scale of joint borrowing scheme. In a model without default risk or in which there are no deadweight losses associated with default, this term would be zero.
We refer to the third term in equation (27) as the *free riding* component. We could have alternatively use the name *moral hazard component*. It is formally given by

\[
\sum_t \mathbb{E}_N \left[ \pi_t^S \frac{\partial q_t^S}{\partial B_t^i} dB_t^R \right] + \sum_t (1 + \rho) \mathbb{E}_D \left[ \pi_t^S \Delta_t^S \frac{\partial q_t^R}{\partial B_t^i} dB_t^R \right]
\]

This third term encapsulates all the effects in which the behavioral responses of country \( R \) to the policy, given by \( \frac{dB_t^R}{dB} \), enter into the change in social welfare. In our model, only the risky country free rides on the safe country. It does so in two different ways. The first term corresponds to the reduction in the price received by country \( S \) due to the increase in borrowing by country \( R \). Because we have shown that \( \frac{\partial q_t^S}{\partial B_t^i} < 0 \) and that \( \frac{dB_t^R}{dB} > 0 \), this term is necessarily negative. The second term corresponds to how the increase in the likelihood of default caused by the behavioral response of the risky country affects the jump in utility of the safe country \( \Delta_t^S \). With joint liability, we expect the term \( \Delta_t^S \) to be negative, since country \( S \) will be in general worse off ex-post in default states having to pay for the remaining debts of country \( R \). However, when \( \lambda = 0 \), the sign of \( \Delta_t^S \) is less clear, since it may be that country \( S \) is better without the joint borrowing scheme. Both the terms of the free riding component are driven by the fact that country \( R \) does not take into the welfare consequences of more expensive borrowing or the welfare change in default states caused to the safe country. Appropriate corrective policies, as those described in section 5 can ameliorate this free riding/moral hazard effects.

We refer to the fourth term in equation (27) as the *inframarginal* part of the *joint liability* component. It is formally given by

\[
\sum_t \mathbb{E}_N \left[ \mathbb{E}_i \left[ \pi_t^i \right] \Omega_t \right] - \sum_t \lambda \theta_t^R \mathbb{E}_D \left[ \pi_t^S \right]
\]

It accounts for the marginal social value of the joint liability feature of the joint borrowing scheme. The first term captures the increase in value of the joint bond at issuance due to the joint liability guarantee. It is given directly by the pricing wedge. The second term is the loss incurred by country \( S \) when it has to repay such additional liability. When there is no joint liability, so \( \lambda = 0 \), both terms disappear. Even though we might that when \( \Omega_t > 0 \) increase the issuance of joint bond can be welfare improving, it is important to consider whether that increase reflects some future payments in the case default, as it is with the case of joint liability. We’ll show in the next section, that sometimes a positive pricing wedge is sufficient to generate first-order gain from increasing the issuance of the joint bond.

Finally, the fifth term in equation (27) contains both the *marginal* part of the *joint liability* component and what we defined as the *default spillover* component. It is formally given by

\[
\sum_t \mathbb{E}_N \left[ \frac{\partial \Omega_t}{\partial B} B \right] - \sum_t (1 + \rho) \mathbb{E}_D \left[ \pi_t^S \Delta_t^S \frac{\partial q_t^R}{\partial B} \right]
\]

Both the values of \( \frac{\partial \Omega_t}{\partial B} \) and \( \Delta_t^S \) depend on the value of \( \lambda \). The change in the pricing wedge caused by
a change in $\bar{B}$ as well as new liability faced by country $S$ when the default region increases with $\bar{B}$ are the marginal components of the joint liability channel. However, even in the case in which $\lambda = 0$, the difference in indirect utilities for the safe country $\Delta^S_{t,D_N}$ in those states that determine the default boundary for the risky country is in general different from zero. We refer to this phenomenon as the default spillover channel. The fact that the future outlook for the safe country changes depending on whether it belongs to the joint borrowing scheme is crucial to generate this effect. In particular, the fact that the risky country becomes excluded from the international markets is directly causing this effect.

Proposition 2 introduces a new result showing that only the first two out of the five channels we have identified have a first-order effect in welfare when $\bar{B} = 0$. This characterization is particularly important, because it provides a simple test that allows us to give a qualitative to whether a joint borrowing scheme is welfare improving.

**Proposition 2. (First-order effects)** The change in social welfare induced by a marginal change in $\bar{B}$ only has two components when the size of the pooled bond is zero. Formally:

$$
\frac{dW}{d\bar{B}} \bigg|_{\bar{B}=0} = \sum_t \mathbb{E}_N \left[ \text{Cov}_i \left[ \Pi^i_t \bigg|_{\bar{B}=0}, \frac{\partial q^i_t}{\partial \bar{B}} B^i_t \bigg|_{\bar{B}=0} \right] \right] + \sum_t \mathbb{E}_N \left[ \mathbb{E}_i \left[ \Pi^i_t \bigg|_{\bar{B}=0} \right] \mathbb{E}_i \left[ \frac{\partial \tilde{q}^i_t}{\partial \bar{B}} B^i_t \bigg|_{\bar{B}=0} \right] \right] \tag{28}
$$

*Proof.* See appendix. □

Exclusively the risk sharing and the default deadweight loss reduction terms remain non-zero. The free riding effects disappear because $\frac{\partial q^S_t}{\partial \bar{B}} = \Delta^S_{t,D_N} = 0$. The joint liability effects go away when $\lambda = 0$. Finally, the default spillover effect also goes away, since the indirect utility of the safe country in default states is identical to the indirect utility in no default states for the same set of shocks, that is, because $\Delta^S_{t,D_N} = 0$.

Under regularity conditions, showing that $\frac{dW}{d\bar{B}} \bigg|_{\bar{B}=0} > 0$ is sufficient to argue that the introduction of a joint borrowing scheme will be welfare improving. Note that the terms in equations (28) are function of the pricing schedules with weights given by the countries own stochastic discount factor. Both set of variables can be reasonably measured. (more to be added)

## 4 Additional results

Although we have emphasized the welfare effect of introducing a joint borrowing scheme, there are additional implications of having an arrangement of this type that our framework allows us to study. We now discuss the optimal determination of ex-ante corrective instruments, which can eliminate free-riding effects, the optimal determinat ion of country shares in the joint bond, and study the possibility of finding strict Pareto improvements,
4.1 Ex-ante corrective instruments

Until now, countries have been free to choose the level of bond issuance freely. As we have characterized in proposition 1, the change in borrowing by country $R$ induced by increasing the scale of the joint borrowing scheme generates a welfare decreasing causes free-riding effect. We now show allowing for an additional set of corrective instruments can eliminate the equilibrium spillovers.

Formally, we allow for a set of time and state contingent taxes/wedges that correct the behavior of the risky country.

**Proposition 3. (Optimal corrective tax)** The optimal path of corrective taxes $\tau^R_t$ that internalize the free-riding effects caused by the joint borrowing scheme sets the following expression to 0 in equation (27):

\[
\mathbb{E}^N \left[ \Pi^S_t \frac{\partial q^S_t}{\partial B^R_t} dB^R_t \right] + (1 + \rho) \mathbb{E}^S \left[ \Pi^S_t \Delta^S_t \frac{\partial q^R_t}{\partial B^S_t} dB^R_t \right]
\]

As expected, $\tau^R_t = 0$ when $B = 0$. Note also that $\tau^S_t = 0$: there is no reason to distort the behavior of the safe country.

4.2 Optimal determination of country shares $\theta^i$

Up to now, we have focused on and we have taken the weights $\{\theta^i\}$ of every country in the pooled bond as given. When introducing the model, we emphasized the importance that the parameters of the joint borrowing scheme must be predetermined before the countries make their borrowing choices, and these cannot affect. We pointed out that a natural choice for the values of $\theta^R$ and $\theta^S$ are the long run shares of GDP’s.

However, there exists well defined value of country shares that will maximize social welfare. The problem it solves is the following:

\[
\max_{\{\theta^R, \theta^S\}} W \left( \{\theta^R, \theta^S\} \right) - \lambda \left( \sum_i \theta^i - 1 \right)
\]

Under appropriate regularity conditions, the optimal set of weights is given by the solution to the following system of equations:

\[
\frac{dW}{d\theta^i} = \lambda, \quad \forall i
\]

\[
\sum_i \theta^i = 1
\]

Intuitively, it is optimal to adjust $\theta^i$ until the contribution of every country to social welfare is equalized.

(to add characterization)
4.3 Pareto improvements

We have focused on whether a joint borrowing scheme maximizes the sum of ex-ante social welfare measured in dollars. However, finding strict Pareto improvements is a much stronger result.

Therefore, we have shown that implementing transfers can generate a Pareto improvement. We now show that this can be the case even without transfers. To do so, we modify the baseline model and introduce a new set of non-tradable goods. By introducing a new relative prices, changes in the scale of the joint borrowing scheme can generate exchange rate movements which can make more plausible that Pareto improvements can be achieved.

(to be completed)

5 Extensions

To simplify the exposition, we assume that $\lambda = 0$ for all extensions. That is, we assume that the joint bond features pooled issuance but not joint liability. The extension to the general case where $\lambda$ can be positive is conceptually straightforward.

5.1 Frictional lending markets

As discussed in our introductions an often used rationale to justify the introduction of joint borrowing agreements is the presence of some form of departure from the frictionless pricing benchmark. In particular, the presence of a liquidity, a collateral, or a safety premium suggests that some form of intervention in borrowing markets may be desirable.

We retain all the assumptions of the baseline model with a single modification. We assume that individually issued bonds by country $S$ require a riskless rate of return $1 + \rho^S$, while the individually issued bonds by country $R$ require a higher riskless rate of return $1 + \rho^R$. We also assume that the joint bond

This formulation allows us to capture several environments. First, different pricing kernels may arise from liquidity problems or market segmentation. Perhaps the bonds of country $R$ are traded in a less liquid market, and investors demand a liquidity premium. Second, it may be that only the bonds in which country $S$ is involved can be used as collateral. We do not take a strong stance in the paper regarding which is more appropriate, since all of them can be captured through different pricing kernels and will endogenous generate a pricing wedge in the joint bond.

Formally, all these important considerations will be subsumed into the pricing wedge $\Omega_t$. As we show in the appendix, when $\lambda = 0$ but $\rho^R > \rho^S$, we can write $\Omega_t$ as

$$\Omega_t = \theta^R \rho^R - \rho^S q_t^R > 0$$
The characterization of propositions 1 and 2 remains valid in this case. Importantly, the term that we denominated the inframarginal term of the joint liability agreement and that we showed was equal to zero up to a first-order when $B = 0$, is going to positive. Similarly with all the term that involve derivatives of $\Omega$.

(to be expanded)

5.2 Rollover risk

By changing the timing about when the bond price scheduled is offered, our model can incorporate self-fulfilling defaults, as in Calvo (1988) and Cole and Kehoe (2000).

From a normative standpoint, the model with rollover risk can be seen as a microfoundation for the deadweight loss of default $\delta$. It generates first-order welfare losses even when $B = 0$. (to be included)

5.3 Symmetric countries

As we have argued, our baseline model featuring one safe and one risky country seems like a sensible choice to study joint borrowing schemes. However, an alternative natural framework features two risky countries that can default. The case with two (or more) countries is somewhat more complicated because it forces us to use a strategic notion of equilibrium. That said, exactly the same effects from propositions 1 and 2 are the welfare relevant ones. (to be included)

5.4 Additional generalizations

Our main characterizations in propositions 1 and 2 are robust to a number of generalizations. We briefly discuss a number of them.

**Epstein-Zin utility** Allowing for Epstein-Zin allows us to disentangle the effects of intertemporal substitution versus risk aversion. In that case, the welfare of the representative agent for country $i$ is given by:

$$V_i^t = \left[ (1 - \hat{\beta}) (C_0^i)^{1 - \frac{1}{\psi}} + \hat{\beta} \left( \mathbb{E} \left[ (V_{i+1}^t)^{1 - \gamma} \right] \right)^{\frac{1 - \frac{1}{\psi}}{1 - \gamma}} \right]^{\frac{1}{1 - \frac{1}{\psi}}} ,$$

where the parameter $\gamma$ is the coefficient of relative risk aversion and $\psi$ represents the elasticity of intertemporal substitution for a given nonstochastic consumption path.

Allowing for a more general utility specification only changes the value of the stochastic discount factor. (to be included)

**Endogenous output** All characterizations remain valid as long as there are no frictions. (to be included)
General pricing kernel A simple way to introduce credit spreads that include non trivial variation in the price of risk is to assume that lenders price assets using a state price density that embeds their risk preference. Under that approach, all the analysis remains valid if we assume that lenders price assets using a risk neutral measure $G$ which can differ from the actual physical measure $F$. This formulation does not allow for feedback from default decisions to risk pricing. For instance,

$$\tilde{q}_1 = \frac{1 - G_i \left( \frac{B_0^i}{B^{i*}} \right)}{1 + \rho}$$

where $F$ differs from $G$. (to be included)

5.5 Complete market benchmark

In the baseline model, we have assumed that country $S$ only has access to the safe bond. Alternatively, we could have assumed that country $S$ has access to a complete set of markets to hedge all risks. In that case, country $S$ values cash flows as a lender. (to be included)

6 Quantitative assessment

Our baseline model extends naturally to an infinite horizon environment. In particular, the problem solved by the risky country is closely related to the canonical sovereign default problem, as in Eaton and Gersovitz (1981) or Aguiar and Gopinath (2006) and Arellano (2008). The problem solved by the safe country is equally close to the canonical individual income fluctuation problem, discussed at length in any modern macroeconomics text, as Ljungqvist and Sargent (2004). Importantly, the safe country/risky country formulation allows us to solve both problems sequentially. First, we solve the problem of the country $R$. Subsequently, we solve the problem of country $S$, given the behavior of country $R$.

Recursive Formulation Risky Country We directly provide the recursive formulation of the problems solved by the safe and the risky country.

The risky country $R$ solves

$$V^R_o (B^R; Y^R) = \max_{\{c,d\}} \{ V^R_c (B^R; Y^R), V^R_d (Y^R) \}$$

where

$$V^R_c (B^R; Y^R) = \max_{B^{R'}} \left\{ U_R (Y^R + B^R - q (B^{R'}, y) B^{R'}) + \beta_R \int_{Y^{R'}} V^R_o (B^{R'}, Y^{R'}) f (Y^{R'}, Y^R) dY^{R'} \right\}$$

$$V^R_d (Y^R) = U_i (Y^{R}_{def}) + \beta_R \int_{Y^{R'}} \left[ \theta V^R_o (0, Y^{R'}) + (1 - \theta) V^R_d (Y^{R'}) \right] f (Y^{R'}, Y^R) dY^{R'}$$
where $Y^R_{def} = (1-\delta) Y^R$. Note that $C^R = Y^R + B^R - q^R (B^{R'}, Y^R) B^{R'}$.

The outcomes of this problem are a) a policy function for the risky country $B^{R'} = B^{R'} (B^R; Y^R)$ and b) a pricing function for the individual debt of the risky country $\tilde{q}^R (B^{R'}, Y^R)$.

\begin{equation}
q^R_t = \phi_i^R q^{RP}_t + (1 - \phi_i^R) \tilde{q}^R_t,
\end{equation}

\begin{equation}
\phi_i^R \equiv \frac{\theta^R B_t}{B_t^R}
\end{equation}

\begin{equation}
q^{RP}_t = \kappa \tilde{q}^R_t + (1 - \kappa) (\tilde{q}^R_t + \chi^R \Omega_t),
\end{equation}

\begin{equation}
\tilde{q}^R_t = \sum_i \theta_i \tilde{q}^R_i + \lambda \theta^R (\tilde{q}^S_t - \tilde{q}^R_t)
\end{equation}

Where

\begin{equation}
\tilde{q}^S_t = \frac{1}{1 + \rho}
\end{equation}

New parameters: $\theta^R = 0.5$, $\lambda = 0.5$, $\kappa = 0.5$, $\chi^R = 1$. $B$ should be something small.

**Recursive Formulation Safe Country** Taking as given the behavior of country $R$, the safe country $S$ solves

\begin{equation}
V^S_o (B^S; B^R, Y^R, Y^S) = \max_{B^S} \left\{ U_S \left( Y^S + B^S - q^S \right) + \beta \int V^S_c \left( B^S; B^R, Y^R, Y^S \right) + \int V^S_d \left( B^S; B^R, Y^R, Y^S \right) \right\}
\end{equation}

\begin{equation}
V^S_c (B^S; B^R, Y^R, Y^S) = \max_{B^S} \left\{ U_S \left( Y^S + B^R - q^R (B^{R'}, Y^R) B^{R'} \right) + \beta \int V^S_c \left( B^S; B^R, Y^R, Y^S \right) + \int V^S_d \left( B^S; B^R, Y^R, Y^S \right) \right\}
\end{equation}

\begin{equation}
V^S_d (B^S; B^R, Y^R, Y^S) = \max_{B^R} \left\{ U_S \left( Y^S + B^S - q^S (B^{R'}, Y^R) B^{R'} \right) + \beta \int \theta V^S_d \left( B^S; B^R, Y^R, Y^S \right) + \int m (1 - \theta) V^S (B^S; B^R) \right\}
\end{equation}

**Solution**

(in progress)

### 7 Conclusion

We have developed a novel systematic analysis of the positive and normative implications of joint borrowing schemes among sovereigns. Using a framework that decouples the implications of pooled issuance and joint liability, we have provided a general decomposition of the welfare effects of introducing a joint borrowing scheme. A change in the scale of a joint borrowing scheme has five first-order effects in welfare. It affects risk-sharing among countries. It directly targets the deadweight losses associated with default. It causes free riding effects. It redistributed resources across periods and states through its joint liability nature. And
it causes a direct default spillover because the countries that decide to default do not take into account the welfare of the ones who don’t do so.

Although our framework has captured the fundamental tradeoffs associated with a joint borrowing scheme, there is scope to further research in this area. For instance, allowing countries to issue debt at different maturities to understand whether joint borrowing agreement are more desirable for long or short maturity bonds seems like a natural avenue for further research. Similarly, a model with richer interactions between the government and financial sectors and the financial and real sectors could generate additional relevant insights. We leave these topics for future research.
Appendix

Proofs: Section 2

We now fully characterize the solution to the problems of both safe and risky countries. We also establish a number of useful properties to show the propositions in the text regarding the value functions/indirect utility of both countries.

Country \( S \)

Date 1

We define the value function/indirect utility of country \( S \) at date 1 when \( D_1^R = 1 \) by \( V_{1D}^S \). Formally,

\[
V_{1D}^S = \max_{B_1^S} \left\{ U_S \left( Y_1^S + q_1^S B_1^S - B_0^S - \lambda \theta R B \right) + \beta_S \int U_S \left( Y_2^S - B_1^S \right) dF_{2|1} \right\}
\]

As described in equation (11) in the text, the country \( S \) borrows at date 1 optimally according to

\[
U'_S(C_1^S) q_1 - \beta_S \int U'_S(Y_2^S - B_1^S) dF_{2|1} = 0
\]

Because \( q_1^S = \frac{1}{1+\rho} \), it is easy to establish that

\[
\frac{\partial V_{1D}^S}{\partial B} = -\lambda \theta R U'_S(C_1^S) \quad \text{and} \quad \frac{\partial V_{1D}^S}{\partial B_0^S} = -U'_S(C_1^S)
\]

Analogously, we define the value function/indirect utility of country \( S \) at date 1 when \( D_1^R = 0 \) by \( V_{1|N}^S \). Formally,

\[
V_{1|N}^S = \max_{B_0^S} \left\{ U_S \left( Y_1^S + q_1^S B_1^S - B_0^S \right) + \beta_S \left( \int \int_{Y_2^S} U_S \left( Y_2^S - B_1^S - \lambda \theta R B \right) dF_{2|1} + \int \int_{Y_2^S} U_S \left( Y_2^S - B_1^S \right) dF_{2|1} \right) \right\}
\]

As described in equation (12) in the text, the country \( S \) borrows at date 0 optimally according to

\[
U'_S(C_1^S) q_1 - \beta_S \int \int_{Y_2^S} U'_S(Y_2^S - B_1^S - \lambda \theta R B) dF_{2|1} - \beta_S \int \int_{Y_2^S} U'_S(Y_2^S - B_1^S) dF = 0
\]

We can write the value of \( \frac{\partial V_{1|N}^S}{\partial B_0^S} \) as follows

\[
\frac{\partial V_{1|N}^S}{\partial B_0^S} = -U'_S(C_1^S)
\]
We can write the value of $\frac{\partial V_{1N}^S}{\partial B}$ as follows

$$
\frac{\partial V_{1N}^S}{\partial B} = \left[ U'_S (C_1^S) q_1 - \beta_S \int \int_{y_2}^{y_3} U'_S (Y^S_2 - B^S_1 - \lambda \theta^R B) dF_{2|1} - \beta_S \int \int_{y_2}^{y_3} U'_S (Y^S_2 - B^S_1) dF \right] \frac{dV_{1N}^S}{dB} 
+ U'_S (C_1^S) \frac{dq^S_1}{dB} B^S_1 - \lambda \theta^R \beta_S \int \int_{y_2}^{y_3} U'_S (Y^S_2 - B^S_1 - \lambda \theta^R B) dF_{2|1} 
+ \beta_S \int (U_S (Y^S_2 - B^S_1 - \lambda \theta^R B) - U_S (Y^S_2 - B^S_1)) f_{2|1}^{R_S} \left( \frac{B^R_1}{\delta} \right) \frac{dB^R_1}{dB} \frac{1}{\delta} dF_{2|1}
$$

Using the optimality condition in equation (12), and exploiting the fact that

$$
\frac{dq^S_1}{dB} B^S_1 = \frac{\partial q^S_1}{\partial B^R} \frac{dB^R_1}{dB} B^S_1 + \frac{\partial q^S_1}{\partial B} B^S_1,
$$

because $\frac{\partial q^S_1}{\partial B^R} = 0$, we can write

$$
\frac{\partial V_{1N}^S}{\partial B} = U'_S (C_1^S) \frac{dq^S_1}{dB} B^S_1 + U'_S (C_1^S) \frac{\partial q^S_1}{\partial B^R} \frac{dB^R_1}{dB} B^S_1 - \lambda \theta^R \beta_S \int \int_{y_2}^{y_3} U'_S (Y^S_2 - B^S_1 - \lambda \theta^R B) dF_{2|1} 
+ \beta_S \int (U_S (Y^S_2 - B^S_1 - \lambda \theta^R B) - U_S (Y^S_2 - B^S_1)) f_{2|1}^{R_S} \left( \frac{B^R_1}{\delta} \right) \frac{dB^R_1}{dB} \frac{1}{\delta} dF_{2|1}
$$

The first and second terms respectively correspond to the direct and indirect (through the behavior of country $R$) pricing impact. The third term corresponds to the infra-marginal increase in the liability of country $S$. The fourth and last term corresponds to a marginal increase in the liability of country $S$, due to an increase in the default region. This last term can be better understood as a first order approximation:

$$
U_S (Y^S_2 - B^S_1 - \lambda \theta^R B) - U_S (Y^S_2 - B^S_1) \approx -U'_S (Y^S_2 - B^S_1) \lambda \theta^R B
$$

Date 0

The value function/indirect utility of country $S$ at date 0 corresponds to $W^S(B)$. Formally,

$$
W^S(B) = \max_{B_{0}^S} U_S (Y_0^S + q_0^S B_{0}^S) + \beta_S \int \int_{y_2}^{y_3} V_{1D}^S dF + \beta_S \int \int_{y_2}^{y_3} V_{1N}^S dF
$$

We can write the value of $\frac{dW^S}{dB}$ as follows

$$
\frac{dW^S}{dB} = U'_S (C_0^S) \left( \frac{dq_0^S}{dB} B_{0}^S + q_0^S \frac{dB_{0}^S}{dB} \right) + \beta_S \int \int_{y_2}^{y_3} \left( \frac{\partial V_{1D}^S}{\partial B} + \frac{\partial V_{1D}^S}{\partial B_{0}^S} \frac{dB_{0}^S}{dB} \right) dF + \int \int_{y_2}^{y_3} \left( \frac{\partial V_{1N}^S}{\partial B} + \frac{\partial V_{1N}^S}{\partial B_{0}^S} \frac{dB_{0}^S}{dB} \right) dF
+ \beta_S \int (V_{1D}^S - V_{1N}^S) f (\hat{Y} (\cdot)) \frac{dY}{dB} dF^S
$$
Using the optimality condition of country $S$ in equation (11), we can write

$$\frac{dW^S}{dB} = U'_S(C_0^S) \frac{\partial q_0^S}{\partial B} B_0^S + U'_S(C_0^S) \frac{\partial q_0^S}{\partial B} B_0^S dB^R_0 dB + \beta_S \left( \int \int y_f^{\gamma} \frac{\partial V^S_{1D}}{\partial B} dF + \int \int y_f^{\gamma} \frac{\partial V^S_{1N}}{\partial B} dF \right)$$

$$+ \beta_S \int (V^S_{1D} - V^S_{1N}) f(Y(\cdot)) \frac{dY}{dB} dF^S$$

The sign $V^S_{1D} - V^S_{1N}$ is not clear. On the one hand, country $S$ is worse off when facing the additional liability $\theta^R \lambda$, which increases the relative value of $V^S_{1N}$. On the other hand, country $R$ may gain from exiting the joint borrowing scheme, which increases the relative value of $V^S_{1D}$. See the additional discussion below and in the text.

Further substituting the values of $\frac{\partial V^S_{1D}}{\partial B}$ and $\frac{\partial V^S_{1N}}{\partial B}$, and dividing by date 0 marginal utility, we can write

$$\frac{dW^S}{U'_S(C_0^S)} = \frac{\partial q_0^S}{\partial B} B_0^S + \frac{\partial q_0^S}{\partial B} B_0^S dB^R_0 dB + \int \int y_f^{\gamma} \left( \Pi^S_t \left( \frac{\partial d_1^S}{\partial B} B^S_1 + \frac{\partial d_1^S}{\partial B} B^S_1 dB^R_1 \right) \right) dF$$

$$- \lambda \theta^R \left( \int \int y_f^{\gamma} \Pi^S_t dF + \int \int y_f^{\gamma} \left( \int y_f^{\gamma} \Pi^S_t dF \right) dF \right)$$

$$+ \lambda \theta^R \left( \int \int y_f^{\gamma} \Pi^S_{1N} \left( Y^S_2 - B^S_1 - \lambda \theta^R \lambda \right) - U_S(Y^S_2 - B^S_1) \frac{d\hat{q}_0^R}{dB} \right)$$

where we define the stochastic discount factor of country $S$ as follows

$$\Pi^S_t = \frac{(\beta_S) U'_S(C_0^S)}{U'_S(C_0^S)} \text{ and } \Pi^S_{1N} = \frac{(\beta_S) U'_S(C_0^S) (D_1^S = 0)}{U'_S(C_0^S)}$$

We also use the fact that the total derivative in the change in prices by country $R$ equally has the information about the change in the likelihood of default, as follows:\textsuperscript{10}

$$\frac{d\hat{q}_0^R}{dB} (1 + \rho) = f(Y(\cdot)) \frac{dY}{dB}$$

$$\frac{d\hat{q}_0^R}{dB} (1 + \rho) = f_{R[S]} (\frac{B^R}{\delta}) \frac{dB^R_1}{dB}$$

Finally, we define the difference in utilities between defaulting or not, normalized by date 0 marginal utility, by

$$\Delta^S_{1N} = \frac{V^S_{1D} - V^S_{1N}}{U'_S(C_0^S) (D_1^S = 0)}.$$ The value of $\Delta^S_{1N}$ is related to the curvature of the utility function. And we also use the fact that $\frac{d\hat{q}_0^R}{dB} = \frac{\partial q_0^R}{\partial B} + \frac{\partial q_0^R}{\partial B} dB^R.$

Collapsing all the terms, we can write

$$\frac{dW^S}{U'_S(C_0^S)} = \sum \left( \mathbb{E}_N \left[ \Pi^S_t \left( \frac{\partial q_0^S}{\partial B} B^S_1 + \frac{\partial q_0^S}{\partial B} B^S_1 dB^R_1 \right) \right] - \lambda \theta^R \mathbb{E}_D \left[ \Pi^S_t \right] (1 + \rho) \mathbb{E}_D \left[ \Pi^S_{1N} \Delta^S_{1N} \left( \frac{\partial q_0^R}{\partial B} + \frac{\partial q_0^R}{\partial B} dB^R \right) \right] \right),$$

which corresponds to equation (25) in the text.

\textsuperscript{10}If lenders recover a positive amount in default, a correction for the recovery rate on default states is needed.
Country R

Date 1

We define the value function/indirect utility of country R at date 1 when \( D_1^R = 1 \) by \( V_{iD}^R (Y_1^R) \). Formally,

\[
V_{iD}^R (Y_1^R) = U_R (Y_1^R (1 - \delta)) + \beta_R \int U_R (Y_2^R (1 - \delta)) dF_{2|1}^R
\]

It is easy to establish that

\[
\frac{\partial V_{iD}^R}{\partial B} = \frac{\partial V_{iD}^R}{\partial B_0} = 0
\]

Analogously, we define the value function/indirect utility of country R at date 1 when \( D_1^R = 0 \) by \( V_{iN}^R (B_0^R, Y_1^R, \overline{B}) \). Formally,

\[
V_{iN}^R = \max_{B_1^R} \left\{ U_R (Y_1^R + q_1^RB_1^R - B_0^R) + \beta_R \left( \int_{Y_1^R}^{\overline{Y}_1} U_R (Y_2^R - B_1^R) dF_{2|1} + \int_{\overline{Y}_1}^{\overline{Y}_2} U_R (Y_2^R (1 - \delta)) dF_{2|1} \right) \right\}
\]

We can write the value of \( \frac{\partial V_{iN}^R}{\partial B_0^R} \) as follows

\[
\frac{\partial V_{iN}^R}{\partial B_0^R} = -U'_R (C_1^R)
\]

We can write the value of \( \frac{\partial V_{iN}^R}{\partial B_1^R} \) as follows

\[
\frac{\partial V_{iN}^R}{\partial B_1^R} = U'_R (C_1^R) \frac{\partial q_1^R}{\partial B_1^R} B_1^R
\]

Where the first line already uses equation (13) to set to zero the term corresponding the difference in utilities between default and no default states. The second line, exploits equation (16), and uses the fact that

\[
\frac{dq_1^R}{dB_1^R} = \frac{\partial q_1^R}{\partial B_1^R} dB_1^R + \frac{\partial q_1^R}{\partial B} dB_1^R
\]

Only the direct price effect through the partial derivative of \( \overline{B} \) remains as first-order.

Date 0

At date 0, we define the value function/indirect utility of country S at date 0 corresponds to \( W^S (\overline{B}) \). Formally,

\[
W^R (\overline{B}) = \max_{B_0^R} U_R (Y_0^R + q_0^RB_0^R) + \beta_R \int \hat{Y}^R dF + \beta_R \int \hat{Y}^R V_{iD}^R dF
\]

Using the optimality condition of country S in equation (11), we can write the value of \( \frac{\partial W^S}{\partial B} \) as follows

\[
\frac{dW^R}{dB} = U'_R (C_0^R) \frac{\partial q_0^R}{\partial B_0^R} B_0^R + \beta_R \int \hat{Y}^R U'_R (C_1^R) \frac{\partial q_1^R}{\partial B} B_1^R dF, \tag{33}
\]

which corresponds to equation (25) in the text.
Equilibrium properties

Comparative statics \( \hat{Y} (B_{0}^{R}, \bar{B}) \)

The default threshold \( \hat{Y} (B_{0}^{R}, \bar{B}) \) is implicitly defined by the equality

\[
V_{1N}^{R} (B_{0}^{R}, Y_{1}^{R}, \bar{B}) - V_{1D}^{R} (B_{0}^{R}, Y_{1}^{R}) = 0
\]

By the implicit function theorem, to show that \( \frac{\partial V_{1N}^{R}}{\partial B} < 0 \), it is sufficient to prove that \( \frac{\partial V_{1N}^{R}}{\partial B} (B_{0}^{R}, Y_{1}^{R}, \bar{B}) > 0 \). Because \( \frac{\partial V_{1N}^{R}}{\partial B} = U_{R}^{'} (C_{1}^{R}) \frac{\partial q_{1}^{R}}{\partial B} B_{0}^{R} \) and given that we show below that \( \frac{\partial q_{1}^{R}}{\partial B} > 0 \), this is sufficient to conclude that \( \frac{\partial V_{1N}^{R}}{\partial B} < 0 \). In a more general case, the condition becomes

\[
\frac{\partial V_{1N}^{R}}{\partial B} > \frac{\partial V_{1N}^{R}}{\partial B}.
\]

Similarly, to show that \( \frac{\partial \hat{Y}}{\partial B_{0}^{R}} > 0 \) it is sufficient to prove that \( \frac{\partial V_{1N}^{R}}{\partial B_{0}^{R}} (B_{0}^{R}, Y_{1}^{R}, \bar{B}) > 0 \). We have already shown that

\[
\frac{\partial V_{1N}^{R}}{\partial B_{0}^{R}} = -U_{R}^{'} (C_{1}^{R}), \text{ so } \frac{\partial \hat{Y}}{\partial B_{0}^{R}} > 0.
\]

Comparative statics country \( R \)

Importantly, \( q_{1}^{R} \) depends on the borrowing and default decision of country \( R \), but not of country \( S \). Therefore, to determine \( \frac{db_{0}^{R}}{db} \), \( \frac{db_{0}^{R}}{db} \) and \( \frac{\partial \hat{Y}}{\partial B_{0}^{R}} \), we solve the system:

\[
\left( q_{1}^{R} + \frac{\partial q_{1}^{R}}{\partial B_{1}^{R}} B_{1}^{R} \right) U^{'} (Y_{1}^{R} + q_{1}^{R} B_{1}^{R} - B_{0}^{R}) = \beta_{R} \int_{\frac{\hat{Y}^{R}}{\bar{B}}}^{\frac{\hat{Y}^{R}}{B_{0}}} U_{R}^{'} (Y_{2}^{R} - B_{1}^{R}) dF_{2|1}^{R}
\]

\[
U_{R}^{'} (Y_{0}^{R} + q_{0}^{R} B_{0}^{R}) \left[ q_{0}^{R} + \frac{\partial q_{0}^{R}}{\partial B_{0}^{R}} B_{0}^{R} \right] = \beta_{R} \int_{\frac{\hat{Y}^{R}}{B_{0}}}^{\frac{\hat{Y}^{R}}{\bar{B}}} U_{R}^{'} (Y_{1}^{R} - B_{0}^{R}) dF_{1}^{R}
\]

\[
V_{1N}^{R} (B_{0}^{R}, Y_{1}^{R}, \bar{B}) - V_{1D}^{R} (B_{0}^{R}, Y_{1}^{R}) = 0
\]

It can be shown, that for sufficiently large values of the EIS, all three comparative statics are positive. There are income, substitution and direct effects. A high EIS makes the substitution effects to dominate. (to include full derivation)

Comparative statics country \( S \)

Whenever country \( R \) has defaulted, the problem of the safe country is independent of \( \bar{B} \). If there has been no default, the values of \( B_{0}^{S} \) and \( B_{1}^{S} \) are given by the solution to equations (11) and (12), written here in more detail as

\[
q_{1}^{S} (\bar{B}) U_{S}^{'} (Y_{1}^{S} + q_{1}^{S} (\bar{B}) B_{1}^{S} - B_{0}^{S}) = \beta_{S} \int_{\frac{\hat{Y}^{S}}{B_{0}}}^{\frac{\hat{Y}^{S}}{\bar{B}}} U_{S}^{'} (Y_{2}^{S} - B_{1}^{S} - \lambda \theta^{R} \bar{B}) dF_{2|1} + \beta_{S} \int_{\frac{\hat{Y}^{S}}{\bar{B}}}^{\frac{\hat{Y}^{S}}{B_{0}}} U_{S}^{'} (Y_{2}^{S} - B_{1}^{S}) dF_{2|1}
\]

\[
q_{0}^{S} (\bar{B}) U_{S}^{'} (Y_{0}^{S} + q_{0}^{S} (\bar{B}) B_{0}^{S}) = \beta_{S} \int_{\frac{\hat{Y}^{S}}{B_{0}}}^{\frac{\hat{Y}^{S}}{\bar{B}}} U_{S}^{'} (Y_{1}^{S} + q_{1}^{S} B_{1}^{S} - B_{0}^{S} - \lambda \theta^{R} \bar{B}) dF_{1} + \beta_{S} \int_{\frac{\hat{Y}^{S}}{\bar{B}}}^{\frac{\hat{Y}^{S}}{B_{0}}} U_{S}^{'} (Y_{1}^{S} + q_{1}^{S} B_{1}^{S} - B_{0}^{S}) dF_{1}
\]

The value of \( \frac{dbh_{0}^{S}}{db} \) and \( \frac{dbh_{0}^{S}}{db} \) are fully determined by implicitly differentiating both equations. need to solve first the properties of the country \( R \) solution, because \( q_{1}^{S} \) is a function of the borrowing and default choices of country \( S \). It can be shown, that for sufficiently large values of the EIS, all three comparative statics are negative. There are income, substitution and direct effects, as in the case of country \( R \). A high EIS makes the substitution effects to dominate. (to include full derivation)
Price schedules

Individual and joint bond pricing

From equations (14) and (20), we can write

\[
\frac{\partial q_i^R}{\partial B} = -f^R \left( \frac{\hat{Y}}{1 + \rho} \right) \frac{\partial \hat{Y}}{\partial B} > 0 \quad \text{and} \quad \frac{\partial q_i^F}{\partial B} = -f^F \left( \frac{\hat{Y}}{1 + \rho} \right) \left( \frac{\partial \hat{Y} (B_0^R; B)}{\partial B} + \frac{\partial \hat{Y} (B_0^F; B)}{\partial B} \right)
\]

\[
\frac{\partial q_i^R}{\partial B} = 0 \quad \text{and} \quad \frac{\partial q_i^F}{\partial B} = -f^R \left( \frac{1}{1 + \rho} \right) \frac{d B_i^R}{d B} < 0
\]

Borrowing scheme

We now describe the properties of the price schedules offered by lenders. We explicitly write the functional dependence of the \( q_i^r \) as \( q_i^r (B_i^r, B_{i-1}^r, B) \). Similarly, we can also write \( q_i^R (B_i^R, B_{i-1}^R, B) \), \( q_i^F (B_i^F, B_{i-1}^F, B) \) and \( \Omega_i (B_i, B_{i-1}) \).

Using the following representation of \( \tilde{q}_i^r \), which follows from equation (4) in the text,

\[
\tilde{q}_i^r = \sum_i \theta_i \tilde{q}_i^r + \Omega_i,
\]

we can write

\[
\frac{\partial \tilde{q}_i^r}{\partial B} = \sum_i \theta_i \frac{\partial \tilde{q}_i^r}{\partial B} + \frac{\partial \Omega_i}{\partial B}
\]

We use the definitions of \( q_i^r \) and \( q_i^p \), restated here

\[
q_i^r = \tilde{q}_i^r + \phi_i (q_i^p - \tilde{q}_i^r)
\]

\[
q_i^p = \kappa q_i^r + (1 - \kappa) (\tilde{q}_i^r + \chi_i \Omega_i)
\]

In that case, we can write the total derivative as

\[
\frac{d q_i^r}{d B} = \frac{\partial q_i^r}{\partial B} + \frac{\partial q_i^p}{\partial B_i} dB_i + \frac{\partial q_i^F}{\partial B_{i-1}} dB_{i-1}
\]

The two key inputs to our normative questions are \( \frac{\partial q_i^r}{\partial B_i} \) and \( \frac{\partial q_i^r}{\partial B_{i-1}} \). We focus on the partial derivative \( \frac{\partial q_i^r}{\partial B_i} \) first:

\[
\frac{\partial q_i^r}{\partial B_i} = \frac{\partial q_i^r}{\partial B_i} + \frac{\partial \phi_i}{\partial B_i} (q_i^p - \tilde{q}_i^r) + \phi_i \left( \frac{\partial q_i^p}{\partial B_i} - \frac{\partial \tilde{q}_i^r}{\partial B_i} \right)
\]

Which implies that

\[
\frac{\partial q_i^r}{\partial B_i} = \frac{\partial q_i^r}{\partial B_i} + \frac{\partial \phi_i}{\partial B_i} (q_i^p - \tilde{q}_i^r) + \phi_i \left( \frac{\partial q_i^p}{\partial B_i} - \frac{\partial \tilde{q}_i^r}{\partial B_i} \right)
\]

\[
= \frac{\partial q_i^r}{\partial B_i} + \theta_i (q_i^p - \tilde{q}_i^r) + \phi_i \left( \frac{\partial q_i^p}{\partial B_i} - \frac{\partial \tilde{q}_i^r}{\partial B_i} \right)
\]

So we can find \( \frac{\partial q_i^p}{\partial B} \) as

\[
\frac{\partial q_i^p}{\partial B} = \kappa \frac{\partial q_i^r}{\partial B} + (1 - \kappa) \left( \frac{\partial q_i^r}{\partial B} + \chi_i \frac{\partial \Omega_i}{\partial B} \right)
\]

\[
= \kappa \frac{\partial q_i^r}{\partial B} + (1 - \kappa) \frac{\partial q_i^r}{\partial B} + (1 - \kappa) \chi_i \frac{\partial \Omega_i}{\partial B}
\]
\[
\frac{\partial q_i^p}{\partial B_i} - \frac{\partial q_i}{\partial B_i} = \kappa \left( \frac{\partial q_i^l}{\partial B_i} - \frac{\partial q_i}{\partial B_i} \right) + (1 - \kappa) \chi' \frac{\partial \Omega_i}{\partial B_i}
\]

Combining results

\[
\frac{\partial q_i^l}{\partial B_i} B_i = \frac{\partial q_i^l}{\partial B_i} B_i^l + \frac{\partial q_i}{\partial B_i} B_i^l (q_i^p - q_i^l) + \phi_i B_i^l \left( \frac{\partial q_i^p}{\partial B_i} - \frac{\partial q_i}{\partial B_i} \right)
\]

\[
= \frac{\partial q_i^l}{\partial B_i} B_i^l + \theta_i \left( \kappa (q_i^l - q_i^l) + (1 - \kappa) \chi' \Omega_i \right) + \theta_i B_i \left( \kappa \left( \frac{\partial q_i^l}{\partial B_i} - \frac{\partial q_i}{\partial B_i} \right) + (1 - \kappa) \chi' \frac{\partial \Omega_i}{\partial B_i} \right)
\]

Where we can use the fact that \(\frac{\partial \Omega_i}{\partial B_i} = 0\).

We want to write

\[
\sum_i \frac{\partial q_i}{\partial B_i} B_i = \sum_i \frac{\partial q_i^l}{\partial B_i} B_i^l + \sum_i \theta_i \left( \kappa (q_i^l - q_i^l) + (1 - \kappa) \chi' \Omega_i \right) + \sum_i \theta_i B_i \left( \kappa \left( \frac{\partial q_i^l}{\partial B_i} - \frac{\partial q_i}{\partial B_i} \right) + (1 - \kappa) \chi' \frac{\partial \Omega_i}{\partial B_i} \right)
\]

\[
= \sum_i \frac{\partial q_i^l}{\partial B_i} B_i^l + \left( \kappa \left( \frac{\partial q_i^l}{\partial B_i} - \sum_i \theta_i \phi_i \right) + (1 - \kappa) \chi' \Omega_i \sum_i \theta_i \phi_i \right) + \sum_i \theta_i B_i \left( \kappa \left( \frac{\partial q_i^l}{\partial B_i} - \sum_i \theta_i \phi_i \right) + (1 - \kappa) \chi' \frac{\partial \Omega_i}{\partial B_i} \sum_i \theta_i \phi_i \right)
\]

This results says that

\[
\sum_i \frac{\partial q_i}{\partial B_i} B_i = \sum_i \frac{\partial q_i^l}{\partial B_i} B_i^l + \Omega_i + B \frac{\partial \Omega_i}{\partial B_i}
\]

Note that we can write \(\Omega_i + B \frac{\partial \Omega_i}{\partial B_i} = \Omega_i \left( 1 + \frac{\partial \Omega_i}{\partial B_i} \right)\). In our baseline model, \(\Omega_i = \lambda \theta \Theta \varepsilon \left[ \frac{1}{1 + \rho} \right] \)

Second, the sensitivity of bond prices faced by country \(i\) with respect to borrowing by country \(-i\) is given by:

\[
\frac{\partial q_i}{\partial B_i} = \phi_i \frac{\partial q_i^p}{\partial B_i}, \quad \text{where} \quad \frac{\partial q_i^p}{\partial B_i} = \kappa \frac{\partial q_i}{\partial B_i} + (1 - \kappa) \chi' \frac{\partial \Omega_i}{\partial B_i}
\]

Combining both

\[
\frac{\partial q_i}{\partial B_i} = \phi_i \left( \kappa \frac{\partial q_i}{\partial B_i} + (1 - \kappa) \chi' \frac{\partial \Omega_i}{\partial B_i} \right)
\]

Note that \(\frac{\partial q_i}{\partial B_i} = 0\) when \(B = 0\).

Third, the sensitivity of bond prices with respect to country \(i\) own borrowing is given by:

\[
\frac{\partial q_i}{\partial B_i} = \frac{\partial q_i^l}{\partial B_i} + \frac{\partial q_i}{\partial B_i} (q_i^p - q_i^l) + \phi_i \left( \frac{\partial q_i^p}{\partial B_i} - \frac{\partial q_i}{\partial B_i} \right)
\]

Using the fact that \(\frac{\partial \phi_i}{\partial B_i} B_i = -\phi_i\), we can write:

\[
\frac{\partial q_i}{\partial B_i} B_i = \frac{\partial q_i}{\partial B_i} B_i^l - \phi_i (q_i^p - q_i^l) + \phi_i \left( \frac{\partial q_i^p}{\partial B_i} B_i^l - \frac{\partial q_i}{\partial B_i} B_i^l \right)
\]

And

\[
q_i + \frac{\partial q_i^l}{\partial B_i} B_i = q_i^l + \frac{\partial q_i}{\partial B_i} B_i^l + \phi_i \left( \frac{\partial q_i^p}{\partial B_i} B_i^l - \frac{\partial q_i}{\partial B_i} B_i^l \right), \quad (34)
\]

where

\[
\frac{\partial q_i^p}{\partial B_i} = \kappa \frac{\partial q_i}{\partial B_i} + (1 - \kappa) \left( \frac{\partial q_i}{\partial B_i} + \chi' \frac{\partial \Omega_i}{\partial B_i} \right)
\]
Equation (34) is the marginal revenue per unit of bond issued by country \( i \). It is a key input to the decisions of country \( R \). As expected, when \( \overline{B} = 0 \), the marginal revenue is identical to the individual one.

For completeness, we show that our formulation of the joint borrowing scheme satisfies \( \sum_i \theta_i^t q_i^{tP} = \tilde{q}_i^t \) :

\[
\sum \theta_i^t q_i^{tP} = \sum \theta_i^t (\kappa \tilde{q}_i^t + (1 - \kappa) (\tilde{q}_i^t + \chi' \Omega_t)) \\
= \kappa \tilde{q}_i^t + (1 - \kappa) (\sum \theta_i^t \tilde{q}_i^t + \Omega_t \sum \theta_i \chi^i) \\
= \kappa \tilde{q}_i^t + (1 - \kappa) \tilde{q}_i^t = \tilde{q}_i^t
\]

**Additional results**

**Regularity conditions** The problem of country \( S \) is a standard problem. It is well-known that such problem is well-behaved for standard preferences. The problem of country \( R \) may fail to be well-behaved due to the default option. However, for standard preferences and an under mild restrictions on the distribution of shocks, the problem of country \( R \) is also well-behaved. (to be extended)

**Case with \( B_i^t < \theta_i^t \overline{B} \)** If we had allowed for \( B_i^t \leq \theta_i^t \overline{B} \), equation (2) would read as \( \phi_i^t = \min \{ 1, \theta_i^t \overline{B} \} \) and \( \tilde{q}_i^t \) as

\[
\tilde{q}_i^t = \frac{1}{1 + \rho} \sum_i \Omega_i \tilde{q}_i^t \text{, where } \Omega_i = \frac{\min \{ B_i^t, \theta_i^t \overline{B} \} \overline{B}}{\sum_i \min \{ B_i^t, \theta_i^t \overline{B} \}}.
\]

**Proofs: Section 3**

**Proposition 1. (Welfare effects of varying the scale \( \overline{B} \) of the joint borrowing scheme)**

a) Equations (25) and (26) are derived in equations (32) and (33) in section 2 of the appendix.

b) Simply adding up equations (25) and (26) in the text, and using the fact that \( \xi^t' U' (C^i_0) \) is constant, because of the Kaldor-Hicks criterion, gives

\[
\frac{dW}{dB} = \sum_i \mathbb{E} \left[ \sum_i \Pi_i^t \frac{\partial q_i^t}{\partial B} B_i^t \right] + \sum_i \mathbb{E} \left[ \left( \Pi_i^t \frac{\partial q_i^t}{\partial B} B_i^t \right) B^t \right] - \sum_i \lambda \theta_i^t \mathbb{E} \left[ \Pi_i^t \right] + \sum_i \left( 1 + \rho \right) \mathbb{E} \left[ \Pi_i^t \left( \Pi_i^t \frac{\partial q_i^t}{\partial B} + \Pi_i^t \frac{\partial q_i^t}{\partial B} \right) \right]
\]

We use the fact that

\[
\mathbb{E} \left[ \Pi_i^t \frac{\partial q_i^t}{\partial B} B_i^t \right] = \mathbb{E} \left[ \Pi_i^t \right] \mathbb{E} \left[ \frac{\partial q_i^t}{\partial B} B_i^t \right] + \mathbb{Cov} \left( \Pi_i^t, \frac{\partial q_i^t}{\partial B} B_i^t \right)
\]

And also lemma 1, that is,

\[
\mathbb{E} \left[ \frac{\partial q_i^t}{\partial B} B_i^t \right] = \sum_i \frac{\partial q_i^t}{\partial B} B_i^t + \Omega_i + \overline{B} \frac{\partial \Omega_i}{\partial B}
\]

Combining both terms, we find that

\[
\frac{dW}{dB} = \sum_i \mathbb{E} \left[ \left( \mathbb{E} \left[ \Pi_i^t \right] \left( \sum_i \frac{\partial q_i^t}{\partial B} B_i^t + \Omega_i + \overline{B} \frac{\partial \Omega_i}{\partial B} \right) \right) + \mathbb{Cov} \left( \Pi_i^t, \frac{\partial q_i^t}{\partial B} B_i^t \right) \right]
\]

**Proposition 2. (First-order effects at \( \overline{B} = 0 \))**

a) (To be included)
Proofs: Section 4

The price of the joint bond when it is priced according to the SDF of country $S$ is given by

$$\tilde{q}_t = \theta R^{1+\rho R} + \theta S^{\tilde{q}_t}$$

The pricing wedge $\Omega_t$ takes the value

$$\Omega_t = \theta R^{1+\rho R} + \theta S^{\tilde{q}_t} - \theta R^{\tilde{q}_t} - \theta S^{\tilde{q}_t}$$

$$= \left(\frac{1+\rho R}{1+\rho S} - 1\right) \theta R^{\tilde{q}_t}$$

$$= \left(\frac{\rho R - \rho S}{1+\rho S}\right) \theta R^{\tilde{q}_t}$$
References


Arellano, Cristina, and Yan Bai. 2013. “Linkages across sovereign debt markets.”


